Special Theory of Relativity

- Up to ~1895, used simple Galilean Transformations
  \[ x' = x - vt \quad t' = t \]
- But observed that the speed of light, \( c \), is always measured to travel at the same speed even if seen from different, moving frames
- \( c = 3 \times 10^8 \text{ m/s} \) is finite and is the fastest speed at which information/energy/particles can travel
- Einstein postulated that the laws of physics are the same in all inertial frames. With \( c=\text{constant} \) he “derived” Lorentz Transformations

\[ \text{sees } v_1 = c \quad \text{moving frame also sees } \]

\[ v_1 = c \quad \text{NOT } v_1 = c - u \]
“Derive” Lorentz Transform

Bounce light off a mirror. Observe in 2 frames:

A velocity = 0 with respect to light source

A’ velocity = v

observe speed of light = c in both frames →

c = distance/time = 2L / t  A

c^2 = 4(L^2 + (x’/2)^2) / t’^2  A’

Assume linear transform (guess)

x’ = G(x + vt)  let x = 0

t’ = G(t + Bx)  so x’=Gvt  t’=Gt

some algebra

(ct’)^2 = 4L^2 + (Gvt)^2  and  L = ct /2 and t’=Gt

gives

(Gt)^2 = t^2(1 + (Gv/c)^2)  or  G^2 = 1/(1 - v^2/c^2)
Lorentz Transformations

Define $\beta = u/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$

\[ x' = \gamma (x + ut) \quad u = \text{velocity of transform} \]
\[ y' = y \quad \text{between frames is in} \]
\[ z' = z \quad \text{x-direction. If do } x' \rightarrow x \text{ then} \]
\[ t' = \gamma (t + \beta x/c) \quad \text{“+” } \rightarrow \text{ “-”}. \text{ Use common sense} \]

can differentiate these to get velocity transforms

\[ v_x' = (v_x - u) / (1 - u v_x / c^2) \]
\[ v_y' = v_y / \gamma / (1 - u v_x / c^2) \]
\[ v_z' = v_z / \gamma / (1 - u v_x / c^2) \]

usually for $v < 0.1c$ non-relativistic (non-Newtonian) expressions are OK. Note that 3D space point is now 4D space-time point $(x,y,z,t)$
Time Dilation

• Saw that \( t' = \gamma t \). The “clock” runs slower for an observer not in the “rest” frame

• muons in atmosphere. Lifetime = \( \tau = 2.2 \times 10^{-6} \) sec \( \quad c\tau = 0.66 \) km
decay path = \( \beta\gamma\tau c \)

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>average in lab</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.005</td>
<td>2.2 ( \mu )s</td>
</tr>
<tr>
<td>0.5</td>
<td>1.15</td>
<td>2.5 ( \mu )s</td>
</tr>
<tr>
<td>0.9</td>
<td>2.29</td>
<td>5.0 ( \mu )s</td>
</tr>
<tr>
<td>0.99</td>
<td>7.09</td>
<td>16 ( \mu )s</td>
</tr>
<tr>
<td>0.999</td>
<td>22.4</td>
<td>49 ( \mu )s</td>
</tr>
</tbody>
</table>
Time Dilation

• Short-lived particles like tau and B. Lifetime $= 10^{-12}$ sec $c \tau = 0.03$ mm

• time dilation gives longer path lengths

• measure “second” vertex, determine “proper time” in rest frame

If measure $L=1.25$ mm and $v = 0.995c$

t(proper) = $L/v\gamma = 0.4$ ps

Twin Paradox. If travel to distant planet at $v \sim c$ then age less on spaceship than in “lab” frame
Adding velocities

• Rocket A has $v = 0.8c$ with respect to DS9. Rocket B have $u = 0.9c$ with respect to Rocket A. What is velocity of B with respect to DS9?

$$V' = \frac{v-u}{1-vu/c^2}$$

$$v' = \frac{.9+.8}{1+.9*.8} = .988c$$

Notes: use common sense on +/-

if $v = c$ and $u = c$  $v' = (c+c)/2 = c$
Adding velocities

Rocket A has v = 0.826c with respect to DS9. Rocket B have u = 0.635c with respect to DS9. What is velocity of A as observed from B?

\[ V' = \frac{v - u}{1 - vu/c^2} \]

\[ v' = \frac{0.826 - 0.635}{1 - 0.826 \times 0.635} = 0.4c \]

If did B from A get -0.4c

\[ \frac{0.4 + 0.635}{1 + 0.4 \times 0.635} = 1.04/1.25 = 0.826 \]
Relativistic Kinematics

- \( E^2 = (pc)^2 + (mc^2)^2 \) \( E \) = total energy \( m = \) mass and \( p = \) momentum

- natural units \( E \) in eV, \( p \) in eV/c, \( m \) in eV/c^2 \( \Rightarrow c = 1 \) effectively. \( E^2 = p^2 + m^2 \)

- kinetic energy \( K = T = E - m \) \( \approx 1/2 \, mv^2 \) if \( v \ll c \)

- Can show: \( \beta = p/E \) and \( \gamma = E/m \) \( \Rightarrow p = \beta \gamma m \) if \( m \neq 0 \) or \( p = E \) if \( m = 0 \)

  (many massless particles, photon, gluon and (almost massless) neutrinos)

- relativistic mass \( m = \gamma m_0 \) a BAD concept
“Derive” Kinematics

• \(dE = -Fdx = -dp/dt*dx = -vdp = -vd(\gamma mv)\)

• assume \(p=\gamma mv\) (need relativistic for \(p\))

• \(d(\gamma mv) = m\gamma dv + mvd\gamma = mdv\gamma^3\)

\[
\int dE = \int mv\gamma^3 dv \\
= \int_{v=0}^{v=\text{final}} mv(1 - v^2/c^2)^{-3/2} dv \\
= \gamma mc^2 - mc^2 \\
= \text{Total Energy - “rest” energy} \\
= \text{Kinetic Energy}
\]
Lorentz Transformations

P and E are components of a 4 vector and so transform by same Lorentz Transformation

Define $\beta = u/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$

$$px' = \gamma (px + uE) \quad u = \text{velocity of transform}$$
$$py' = py \quad \text{between frames is in}$$
$$pz' = pz \quad \text{x-direction. If do } x' \rightarrow x \text{ then}$$
$$E' = \gamma (E + \beta px/c) \quad \text{“+” } \rightarrow \text{ “-”. common sense}$$

Examples with particle with mass=m at rest “transformed” to moving with velocity v
What are the momentum, kinetic, and total energies of a proton with \( v = 0.86c \)?

- \( v = 0.86c \) \[ \gamma = \frac{1}{\sqrt{1 - 0.86 \cdot 0.86}} = 1.96 \]

- \( E = \gamma m = 1.96 \cdot 938 \text{ MeV/c}^2 = 1840 \text{ MeV} \)
- \( T = E - mc^2 = 1840 - 938 = 900 \text{ MeV} \)

- \( p = \beta E = \gamma \beta m = 0.86 \cdot 1840 \text{ MeV} = 1580 \text{ MeV/c} \) or \( p = (E^2 - m^2)^{0.5} \)

Note units: MeV, MeV/c and MeV/c^2. Usually never have to use \( c = 300,000 \text{ km/s} \) in calculation.
• Accelerate electron to 0.99c and then to 0.999c. How much energy is added at each step?

\[
\begin{align*}
\gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
\gamma_1 &= \frac{1}{\sqrt{1 - \left(\frac{0.99c}{c}\right)^2}} = 7.1 \\
\gamma_2 &= \frac{1}{\sqrt{1 - \left(\frac{0.999c}{c}\right)^2}} = 22.4
\end{align*}
\]

• \(E = \gamma m = 7.1 \times 0.511\text{ MeV} = 3.6\text{ MeV}\)
• \(= 22.4 \times 0.511\text{ MeV} = 11.4\text{ MeV}\)

• step 1 adds 3.1 MeV and step 2 adds 7.8 MeV even though velocity change in step 2 is only 0.9%
(px,py,px,E) are components of a 4-vector which has same Lorentz transformation

\[ \begin{align*}
px' &= \gamma (px + uE/c^2) & u &= \text{velocity of transform between frames is in x-direction. If do } px' \rightarrow px \\
py' &= py \\
pz' &= pz \\
E' &= \gamma (E + upx)
\end{align*} \]

“+” \rightarrow “-” Use common sense also let \( c = 1 \)

Frame 1

Frame 2 (cm)

Before and after scatter
Some quantities are invariant when going from one frame to another:

py and pz are “transverse” momentum

\( M_{\text{total}} = \) Invariant mass of system derived from \( E(\text{total}) \) and \( P(\text{total}) \) as if just one particle

How to get to C.M. system? Think as if 1 particle

\[
E(\text{total}) = E_1 + E_2 \quad P_x(\text{total}) = p_{x1} + p_{x2} \quad \text{(etc)}
\]

\[
M(\text{total})^2 = E(\text{total})^2 - P(\text{total})^2
\]

\[
\gamma(\text{c.m.}) = \frac{E(\text{total})}{M(\text{total})} \quad \text{and} \quad \beta(\text{c.m.}) = \frac{P(\text{total})}{E(\text{total})}
\]
Particle production
convert kinetic energy into mass - new particles
assume 2 particles 1 and 2 both mass = m
Lab or fixed target  \[ E(\text{total}) = E_1 + E_2 = E_1 + m^2 \]
\[ P(\text{total}) = p_1 \quad \rightarrow \quad M(\text{total})^2 = E(\text{total})^2 - P(\text{total})^2 \]
\[ M(\text{total}) = (E_1^2 + 2E_1m + m^2 - p_1^2)^{0.5} \]
\[ M(\text{total}) = (2m^2 + 2E_1m)^{0.5} \sim (2E_1m)^{0.5} \]
CM: \( E_1 = E_2 \quad E(\text{total}) = E_1+E_2 \quad \text{and} \quad P(\text{total}) = 0 \)
\[ M(\text{total}) = 2E_1 \]
\[ p + p \rightarrow p + p + p + p + \bar{p} \]

What is the minimum energy to make a proton-antiproton pair?

- In all frames, \( M(\text{total}) \) (invariant mass) at threshold is equal to \( 4m_p \) (think of cm frame, all at rest)

\[ \text{Lab} \quad M(\text{total}) = (E_1^2 + 2E_1m + m^2 - m_p^2)^{0.5} \]

\[ M(\text{total}) = (2m^2 + 2E_1m)^{0.5} = 4m \]

\[ E_1 = \frac{(16m_p^2m_p^2 - 2m_p^2m_p^2)}{2m_p} = 7m_p \]

\[ \text{CM: } M(\text{total}) = 2E_1 = 4m_p \text{ or } E_1 \text{ and } E_2 \text{ each } = 2m_p \]

at threshold all at rest in c.m. after reaction

1 2 at rest

\[ p_1 = p \]

Lab

CM
Transform examples

- Trivial: at rest $E = m$ $p=0$. “boost” velocity $= v$
  
  \[ E' = \gamma(E + \beta p) = \gamma m \]
  \[ p' = \gamma(p + \beta E) = \gamma \beta m \]

- moving with velocity $v = p/E$ and then boost velocity $= u$ (letting $c=1$)
  
  \[ E' = \gamma(E + pu) \]
  \[ p' = \gamma(p + Eu) \]
  
  calculate $v' = p'/E' = (p + Eu)/(E+pu)$
  
  \[ = (p/E + u)/(1+up/E) = (v+u)/(1+vu) \]

- “prove” velocity addition formula
• A p=1 GeV proton hits an electron at rest. What is the maximum pt and E of the electron after the reaction?

• Elastic collision. In cm frame, the energy and momentum before/after collision are the same. Direction changes. 90 deg = max pt 180 deg = max energy

• $\beta_{cm} = P_{tot}/E_{tot} = P_{p}/(E_{p} + m_e)$

• $P_{cm} = \gamma_{cm}\beta_{cm}*m_e$ (transform electron to cm)

• $E_{cm} = \gamma_{cm}*m_e$ ("easy" as at rest in lab)

• $p_{t\text{max}} = P_{cm}$ as elastic scatter same pt in lab

• $E_{max} = \gamma_{cm}(E_{cm} + \beta_{cm}P_{cm})$ 180 deg scatter
  $= \gamma_{cm}(\gamma_{cm}*m_e + \gamma\beta\beta*m_e)$
  $= \gamma*\gamma*m_e(1 + \beta*\beta)$
• p=1 GeV proton (or electron) hits a stationary electron (or proton) 
mp = .94 GeV  me = .5 MeV

<table>
<thead>
<tr>
<th>incoming</th>
<th>target</th>
<th>$\beta_{cm}$</th>
<th>$\gamma_{cm}$</th>
<th>Ptmax</th>
<th>Emax</th>
</tr>
</thead>
<tbody>
<tr>
<td>p</td>
<td>e</td>
<td>.7</td>
<td>1.5</td>
<td>.4 MeV</td>
<td>1.7 MeV</td>
</tr>
<tr>
<td>p</td>
<td>p</td>
<td>.4</td>
<td>1.2</td>
<td>.4 GeV</td>
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</tr>
<tr>
<td>e</td>
<td>p</td>
<td>.5</td>
<td>1.2</td>
<td>.5 GeV</td>
<td>1.7 GeV</td>
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<tr>
<td>e</td>
<td>e</td>
<td>.9995</td>
<td>30</td>
<td>15 MeV</td>
<td>1 GeV</td>
</tr>
</tbody>
</table>

• Emax is maximum energy transferred to stationary particle. Ptmax is maximum momentum of (either) outgoing particle transverse to beam. Ptmax gives you the maximum scattering angle

• a proton can’t transfer much energy to the electron as need to conserve E and P. An electron scattering off another electron can’t have much Pt as need to conserve E and P.