Northern Illinois University, PHY 571, Fall 2006

Part II: Special Relativity

Last updated on September 26, 2006 (report errors to piot@fnal.gov)
EM field of point charge moving at constant velocity

Start with Maxwell’s equations:

\[ \nabla \cdot \vec{D} = \rho, \quad \nabla \cdot \vec{H} = 0, \]
\[ \nabla \times \vec{E} + \partial_t \vec{B} = 0, \quad \text{and} \quad \nabla \times \vec{H} - \partial_t \vec{D} = \vec{J}. \]

Write in terms of electromagnetic potentials, \( \vec{A} \) and \( \Phi \):

\[ \vec{B} = \nabla \times \vec{A} \Rightarrow \nabla \times (\vec{E} + \partial_t \vec{A}) = 0 \Rightarrow \vec{E} = -\nabla \Phi - \partial_t \vec{A} \]
\[
\frac{1}{\mu} \nabla \times \vec{B} - \epsilon \partial_t \vec{E} = \vec{J} \Rightarrow \nabla \times \vec{B} - \mu \epsilon \partial_t \vec{E} = \mu \vec{J} \]
\[ \Rightarrow \nabla \times (\nabla \times \vec{A}) + \mu \epsilon (\nabla \partial_t \Phi + \partial_t^2 \vec{A}) = \mu \vec{J} \]

Note \( \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \)

*for pretty movies of moving charge check Shintake-san’s homepage SCSS-FEL: http://www-xfel.spring8.or.jp*
\[-\nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A} + \mu \epsilon \partial_t \Phi) + \mu \epsilon \partial_t^2 \vec{A} = \mu \vec{J}\]

\[\nabla \cdot \vec{A} + \mu \epsilon \partial_t \Phi = 0\text{ in Lorenz gauge.}\]

\[\nabla^2 \vec{A} - \mu \epsilon \partial_t^2 \vec{A} = -\mu \vec{J}\quad [\text{JDJ, Eq. (6.16)}]\]

\[\nabla \cdot \vec{D} = \rho \Rightarrow -\nabla^2 \Phi - \partial_t \nabla \cdot \vec{A} = \frac{\rho}{\epsilon}\]

\[\nabla^2 \Phi - \mu \epsilon \partial_t^2 \Phi = -\frac{\rho}{\epsilon}\quad [\text{JDJ, Eq. (6.15)}]\]

For a source moving at constant velocity, \(\vec{v}\): \(\rho = \rho(\vec{x} - \vec{v}t)\) and \(\vec{J} = \vec{v} \rho(\vec{x} - \vec{v}t)\). We then have to solve a set of inhomogeneous d’Alembert equations: \(\Box f = g(\vec{x} - \vec{v}t)\).
Consider the case $\vec{v} = vz \Rightarrow f(\vec{x} - \vec{v}t) = (x, y, z - vt) = f(x, y, \zeta)$ with $\zeta \equiv z - vt$. Then

\[
\begin{align*}
\partial_z f & \rightarrow \frac{\partial \zeta}{\partial z} \partial_\zeta f = \partial_\zeta f \quad (3) \\
\partial_t f & \rightarrow \frac{\partial \zeta}{\partial t} \partial_\zeta f = -v \partial_\zeta f \quad (4)
\end{align*}
\]

$\Rightarrow \Box f \rightarrow \left( \partial_x^2 + \partial_y^2 + \partial_\zeta^2 - \mu \epsilon v^2 \partial_\zeta^2 \right) f = \left( \partial_x^2 + \partial_y^2 + \gamma^{-2} \partial_\zeta^2 \right) f.$ \quad (5)

with $\gamma \equiv \frac{1}{\sqrt{1 - \mu \epsilon v^2}}$. Let $z' = \gamma \zeta \Rightarrow \partial_\zeta = \frac{\partial z'}{\partial \zeta} \partial_{z'} = \gamma \partial_{z'}$:

\[
\left( \partial_x^2 + \partial_y^2 + \partial_{z'}^2 \right) f(x, y, \gamma^{-1} z') = g(x, y, \gamma^{-1} z'). \quad (6)
\]

Point charge $\Rightarrow \rho(\vec{x} - \vec{v}t) \rightarrow \delta(x)\delta(y)\delta(\gamma^{-1} z') = \gamma \delta(x)\delta(y)\delta(z') = \gamma \delta(\vec{x}').$
Results:
\[ A \rightarrow A \hat{z} \ (A_x = A_y = 0); \]
\[ \nabla^2_{x'} A = -\gamma \mu q v \delta(x'), \quad \nabla^2_{x'} \Phi = -\frac{q}{\epsilon} \delta(x'). \]  \hspace{1cm} (7)

Solve by inspection:
\[ \nabla^2_{x'} \left( \frac{1}{|x'|} \right) = -4\pi \delta(x') \Rightarrow \begin{cases} A = \frac{\gamma \mu q v}{4\pi R}, \\ \Phi = \frac{\gamma q}{4\pi \epsilon R}, \end{cases} \]  \hspace{1cm} (8)

where \( R \equiv \sqrt{x^2 + y^2 + \gamma^2 (z - vt)^2}. \)

Now we can calculate \( \overrightarrow{E} = -\nabla \Phi - \partial_t \overrightarrow{A}: \)
\[ \overrightarrow{E} = -\frac{\gamma q}{4\pi \epsilon} \left( \hat{\nabla} + \mu \epsilon v \partial_t \hat{z} \right) \frac{1}{R} \]
\[ = \frac{\gamma q}{4\pi \epsilon R^3} \left[ x \hat{x} + y \hat{y} + \gamma^2 (z - vt)(1 - \mu \epsilon v^2) \hat{z} \right] \]  \hspace{1cm} (9)
\[ \vec{E} = \frac{\gamma q}{4\pi \epsilon R^3} [x\hat{x} + y\hat{y} + (z - vt)\hat{z}] \]  

(10)

Convert to spherical coordinates:

\[ x^2 + y^2 = r^2 \sin^2 \theta, \quad z - vt = r \cos \theta. \]

\[ \Rightarrow R^2 = r^2 (\sin^2 \theta + \gamma^2 \cos^2 \theta) \]
\[ = \gamma^2 r^2 \left( 1 + \frac{1 - \gamma^2}{\gamma^2} \sin^2 \theta \right) = \gamma^2 r^2 (1 - \mu \epsilon v^2 \sin^2 \theta), \]

\[ \Rightarrow E = \frac{\gamma q}{4\pi \epsilon \gamma^3 r^3 (1 - \mu \epsilon v^2 \sin^2 \theta)^{3/2}} \]
\[ = \frac{q}{4\pi \epsilon r^2 (1 - \mu \epsilon v^2 \sin^2 \theta)^{3/2}}. \]

(11)
Note: In vacuum, take $\mu \varepsilon \to \mu_0 \varepsilon_0 = c^{-2}$, and then

$$\vec{E} = \frac{q}{4\pi \varepsilon r^2} \frac{\vec{r}}{\gamma^2 (1 - \beta^2 \sin^2 \theta)^{3/2}}.$$  [JDJ, Eq. (11.154)]  \hspace{1cm} (12)

Note that $E(\pi/2)/E(0) = \gamma^3 \Rightarrow$ field lines are “squashed” orthogonal to the direction of motion.

Also we can find $\vec{B} = \vec{\nabla} \times \vec{A}$:

$$\vec{A} = \mu \varepsilon \Phi \vec{v} \Rightarrow \vec{B} = \mu \varepsilon \vec{\nabla} \times (\Phi \vec{v}) = \mu \varepsilon [\vec{\nabla} \Phi \times \vec{v} + \Phi \vec{\nabla} \times \vec{v}]$$

$$\Rightarrow \vec{B} = \mu \varepsilon \vec{\nabla} \Phi \times \vec{v}.$$  \hspace{1cm} (13)

$$\vec{v} \times \vec{E} = -\vec{v} \times (\vec{\nabla} \Phi + \partial_t \vec{A}) = \vec{\nabla} \Phi \times \vec{v}.$$  

$$\vec{B} = \mu \varepsilon \vec{v} \times \vec{E}, \text{ or } \vec{B} = \frac{\mu}{4\pi R^3} \gamma q \vec{v} (x\hat{y} - y\hat{x}).$$
Further reductions [toward JDJ Eq. (11.152)]:

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2} \frac{\hat{r}}{\gamma^2 (1 - \beta^2 \sin^2 \theta)^{3/2}}$$

(14)

$$\sin \theta = \frac{b}{r} = \frac{b}{\sqrt{b^2 + v^2 t^2}}.$$

$$1 - \beta^2 \sin^2 \theta = 1 - \frac{\beta^2 b^2}{b^2 + (vt)^2} = \frac{b^2 + v^2 t^2 - \beta^2 b^2}{b^2 + v^2 t^2} = \frac{(1 - \beta^2) b^2 + v^2 t^2}{b^2 + v^2 t^2}$$

$$1 - \beta^2 \sin^2 \theta = \frac{b^2 + \gamma^2 v^2 t^2}{\gamma^2 r^2} \Rightarrow \gamma r \sqrt{1 - \beta^2 \sin^2 \theta} = \sqrt{b^2 + \gamma^2 v^2 t^2}$$

Finally

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 (b^2 + \gamma^2 v^2 t^2)^{3/2}} \Rightarrow \vec{E}_\perp = \frac{q}{4\pi\varepsilon_0 (b^2 + \gamma^2 v^2 t^2)^{3/2}} \gamma b \hat{x}.$$

(15)
Consider a charge $q_0$ comoving with $q$ at velocity $\vec{v}$. The force imparted to $q_0$ by $q$ is

$$F = q_0(\vec{E} + \vec{v} \times \vec{B})$$

$$= q_0[\vec{E} + \mu \epsilon \vec{v} \times (\vec{v} \times \vec{E})]$$

$$\Rightarrow \vec{F} = q_0 \left[ (1 - \mu \epsilon v^2) \vec{E} + \mu \epsilon v^2 E_z \hat{z} \right]$$

$$= q_0 \left( \frac{1}{\gamma^2} \vec{E} + \frac{\gamma^2 - 1}{\gamma^2} E_z \hat{z} \right) = q_0 \left[ \frac{1}{\gamma^2} (\vec{E} - E_z \hat{z}) + E_z \hat{z} \right]$$

$$\Rightarrow \vec{F} = q_0 \left[ \frac{1}{\gamma^2} \vec{E} \perp + \vec{E} \parallel \right] \quad (16)$$

The self-magnetic field of $q$ cancels its self-electric field to within a factor $1/\gamma^2$. 


The squashing of the E-field of a moving charge, as it corresponds to the equation of motion, is suggestive of the Lorentz contraction, and thus indicative that electrodynamics is invariant under Lorentz transformations.

Invariance of proper time:
spherical waves propagate such that \((\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2 = c^2\). If \(c\) is the same in all inertial reference frames (postulate), then

\[
\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2 = c^2
\]
So, we write:

\[ c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 \quad \text{for photons.} \quad (17) \]

This holds true in any inertial coordinate system. More generally we can define the proper time:

\[ d\tau^2 \equiv dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2). \quad (18) \]

In SR, the proper time is an invariant – all inertial observers measure the same \( d\tau \). Note that:

\[ d\tau^2 = dt^2(1 - \beta^2) = \frac{1}{\gamma^2}dt^2; \quad (19) \]

\( \vec{\beta} \equiv \frac{1}{c}\vec{v}; \quad \vec{v} = \text{velocity measured in lab frame } (\mathcal{O}), \quad dt = \text{period between “ticks” of clock in lab frame.} \)

When \( \vec{v} = 0 \), \( d\tau = dt \Rightarrow d\tau = \text{period between “ticks” of clock comoving with } \mathcal{O}'. \) Every inertial observer measure the same value for this time interval: it is a scalar – a fixed physical quantity!
particle in O emits light
spherical waves propagate
v of O' measured by O

left: notation for previous slides. right: light cone, [AB] is time-like [AC] is space-like.
If $\delta t$ represents the period between ticks of $O'$'s clock, then $O$ sees it ticks with period:

$$dt = \gamma \delta t \quad (20)$$

This is "time dilatation": $O$ thinks $O'$'s clock runs slow.

**Minkowski metric and Lorentz transformations:**

Let $x^0 \equiv ct$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$ [so $\vec{x}^i \equiv \vec{X}$ (i=1,2,3)]. Then we can write:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad (21)$$

with $\alpha, \beta = 0, 1, 2, 3$ and $g_{\alpha\beta}$ is the Minkowski metric:

$$g_{\alpha\beta} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \quad (22)$$

standard convention: Use Greek indices to represent sums from 0-3 and Latin indices for sum from 1-3.
The Lorentz transformation matrix from stationary observer $O$ to moving observer $O'$ is the “boost matrix” [JDJ, Eq.(11.98)] ($\Lambda^\alpha_\gamma \Lambda^\beta_\delta g_{\alpha\beta} = g_{\gamma\delta}$):

$$\Lambda^\nu_\mu = \begin{pmatrix}
\gamma & -\gamma \beta_x & -\gamma \beta_y & -\gamma \beta_z \\
-\gamma \beta_x & 1 + \left(\frac{\beta_x}{\beta}\right)^2 (\gamma - 1) & \frac{\beta_x \beta_y}{\beta^2} (\gamma - 1) & \frac{\beta_x \beta_z}{\beta^2} (\gamma - 1) \\
-\gamma \beta_y & \frac{\beta_x \beta_y}{\beta^2} (\gamma - 1) & 1 + \left(\frac{\beta_y}{\beta}\right)^2 (\gamma - 1) & \frac{\beta_y \beta_z}{\beta^2} (\gamma - 1) \\
-\gamma \beta_z & \frac{\beta_x \beta_z}{\beta^2} (\gamma - 1) & \frac{\beta_y \beta_z}{\beta^2} (\gamma - 1) & 1 + \left(\frac{\beta_z}{\beta}\right)^2 (\gamma - 1)
\end{pmatrix} \tag{23}$$

provided the coordinates of $O$ and $O'$ are aligned. The the Lorentz transformation from $O$ and $O'$ is:

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta. \tag{24}$$

Note $\Lambda^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}$. If the coordinate axes are not aligned then the transformation is the product of $\Lambda^\alpha_\beta$ and a rotation matrix.
The principle of SR is: All laws of physics must be invariant under Lorentz transformations. “Invariant” ↔ Laws retain the same mathematical form and numerical constant (scalar) retain the same value.

Particle dynamics in SR

Define the “4-velocity”: \( u^\alpha \equiv \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds} \):

\[
\begin{align*}
\quad u^0 &= c\frac{dt}{d\tau} = \gamma c \\
\quad u^i &= \frac{1}{c} \frac{dx^i}{d\tau} = c\frac{dt}{d\tau} \frac{dx^i}{dt} = \gamma \beta^i
\end{align*}
\]

Then

\[
\quad u_\alpha^\alpha = g_{\alpha\beta}u^\beta u^\alpha = \gamma^2 - \gamma^2 \beta^2 = c^2
\]

is an invariant.
Moreover since $d\tau$ is an invariant and $x^\alpha$ conforms to Lorentz transformation, then

$$u'^\alpha = \Lambda^\alpha_\beta u^\beta$$

(27)

$\Rightarrow u^\alpha$ satisfies the Principle of SR.

Define the 4-momentum of a particle:

$$P_\alpha \equiv mu^\alpha$$

(28)

$\Rightarrow P^0 = \gamma mc = E/c, \ P^i = p^i;\ E = \text{total energy, } p^i = \text{ordinary 3-momentum, } m = \text{particle's rest mass. Then}$

$$P_\alpha P^\alpha = m^2 u_\alpha u^\alpha = m^2 c^2 = E/c^2$$

(29)

is an invariant. The fundamental dynamical law for particle interactions in SR is that 4-momentum is conserved in any Lorentz frame.
Note that

\begin{equation}
P'^\alpha = \Lambda^\alpha_\beta P^\beta \tag{30}
\end{equation}

also one has:

\begin{equation}
P_\alpha P^\alpha = g_{\alpha\beta}P^\beta P^\alpha = \frac{E^2}{c^2} - p^2 \tag{31}
\end{equation}

\begin{equation}
\frac{E^2}{c^2} - p^2 = (mc)^2 \tag{32}
\end{equation}

\Rightarrow \quad E = \sqrt{(pc)^2 + (mc^2)^2}.

The kinetic energy of a particle is \( T = E - mc^2 \):

\begin{equation}
T = \sqrt{(pc)^2 + (mc^2)^2} - mc^2 \tag{33}
\end{equation}
Example: Consider the reaction (one neutron at rest)

\[ n + n \rightarrow n + n + n + \bar{n} \]

What is the minimum required energy for the incoming \( n \) that will enable the reaction to proceed?

At threshold the four neutrons are at rest in the lab frame, so that the 4-momentum conservation requires:

\[
P_1^\alpha + P_2^\alpha = P_f^\alpha \quad \Rightarrow \quad (P_1^\alpha + P_2^\alpha)(P_{1\alpha} + P_{2\alpha}) = P_f^\alpha P_{f\alpha} = 16(m_n c)^2
\]

\[
P_1^\alpha P_{1\alpha} + 2P_1^\alpha P_{2\alpha} + P_2^\alpha P_{2\alpha} = 2(m_n c)^2 + 2P_1^\alpha P_{2\alpha}
\quad \Rightarrow \quad P_1^\alpha P_{2\alpha} = 7(m_n c)^2. \quad (35)
\]

\[
P_1^\alpha P_{2\alpha} = g_{\alpha\beta}P_1^\alpha P_2^\beta = g_{00}P_1^0 P_2^0 = m_n c \frac{E}{c}
\quad E = 7m_n c^2. \quad (36)
\]
Photon emission and absorption:

Let $u_{e,a}^\alpha = 4$-velocity of emitter, absorber, respectively. $E_{e,a} = photon$ energy measured by emitter, absorber, respectively. $P^\alpha = 4$-momentum of photon.

Then look at

$$P_\alpha u^\alpha = g_{\alpha\beta} P^\beta u^\alpha$$

$$= P^0 u^0 - P^i u^i = c P^0 = E.$$ 

1st term $u^0 = c$, 2nd term $u^i = 0$ in either emitter’s or absorber’s frame.
So \( E = p_\alpha u^\alpha \) is the photon energy measured by an observer with 4-velocity \( u^\alpha \). The expression is the same in any frame, including accelerating frame! So:

\[
E_e = P_\alpha u^\alpha_e \quad \text{and}, \quad E_a = P_\alpha u^\alpha_a
\]

Example: “Absorber” is rotating with angular velocity \( \Omega \) on a circle of radius \( R_A \). Emitter is stationary – Let’s find \( E_a/E_e \)

In emitter’s frame: \( c^2d\tau = g_{\alpha\beta}dx^\alpha dx^\beta \), the emitter is stationary so \( u^\alpha_e = (c, 0, 0, 0) \).

In absorber frame:

\[
\begin{align*}
(c^2(d\tau)^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\
&= c^2dt^2 - v^2dt^2 = c^2dt^2 - R_A^2 d\phi^2 \\
d\tau^2 &= dt^2 - \frac{R_A^2}{c^2} d\phi^2
\end{align*}
\]

(37)
From previous slide (two slides ago) we have:

\[
\frac{E_a}{E_e} = \frac{P_\alpha u_\alpha^a}{P_\alpha u_\alpha^e} = \frac{P_0 u_\alpha^0 - \vec{p} \hat{u}_a^i}{P_0 c} = \frac{P_0 u_\alpha^0 - |\vec{p}| |\hat{u}_a^i| \cos \theta}{P_0 c}
\] (38)

But \(\cos \theta = \sin \phi\), and for photons \(P_\alpha P_\alpha = (P^0)^2 - |\vec{P}|^2 = 0 \Rightarrow |\vec{P}| = P^0\). Thus

\[
\frac{E_a}{E_e} = \frac{u_\alpha^0 - |\vec{u}_a^i| \sin \phi}{c}.
\] (39)

But,

\[
|\vec{u}_a^i| = R_A \frac{d\phi}{d\tau} = \frac{R_A \Omega}{\sqrt{1 - (R_A \Omega/c)^2}}; \quad u_\alpha^0 = \frac{c}{\sqrt{1 - (R_A \Omega/c)^2}}
\] (40)
Doppler shift ($\phi = 90^\circ$):

\[
\frac{\lambda_e}{\lambda_a} = \frac{1 - (R_A \Omega/c)}{\sqrt{1 - (R_A \Omega/c)^2}} = \sqrt{\frac{1 - (R_A \Omega/c)}{1 + (R_A \Omega/c)}}
\]  

(42)
Covariance of Electrodynamics

We wish to proceed in keeping with Jackson’s notation, which involves switching from SI units to Gaussian units.

\[
\begin{align*}
\nabla \cdot \mathbf{D} & = \rho \\
\nabla \times \mathbf{H} - \partial_t \mathbf{D} & = \mathbf{J} \\
\nabla \times \mathbf{E} + \partial_t \mathbf{B} & = 0 \\
\n\nabla \mathbf{B} & = 0 \\
\mathbf{F} & = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0
\end{align*}
\]

(43)

Conversions:

\[
\begin{align*}
\frac{\mathbf{E}}{\sqrt{4\pi\varepsilon_0}} & = \mathbf{E}^{SI}; \quad \sqrt{\frac{\varepsilon_0}{4\pi}} \mathbf{D} & = \mathbf{D}^{SI}; \quad \sqrt{4\pi\varepsilon_0}\rho^G(\mathbf{J}^G, q^G) = \rho^{SI}(\mathbf{J}^{SI}, q^{SI}); \\
\sqrt{\frac{\mu_0}{4\pi}} \mathbf{B}^G & = \mathbf{B}^{SI}; \quad \frac{\mathbf{H}^G}{\sqrt{4\pi\mu_0}} & = \mathbf{H}^{SI}; \quad \varepsilon_0\epsilon^G = \epsilon^{SI}; \quad \mu_0\mu^G = \mu^{SI}; \quad c = (\mu_0\varepsilon_0)^{-1/2}.
\end{align*}
\]
As one check, look at the Lorentz force:

\[
\vec{F}^G = q^G (\vec{E}^G + \frac{1}{c} \vec{v} \times \vec{B}^G)
\]

\[
\Rightarrow \vec{F}^{SI} = \frac{q^{SI}}{\sqrt{4\pi\varepsilon_0}} \left[ \sqrt{4\pi\varepsilon_0} \vec{E}^{SI} + \sqrt{\mu_0\varepsilon_0} \vec{v} \times \sqrt{\frac{4\pi}{\mu_0}} \vec{B}^{SI} \right]
\]

\[
= q^{SI} (\vec{E}^{SI} + \vec{v} \times \vec{B}^{SI}).
\]

The conversion from “Maxwell G” to “Maxwell SI” works the same way. So we do have a prescription to go from Gaussian results to SI results and vice versa.
Current density as a 4-vector:
Consider a system of particles with positions $\vec{x}_n(t)$ and charges $q_n$. 
The current and charge densities are:

$$\vec{J}(\vec{x}, t) = \sum_n q_n \delta^3(\vec{x} - \vec{x}_n(t)) \vec{x}_n(t),$$
$$\rho(\vec{x}, t) = \sum_n q_n \delta^3(\vec{x} - \vec{x}_n(t))$$

Note that for any smooth function $f(\vec{x})$, $\delta^3$ acts as:

$$\int_{-\infty}^{\infty} f(\vec{x}) \delta^3(\vec{x} - \vec{y}) = f(\vec{y})$$

(44)

if we define $J^0 \equiv c \rho$ and $J^i(\vec{x}) = \sum_n q_n \delta^3(x^i - x^i_n(t))d_t x^i_n(t)$, then using $\delta^4$ function we can write:

$$J^\alpha(x) = \int \sum_n q_n \delta^4(x^\alpha - x^\alpha_n(t)) dx^0 \frac{dx^\alpha_n(t)}{dt}$$

(45)

$J^\alpha$ is a function of $x^\alpha \rightarrow$ it is a Lorentz invariant; $J^\alpha$ is a 4-vector. $J^\alpha \equiv (c \rho, \vec{J})$. Also note $J^\alpha \equiv \rho u^\alpha$
Equation of charge continuity:

\[
\vec{\nabla} \cdot \vec{J}(\vec{x}, t) = \sum_n q_n \frac{\partial}{\partial x^i_n} \delta^3(\vec{x} - \vec{x}_n(t)) \frac{dx^i_n(t)}{dt} \\
= - \sum_n q_n \frac{\partial}{\partial x^i_n} \delta^3(\vec{x} - \vec{x}_n(t)) \frac{dx^i_n(t)}{dt} \\
= - \sum_n q_n \partial_t \delta^3(\vec{x} - \vec{x}_n(t)) \\
= - \partial_t \rho(\vec{x}, t) = - \partial_0 [c \rho(\vec{x}, t)].
\]

(46)

So the equation of charge continuity writes as \( \partial^\alpha J_\alpha = 0 \)
4-gradient

In the previous slide we use the operator $\partial^\alpha$. It is defined as

$$\partial^\alpha \equiv \frac{\partial}{\partial x^\alpha}.$$  \hspace{1cm} (47)

This operator transforms as:

$$\partial'^\mu = \frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\nu} \frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^{\nu}_{\mu} \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^{\nu}_{\mu} \partial^\nu.$$  \hspace{1cm} (48)

Note that $\partial^\mu = (\partial_0, \vec{\nabla})$.

We can "upper" the indice and define

$$\partial^\mu = g^{\mu\nu} \partial^\nu = (\partial_0, -\vec{\nabla})$$ \hspace{1cm} (49)

Finally we can define the d’Alembertian: $\Box \equiv \partial^\alpha \partial_\alpha$.
Potential as a 4-vector:

\[ A^\alpha \equiv (\phi, \vec{A}) \]  

Lorentz Gauge then write \( \partial_\alpha A^\alpha = 0 \). We also have

\[ \Box A^\alpha = \frac{4\pi}{c} J^\alpha, \]  

or in SI units

\[ \Box A^\alpha = \mu_0 J^\alpha, \quad [\text{SI}] \]  

this is the equation we wrote when deriving the field induced by a charge moving at constant velocity.
Returning to Maxwell Equation

Define the matrix $F^{\alpha \beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = g^{\alpha \delta} \partial_\delta A^\beta - g^{\beta \delta} \partial_\delta A^\alpha$:

$$F^{\alpha \beta} = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{pmatrix}$$  \hfill (53)

Look at:

$$\partial_\alpha F^{\alpha \beta} = \partial_0 F^{0 \beta} + \partial_1 F^{1 \beta} + \partial_2 F^{2 \beta} + \partial_3 F^{3 \beta}:$$

$$\partial_\alpha F^{\alpha 0} = \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30}$$

$$= \partial_i E^i = \vec{\nabla} \cdot \vec{E} = 4\pi \rho = \frac{4\pi}{c} J^0.$$  \hfill (54)

Similarly,

$$\partial_\alpha F^{\alpha 1} = \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31}$$

$$= \frac{1}{c} \partial_t (-E_x) + \partial_x (0) + \partial_y (-B_z) - \partial_z (B_y) = -\frac{1}{c} \partial_t (E_x) + [\vec{\nabla} \times \vec{B}]_x$$

$$= [\vec{\nabla} \times \vec{B}]_x - \frac{1}{c} \partial_t E_x = \frac{4\pi}{c} J^1$$  \hfill (55)
...The same for component 2, and 3. So we cast these equations under:

\[ \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \]  

(56)

This corresponds to the inhomogeneous Maxwell's equations. In SI units, \(F^{\alpha\beta}\) is obtained by replacing \(\overrightarrow{E}\) by \(\overrightarrow{E}/c\).

How do we get the homogenous Maxwell's equations? Let's introduce the Levi-Civita (rank 4) tensor as:

\[
\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 
+1 & \text{if } \alpha, \beta, \gamma, \delta \text{ are even permutation of 0,1,2,3} \\
-1 & \text{if } \alpha, \beta, \gamma, \delta \text{ are odd permutation of 0,1,2,3} \\
0 & \text{otherwise}
\end{cases},
\]  

(57)

and consider the quantity \(\epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\delta\gamma}\), with \(F_{\delta\gamma} = g_{\gamma\alpha} g_{\delta\beta} F^{\alpha\beta}\).
\[ F_{\gamma\delta} = \begin{pmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
-E_z & -B_y & B_x & 0
\end{pmatrix} \] (58)

\( F_{\gamma\delta} \) is obtained from \( F^{\alpha\beta} \) by doing the change \( \vec{E} \rightarrow -\vec{E} \). Now consider the component "0" of the 4-vector \( \epsilon^{\alpha\beta\gamma\delta} \partial_\beta F_{\gamma\delta} \):

\[
\epsilon^{0\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = \epsilon^{0123} \partial_1 F_{23} + \epsilon^{0132} \partial_1 F_{32} + \epsilon^{0213} \partial_2 F_{13} + \epsilon^{0231} \partial_2 F_{31} + \epsilon^{0312} \partial_3 F_{12} + \epsilon^{0321} \partial_3 F_{21}
\]

\[
= \partial_1 F_{23} - \partial_1 F_{32} - \partial_2 F_{13} + \partial_2 F_{31} + \partial_3 F_{12} - \partial_3 F_{21}
\]

\[
= \partial_x(-B_x) - \partial_x(B_x) - \partial_y(B_y) + \partial_y(-B_y) + \partial_z(-B_z) - \partial_z(B_z)
\]

\[
= -2 \vec{\nabla} \cdot \vec{B} (= 0)
\] (59)
now let’s compute the ”1” component

\[
\epsilon^{1\beta\gamma\delta} \partial_\beta F_{\gamma\delta} = \epsilon^{1023} \partial_0 F_{23} + \epsilon^{1032} \partial_0 F_{32} + \epsilon^{1302} \partial_3 F_{02} + \epsilon^{1320} \partial_3 F_{20} \\
+ \epsilon^{1203} \partial_2 F_{03} + \epsilon^{1230} \partial_2 F_{30}
\]

\[
= -\partial_0 F_{23} + \partial_0 F_{32} - \partial_3 F_{02} + \partial_3 F_{20} + \partial_2 F_{03} - \partial_2 F_{30}
\]

\[
= 2(D_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20})
\]

\[
= 2 \left( \frac{1}{c} \partial_t B_x - \partial_z E_y + \partial_y E_z \right)
\]

\[
= 2 \left[ (\vec{\nabla} \times \vec{E})_x + \frac{1}{c} \partial_t B_x \right] (= 0)
\]

(60)

It is common to define the dual tensor of \( F_{\gamma\delta} \) as \( \mathcal{F}^{\alpha\beta} \equiv \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \). With such a definition the homogeneous Maxwell equations can be casted in the expression:

\[
\partial_\alpha \mathcal{F}^{\alpha\beta} = 0.
\]

(61)

Note: \( \mathcal{F}_{\alpha\beta} = F_{\alpha\beta}(\vec{E} \rightarrow \vec{B}, \vec{B} \rightarrow -\vec{E}) \).
To include $\vec{H}$ and $\vec{D}$, one defines the tensor $G^{\alpha\beta} = \mathcal{F}^{\alpha\beta}(\vec{E} \rightarrow \vec{D}, \vec{B} \rightarrow \vec{H})$, and then Maxwell’s equations write:

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta,$$

and

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0.$$  \hspace{1cm} (62)

Due to covariance of $\mathcal{F}^{\alpha\beta}$, it is a tensor, the calculation of em field from one Lorentz frame to another is made easy. Just consider:

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta},$$  \hspace{1cm} (63)

or in matrix notation

$$F' = \tilde{\Lambda} F \Lambda = \Lambda F \Lambda.$$  \hspace{1cm} (64)
Example: Consider a boost along the \( \hat{z} \)-axis, then

\[
\Lambda = \begin{pmatrix}
\gamma & 0 & 0 & -\beta \gamma \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\gamma \beta & 0 & 0 & \gamma
\end{pmatrix}
\] (65)

Plug the \( F \) matrix associated to \( F^\gamma \delta \) in Eq. 64, the matrix multiplication yields:

\[
F^\gamma \delta = \begin{pmatrix}
0 & \gamma(E_x - \beta B_y) & \gamma(E_y + \beta B_x) & E_z \\
-\gamma(E_x - \beta B_y) & 0 & B_z & -\gamma(B_y - \beta E_x) \\
-\gamma(E_y + \beta B_x) & -B_z & 0 & \gamma(B_x + \beta E_y) \\
-E_z & \gamma(B_y - \beta E_x) & -\gamma(B_x + \beta E_y) & 0
\end{pmatrix}
\] (66)

by inspection we obtain the same equation as [JDJ, Eq. (11.148)].
Fundamental Invariant of the electromagnetic field tensor: *

Note that the quantities

\[ F^{\mu\nu} F_{\mu\nu} = 2(E^2 - B^2), \text{ and } F^{\mu\nu} F_{\mu\nu} = 4 \vec{E}.\vec{B}, \] (67)

are invariants. Usually one redefines these two invariants as:

\[ \mathcal{I}_1 \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(B^2 - E^2), \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\vec{E}.\vec{B}. \] (68)

Note that these invariants may be rewritten as:

\[ \mathcal{I}_1 \equiv -\frac{1}{4} \text{tr}(F^2) \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4} \text{tr}(FF), \] (69)

where \( F \equiv F^\nu_\mu = F^\mu_\alpha g_{\alpha\nu} \) and \( \mathcal{F} \equiv \mathcal{F}^\nu_\mu = \mathcal{F}^\mu_\alpha g_{\alpha\nu} \).

Finally note the identities:

\[ FF = FF = -\mathcal{I}_2 I, \text{ and } F^2 - \mathcal{F}^2 = -2\mathcal{I}_1 I \] (70)

*adapted from G. Muñoz, Am. J. Phys. 65 (5), May 1997
Eigenvalues of $F$ (for later!):

Look for eigenvalue $\lambda$ associated to eigenvector $\Psi$:

$$F\Psi = \lambda\Psi \Rightarrow FF\Psi = \lambda F\Psi \Rightarrow F\Psi = -\frac{I_2}{\lambda}\Psi.$$ (71)

$$(F^2 - F^2)\Psi = -2I_1\Psi = [\lambda^2 - (I_2/\lambda)^2]\Psi,$$ (72)

So characteristic polynomial is: $\lambda^4 + 2I_1\lambda^2 - I_2^2 = 0$.

Solutions are:

$$\lambda_{\pm} = \sqrt[4]{I_1^2 + I_2^2 \pm I_1}$$ (73)

$\lambda_1 = -\lambda_2 = \lambda_-, \lambda_3 = -\lambda_4 = i\lambda_+.$
Equation of motion:
The equation describing the dynamics of a relativistics particle of mass $m$ and charge $q$ moving under the influence of em field $F_{\alpha\beta}$ is:

$$\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^\alpha_{\beta} u^\beta. \quad (74)$$

with $u^\alpha = (\gamma c, \gamma \vec{v})$. Note that this is equivalent to introducing the "quadri-force"

$$f^\mu = F^{\mu\nu} u_\nu. \quad (75)$$