LECTURES ON SPECIAL RELATIVITY

These lectures are similar in content to, but somewhat different in exposition from, the material in Jim Hartle’s ‘Lectures in Introductory Gravitation and Relativity.’

Newtonian physics

Newtonian mechanics implements two basic concepts, viz:
1) our ordinary intuitive notion of immutable space and time; and
2) the idea that only relative motion is fundamental; in particular, it is assumed that there exist a preferred set of observers, inertial observers, moving relative to one another with constant velocity, who see the same physical laws.

The geometry of space is defined uniquely by the assumption that the distance \( dr \) between two nearby points \( \mathbf{r}_1 = (x, y, z) \) and \( \mathbf{r}_2 = (x + \Delta x, y + \Delta y, z + \Delta z) \) satisfies

\[
 dr^2 = dx^2 + dy^2 + dz^2.
\]

In particular, all of Euclidean geometry, including such ideas that (i) initially parallel lines remain parallel and (ii) the angles of a triangle sum up to 180°, can be derived from this relation.

This notion of distance is invariant with respect to two sorts of variations, viz:

- **translational invariance**, e.g., \( dr \) is invariant with respect to a transformation
  \[
  x = x' + k, \quad y = y', \quad \text{and} \quad z = z',
  \]

since \( dx = d(x' + k) = dx' \).

- **rotational invariance**, e.g., \( dr \) is invariant with respect to a transformation
  \[
  x = x' \cos \theta - y' \sin \theta \quad y = x' \sin \theta + y' \cos \theta, \quad \text{and} \quad z = z',
  \]

which follows because the relations

\[
 dx = \cos \theta dx' - \sin \theta dy' \quad \text{and} \quad dy = \sin \theta dx' + \cos \theta dy'
\]

imply that

\[
 dx^2 + dy^2 = dx'^2 + dy'^2.
\]

Mathematically, the set of six transformations defined by these three translations and three rotations forms a group, the so-called Galilean group.

One also has an additional invariance with respect to time translation, i.e., \( t = t' + k \), but this invariance is treated on a different footing than the spatial invariances since space and time are viewed as distinct entities.

It is assumed Newtonianly that interactions between objects typically involve a force field, e.g., the gravitational field or the electric field. This assumption, known as ‘action at a distance,’ implies that objects can effect one another even though they do not touch physically. It is also assumed implicitly that this action at a distance involves ‘instantaneous propagation,’ i.e., the information about an object at \( \mathbf{r}_1 \) propagates instantaneously to another object at \( \mathbf{r}_2 \). (This picture, which underlies electrostatics, is different from the picture underlying electrodynamics, where one supposes correctly that this information propagates with a finite speed, namely the speed of light \( c \).)

It is assumed that the fundamental laws of physics all derive from an Action Principle, i.e., that the equations of motion can be derived by demanding that the value of some action \( S \) be extremal. The action can be written as

\[
 S = \int dt L(\mathbf{r}, \dot{\mathbf{r}}, t)
\]
for a particle or
\[ S = \int dt \int d^3r \mathcal{L}(r, \dot{r}, t) \] (7)
for a field. For the case of a particle, this leads to the Lagrange equations
\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{r}_i} = \frac{\partial L}{\partial r_i}. \] (8)
For the simple case of a free particle,
\[ L = \frac{1}{2} m \dot{r}^2, \quad \text{for which} \quad \frac{d}{dt} (m \dot{r}) = 0. \] (9)

CRISIS!!! Newtonian physics is wrong experimentally!

1) The Michelson-Morley experiment: (in a vacuum) all observers measure the same speed of light \( c \). This contradicts Galilean invariance, which predicts that, for two observers moving with relative velocity \( V \) in the \( x \)-direction,
\[ x = x' + vt, \quad y = y', \quad z = z', \quad \text{and} \quad t = t' \] (10)
so that
\[ \frac{dx}{dt} = \frac{dx'}{dt} + v \quad \Rightarrow \quad c = c' + v. \] (11)

2) Electromagnetism is not invariant with respect to Galilean transformations! e.g.,
- particles interact via retarded potentials (so-called Liénard-Wiechert potentials)
- Galilean relativity is inconsistent with the propagation of electromagnetic radiation

Special relativity is an alternative to Newtonian physics which still assumes that relative motion is fundamental, but incorporates explicitly the assumption, verified experimentally, that all observers measure the same speed of light.

Since the assumptions of (1) absolute space and time and (2) relative motion being fundamental imply that all observers must measure the same speed of light, the assumptions of (1) relative motion and (2) constant speed of light must contradict the notion of absolute space and time!

The symmetries of special relativistic spacetime

Special relativity is based on the idea that the fundamental arena of physics is a four-dimensional spacetime, characterised by an invariant four-dimensional spacetime interval
\[ ds^2 \equiv -c^2 dt^2 + dx^2 + dy^2 + dz^2. \] (12)

This notion of distance can be used to derive a theory of geometry every bit as systematic as the Euclidean geometry predicated on \( dr^2 = dx^2 + dy^2 + dz^2 \). It is often called Minkowskian geometry and the spacetime equipped with this notion of distance is called Minkowski space.

This spacetime admits ten different symmetries, viz:
1) translational invariance in the \( x, y, z, \) and \( t \) directions (4 symmetries);
2) space-rotational invariance, i.e., invariance with respect to rotations in the \( x - y, y - z, \) and \( z - x \) planes (3 symmetries); and
3) spacetime rotational invariance (also called boosts), which correspond to imaginary rotations in the \( x - t, y - t, \) and \( z - t \) planes, e.g.,
\[ ct = x' \sinh \psi + ct' \cosh \psi, \quad x = x' \cosh \psi + ct' \sinh \psi, \quad y = y', \quad \text{and} \quad z = z' \] (13)
leaves the interval $ds$ invariant.

This collection of symmetries also forms a group, which is known as the Lorentz group.

How does the motion of the origin $x' = 0$ appear in a new frame of reference which is moving relative to the original frame of reference with velocity $v$ in the $x$-direction? Note that the preceding relation implies that

$$\frac{x}{ct} = \frac{v}{c} = \tanh \psi.$$  

But the relation \( \cosh^2 \psi - \sinh^2 \psi = 1 \) then implies that

$$\sinh \psi = \frac{v/c}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad \cosh \psi = \frac{1}{\sqrt{1 - v^2/c^2}},$$

so that

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}, \quad x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}, \quad y = y', \quad \text{and} \quad z = z'.$$  

This is an example of a so-called Lorentz transformation.

Note that, for $v \ll c$, one recovers approximately Galilean transformations, for which

$$x \approx x' + vt', \quad y = y', \quad z = z', \quad \text{and} \quad t = t'.$$

This is important because of the overwhelming experimental evidence that Newtonian physics is almost exactly correct for $v \ll c$.

The Lorentz group implies phenomena like length contraction and time dilation.

Suppose, e.g., that there is a rod at rest in the $K$-system but which is moving with velocity $v$ in the $x$-direction in the $K'$-system. To determine the length of the rod, an observer in the $K'$-system must determine the location of the two ends of the rod, $x'_1$ and $x'_2$, at the same instant $t'$. This means that

$$x_1 = \frac{x'_1 + vt'}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad x_2 = \frac{x'_2 + vt'}{\sqrt{1 - v^2/c^2}} \implies \Delta x = \frac{\Delta x'}{\sqrt{1 - v^2/c^2}}.$$  

If $\ell_0$ is the proper length of the rod, i.e., the length as measured in a frame of reference in which the rod is at rest, then the length $\ell$ as measured in the moving frame will satisfy

$$\ell(v) = \ell_0 \sqrt{1 - v^2/c^2},$$

i.e., in a moving reference frame the rod is shorter! Note also that, because there is no contraction in the two spatial directions orthogonal to the motion, the spatial volume contracts by an identical amount: $V(v) = V_0 \sqrt{1 - v^2/c^2}$.

In a similar fashion, to determine the duration of some process, one must measure the initial and final times at some fixed point in space. This means that

$$t_1 = \frac{t'_1 + vx'/c^2}{\sqrt{1 - v^2/c^2}}, \quad \text{and} \quad t_2 = \frac{t'_2 + vx'/c^2}{\sqrt{1 - v^2/c^2}} \implies \Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}.$$  

Equation (20) has been derived assuming that $v$ is constant, but that was by no means necessary. The entire calculation can be reformulated infinitesimally, in which one case, by integrating over a finite interval, one infers that

$$t'_2 - t'_1 = \int_{t_1}^{t_2} dt \sqrt{1 - v^2(t)/c^2}$$

7
Spacelike, timelike, and null intervals

Note that the four-dimensional interval

\[ ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]  \hspace{1cm} (22)

is not of uniform sign. One can of course write

\[ ds^2 = -c^2 dt^2 \left( 1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2} \right) = -c^2 dt^2 \left( 1 - \frac{v^2}{c^2} \right). \]  \hspace{1cm} (23)

It thus follows that:

- if \( v^2 < c^2 \), then \( ds^2 < 0 \), which is called a timelike interval; but
- if \( v^2 > c^2 \), then \( ds^2 > 0 \), which is called a spacelike interval; and
- if \( v^2 = c^2 \), then \( ds^2 = 0 \), which is called a null interval, i.e., ‘light never goes anywhere’.

All objects with nonzero mass must follow trajectories corresponding to timelike intervals, for which the locally measured speed \( v < c \).

Since all inertial observers agree on the four-dimensional interval \( ds \), it follows that all observers measure the same speed of light \( c \).

For \( v^2 < c^2 \), one can write

\[ c^2 d\tau^2 = c^2 dt^2 \left( 1 - \frac{v^2}{c^2} \right) \quad \Rightarrow \quad d\tau = dt \sqrt{1 - \frac{v^2}{c^2}}, \]  \hspace{1cm} (24)

where \( d\tau \) is the proper time as measured in a system at rest.

A useful convention: choose a set of units in which \( c = 1 \), i.e., measure time in units of \( ct \). (Later on, units will also be so chosen that \( G = \hbar = 1 \).)

A proof that all inertial observers agree on the magnitude of the four-dimensional volume element \( d\Omega = dt d^3x \):

It is clear that

\[ d^3x dt = d^3x \frac{dt}{d\tau} d\tau, \]  \hspace{1cm} (25)

where \( d\tau \) is an invariant for all inertial observers. It suffices, therefore, to show that \( d^3x (dt/d\tau) \) is invariant.

In a frame of reference with \( \vec{v} = 0 \),

\[ d^3x \frac{dt}{d\tau} \rightarrow d^3x_R = dx dy dz. \]  \hspace{1cm} (26)

In a frame of reference moving with velocity \( v \) in the \( x \)-direction,

\[ dx dy dz \rightarrow \sqrt{1 - v^2} dx dy dz \quad \text{and} \quad \frac{dt}{d\tau} \rightarrow \frac{1}{\sqrt{1 - v^2}} \quad \Rightarrow \quad d^3x \frac{dt}{d\tau} \rightarrow d^3x_R. \]  \hspace{1cm} (27)

The equation of motion for a free particle derived in Newtonian language

The motion of a free particle derives from an action

\[ S = -m \int d\tau = -m \int dt \frac{d\tau}{dt} = -m \int dt \sqrt{1 - v^2} \quad \Rightarrow \quad L = -m \sqrt{1 - v^2} \approx -m + \frac{1}{2} mv^2, \]  \hspace{1cm} (28)
i.e., the equation of motion derives from the demand that the proper time/length be extremised!
The canonical momentum $\vec{p}$ satisfies
\[ \vec{p} \equiv \frac{\partial L}{\partial \vec{v}} = \frac{mv}{\sqrt{1-v^2}} \quad \Rightarrow \quad \frac{d\vec{p}}{dt} = 0 = \frac{d}{dt} \frac{mv}{\sqrt{1-v^2}}. \] (29)
The Hamiltonian function, which is equally numerically to the energy $E$, satisfies
\[ H = \vec{p} \cdot \vec{v} - L = \sqrt{m^2 + \vec{p}^2} \approx m + \frac{1}{2m} \quad \text{for} \; v \ll c. \] (30)
A massive object with $v \rightarrow c$ has energy $E \rightarrow \infty$.

**Four-vectors**

*Why does one care about vectors?* When written in vectorial form, the equations of motion formulated in the context of Newtonian physics are invariant with respect to Galilean transformations. In a similar fashion, the laws of special relativity can be written in terms of suitably defined vectors in such a fashion that they are invariant under Lorentz transformations.

In Minkowski space, a vector is a collection of four numbers which transforms like the spatial coordinates $x^i = (t, x, y, z)$ under a Lorentz transformation, i.e.,
\[ A^i = \frac{A'^{i} + vA'^{x}}{\sqrt{1-v^2}}, \quad A^x = \frac{A'^{x} + vA'^{t}}{\sqrt{1-v^2}}, \quad A^y = A'^{y}, \quad \text{and} \quad A^z = A'^{z}. \] (31)

Given this definition, one can interpret
\[ ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \] (32)
as an equation involving four-vectors, writing
\[ ds^2 = \sum_{i=0}^{3} \sum_{j=0}^{3} g_{ij} dx^i dx^j, \] (33)
where
\[ g_{ij} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \] (34)
$g_{ij}$ is denoted the spacetime metric since it serves to define distances between nearby points.

It is conventional to implement the so-called summation convention, whereby there is an implicit summation when the same index appears twice, *i.e.*, one should write
\[ ds^2 = g_{ij} dx^i dx^j. \] (35)
Pragmatically, the quantity $dx^i$ can be viewed as a column matrix and $g_{ij}$ can be viewed as a square matrix. More formally, $dx^i$ and $g_{ij}$ are examples of so-called tensors.

Technically, $dx^i$ is termed a contravariant vector or a contravariant first rank tensor. $g_{ij}$ is termed a covariant second rank tensor. (Contravariant means the indices $i, j, etc.$ live upstairs; covariant means that the indices live downstairs. First rank means there is one index; second rank means there are two indices.)

The quantity $dx_i \equiv dx^i g_{ij}$ can be viewed as a row matrix. It is a simple example of a covariant vector or a covariant first rank tensor. Note that if $dx^i$ has components $t, x, y,$ and $z$, then $dx_i$ has components $-t, x, y,$ and $z$. 

9
Figure 1: a spacetime diagram

A covariant vector is a collection of four numbers which transform like $dx_i = g_{ij}dx^j$ under Lorentz transformations, i.e.,

$$A_t = \frac{A'_t - vA'_x}{\sqrt{1-v^2}}, \quad A_x = \frac{A'_x - vA'_t}{\sqrt{1-v^2}}, \quad A_y = A'_y, \quad \text{and} \quad A_z = A'_z. \quad (36)$$

The length of $dx^i$ satisfies $ds^2 = g_{ij}dx^idx^j$ so, by analogy, for a general vector $A^i$,

$$|A|^2 \equiv g_{ij}A^iA^j = A^iA^i = -(A^t)^2 + (A^x)^2 + (A^y)^2 + (A^z)^2. \quad (37)$$

Analogously, one can define a scalar product

$$\tilde{A} \cdot \tilde{B} \equiv g_{ij}A^iB^j = A_iB^i = A^jB_j. \quad (38)$$

This is the analogue of the dot product in ordinary Euclidean space, i.e.,

$$\tilde{A} \cdot \tilde{B} = \delta_{\alpha\beta}A^\alpha B^\beta = A^xB^x + A^yB^y + A^zB^z. \quad (39)$$

Note that $\tilde{A} \cdot \tilde{B}$ is invariant, i.e., transforms as a scalar, under Lorentz transformations.

**Spacetime diagrams**

Kinematics involves specifying the trajectories of objects through spacetime, parameterised appropriately. The natural parameter is *proper time*, i.e., time as measured by a clock that the object carries along with it. This gives rise to the concept of a *worldline* moving through spacetime, which corresponds to specifying four coordinates as functions of proper time $\tau$, i.e.,

$$x^i = x^i(\tau). \quad (40)$$

A useful fashion in which to visualise a particle trajectory is in the context of a *spacetime diagram*. Suppose, for simplicity, that one works in a frame of reference where the only spatial motion is in
the $x$-direction. It is then natural to generate a plot of the $ct - x$ plane, to which the spacetime motion is necessarily restricted.

In such a plot, trajectories with $v = c$ correspond to straight lines with slope $\pm 1$. If one were somehow to plot all three spatial dimensions, these lines would entail cuts through a cone, so that these trajectories of slope unity are termed the light cone.

A particle which is at rest in the specified frame of reference has slope $\rightarrow \infty$, i.e., it corresponds to a vertical line. Physical trajectories with $v < c$ which pass through $x = 0$ at $t = 0$ must always travel with slope of magnitude greater than unity, i.e., travel from the origin into the absolute future and arrive at the origin from the absolute past. Trajectories with slopes of magnitude less than unity are unphysical since they would correspond to particles travelling with speeds $v > c$.

Suppose now that one considers a different frame of reference moving relative to the original frame of reference with speed $v$ in the $x$-direction, but which agrees as to the location of the origin $t = x = 0$. Spacetime coordinates in that new frame can be identified in the original spacetime diagram if one recognises that the new and old coordinates $t, x$ and $t', x'$ are related by a Lorentz transformation, which corresponds to an imaginary rotation. If the Lorentz transformation were an honest rotation, all that would happen is that the new frame of reference involves a uniform rotation about the origin by some angle $\theta$. Given, however, that one is dealing with an imaginary rotation, things are a bit more subtle. Specifically, what one finds is that the $ct'$ and $x'$ axes have been rotated towards one another in such a fashion that the angle between them, as computed in the original frame of reference, is smaller than $90^\circ$.

Horizontal lines in a spacetime diagram correspond to surfaces of simultaneity. In other words, all points on these surfaces correspond in the specified frame of reference to the same instant of coordinate time $t$. If, in the original spacetime diagram, one exhibits surfaces of simultaneity as defined in another inertial frame, these surfaces will not in general correspond to horizontal lines. What this means is that different observers moving relative to one another will not necessarily agree as to whether two events happened at the same instant of time or even which of two events happened first.
Special relativistic kinematics

The ordinary three-velocity \( \vec{v} = d\vec{x}/dt \), with components \( v^\alpha = dx^\alpha/dt \), as defined in Newtonian mechanics, does not transform as a vector in Minkowski space and, as such, is not a natural object to consider in special relativity. Instead, it is natural to define a four-velocity

\[
  u^i = \frac{dx^i}{d\tau},
\]

(41)

Because \( dx^i \) transforms as a vector under Lorentz transformations and \( d\tau \) is invariant under Lorentz transformations, \( u^i \) clearly transforms as a vector. \( u^i \) is easily related to the ordinary three-velocity in a specific frame of reference.

One of the fundamental assumptions of special relativity is that rates of all moving clocks depend only on their velocity, so that an accelerated clock will be time dilated in the same way as a uniformly moving one with the same instantaneous speed. Thus, using \( dt = d\tau/\sqrt{1-v^2} \), one has

\[
  u^t = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}},
\]

(42)

and, e.g.,

\[
  u^x = \frac{dx}{d\tau} = \frac{dx}{dt} \frac{dt}{d\tau} = \frac{v^x}{\sqrt{1-v^2}}.
\]

(43)

The four-velocity thus has components

\[
  u^i = (\gamma, \gamma \vec{v}) \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1-v^2}}.
\]

(44)

Note that the scalar product of \( \vec{u} \) with itself is

\[
  \vec{u} \cdot \vec{u} = -1.
\]

(45)

Motion of a free particle

In the absence of forces, particles move with constant speed. This is equivalent to the statement that

\[
  \frac{d\vec{u}}{d\tau} = 0,
\]

(46)

where \( d\vec{u}/d\tau \) defines the four-acceleration.

How should this be generalised to an analogue of Newton’s Second Law \( \vec{F} = m\vec{a} \)? Any plausible rule must satisfy three criteria, viz:

1) It must satisfy the principle of relativity, i.e., take the same form in every inertial frame.
2) It must reduce to \( d\vec{u}/d\tau = 0 \) for vanishing ‘force’.
3) It must reduce to \( \vec{F} = m\vec{a} \) for speeds \( v \ll 1 \).

The obvious candidate is

\[
  m \frac{d\vec{u}}{d\tau} = \vec{f},
\]

(47)

where \( m \) is a constant characterising the particle, called the rest mass, and \( \vec{f} \) is the four-force. This clearly satisfies (1) and (2). For appropriate choices of \( \vec{f} \), (3) is also satisfied.

By analogy with Newtonian mechanics, one is then led to identify the relativistic energy-momentum four vector

\[
  \vec{p} = m\vec{u},
\]

(48)
which has components
\[ p' = \frac{m}{\sqrt{1 - v^2}} \quad \text{and} \quad \mathbf{p} = \frac{m \mathbf{v}}{\sqrt{1 - v^2}}. \] (49)

For small velocities, \( v \ll 1 \), one has
\[ p' \rightarrow m + \frac{1}{2} m v^2 \quad \text{and} \quad \mathbf{p} \rightarrow m \mathbf{v}, \] (50)
i.e., \( \mathbf{p} \) reduces to the usual momentum and \( p' \) yields the kinetic energy plus rest mass.

It follows from the definition of \( \mathbf{p} \) that
\[ \mathbf{p} \cdot \mathbf{p} = -m^2. \] (51)

Particles with \( m = 0 \) like the photon have both energy and momentum and, as such, an energy-momentum four-vector. They thus satisfy
\[ \mathbf{p} \cdot \mathbf{p} = 0. \] (52)

They can only satisfy \( p' = m/\sqrt{1 - v^2} \) and \( \mathbf{p} = m \mathbf{v}/\sqrt{1 - v^2} \) in the singular limit that \( v \rightarrow 1 \), so one can infer that all massless particles travel with the speed of light.

**Equations of motion from an action principle**

In spacetime, a free particle follows that worldline between two points which extremises the spacetime interval.

Consider an arbitrary path between points A and B. The elapsed proper time along this path is
\[ \tau_{AB} = \int_{A}^{B} d\tau = \int_{A}^{B} \sqrt{dt^2 - dx^2 - dy^2 - dz^2}. \] (53)

Suppose now that instead of the proper time the path is parameterised by another parameter \( \lambda \), so chosen that it takes a fixed value, say \( \lambda = 0 \), at point A and another fixed value, say \( \lambda = 1 \), at point B. The path is then specified by giving the coordinates as a function of \( \lambda \), i.e., \( x^i = x^i(\lambda) \).

One thus has
\[ \tau_{AB} = \int_{0}^{1} d\lambda \left[ \left( \frac{dt}{d\lambda} \right)^2 - \left( \frac{dx}{d\lambda} \right)^2 - \left( \frac{dy}{d\lambda} \right)^2 - \left( \frac{dz}{d\lambda} \right)^2 \right]^{1/2} = \int_{0}^{1} d\lambda \left[ -g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right]^{1/2}. \] (54)

What one wants is to determine the path (or paths) which extremise \( \tau_{AB} \), i.e., for which a small variation \( \delta x^i(\lambda) \) produces a vanishing variation in elapsed proper time, i.e., \( \delta \tau_{AB} = 0 \).

This is a standard problem from classical mechanics, which implies the Lagrange equations
\[ \frac{d}{d\lambda} \left( \frac{\partial L}{\partial (dx^i/d\lambda)} \right) - \frac{\partial L}{\partial x^i} = 0 \quad \text{with} \quad L = \left[ -g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} \right]^{1/2}. \] (55)

This yields
\[ \frac{d}{d\lambda} \left[ \left( -g_{ik} \frac{dx^k}{d\lambda} \frac{dx^i}{d\lambda} \right)^{-1/2} \frac{dx^i}{d\lambda} \right] = 0. \] (56)

But \( [-g_{jk}(dx^j/d\lambda)(dx^k/d\lambda)]^{1/2} = d\tau/d\lambda \), so that, multiplying by \( d\lambda/d\tau \) and raising the index \( i \), one has
\[ \frac{d^2 x^i}{d\tau^2} = 0. \] (57)
What an observer observes

An observer carries along four orthogonal unit vectors, \( \vec{e}_0, \vec{e}_1, \vec{e}_2, \) and \( \vec{e}_3 \), which define, respectively, his/her time direction and three spatial directions relative to which the observer makes observations. The timelike unit vector \( \vec{e}_0 \) will be tangent to the observer’s world line since that is the spacetime direction in which his clocks are moving. Thus \( \vec{e}_0 = \vec{u} \).

All the results of all measurements of an observer can be computed in terms of the projections of physical quantities onto the orthogonal unit vectors which define the inertial frame of the observer.

For example, it is easily seen that the measured energy of a particle is

\[
E = -\vec{p} \cdot \vec{e}_0 = -\vec{p} \cdot \vec{u},
\]

where \( \vec{p} \) is the four-momentum of the particle and \( \vec{u} \) is the four-velocity of the observer. Note that the energy \( E \) defined in this fashion is a scalar, i.e., invariant under Lorentz transformations, so that this result holds in any frame of reference. Suppose, therefore, that one considers a frame of reference in which the particle is at rest and the observer moves with velocity \( \vec{u} \) in the \( x \)-direction. One then has

\[
\vec{p} = (m, 0, 0, 0) \quad \text{and} \quad \vec{e}_0 = \vec{u} = (\gamma, \gamma v, 0, 0),
\]

where once again \( \gamma = 1/\sqrt{1 - v^2} \). It thus follows that

\[
E = -\vec{p} \cdot \vec{e}_0 = g_{ij} \vec{p}^i \vec{e}_0^j = -g_{\mu\gamma} m \gamma = \frac{m}{\sqrt{1 - v^2}},
\]

which is obviously the correct answer.

A less trivial example: In one particular inertial frame a particle is moving to the right with \( v = 1/2 \). An observer whose world line intersects the particle world line moves with speed \( 4/5 \). What energy does the observer measure? Velocity \( v = 1/2 \) corresponds to \( \gamma = 1/\sqrt{1 - v^2} = \sqrt{3}/3 \); velocity \( v = 4/5 \) yields \( \gamma = 5/3 \). Thus, the particle four-momentum \( \vec{p} \) and the observer four-velocity \( \vec{u} \) satisfy

\[
\vec{p} = m \left( \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, 0 \right) \quad \text{and} \quad \vec{e}_0 = \vec{u} = \left( \frac{5}{3}, \frac{4}{3}, 0, 0 \right).
\]

It follows that

\[
E = -\vec{p} \cdot \vec{e}_0 = -m \left[ -\left( \frac{2}{\sqrt{3}} \right) \left( \frac{5}{3} \right) + \left( \frac{1}{\sqrt{3}} \right) \left( \frac{4}{3} \right) \right] = \frac{2m}{\sqrt{3}}.
\]

In a similar fashion, the measured spatial momentum in the \( e_\alpha \) direction satisfies \( P = \vec{p} \cdot \vec{e}_\alpha \).

An even less trivial example: Show that it is impossible for an isolated free electron to admit or absorb a photon. Conservation of four-momentum implies that

\[
\vec{p}_\gamma + \vec{p}_e = \vec{p}_e',
\]

where \( \vec{p}_\gamma \) represents the photon four-momentum and \( \vec{p}_e \) and \( \vec{p}_e' \) represent the initial and final electron momenta. Squaring this equality yields

\[
\vec{p}_\gamma \cdot \vec{p}_\gamma + 2\vec{p}_e \cdot \vec{p}_\gamma + \vec{p}_e \cdot \vec{p}_e = \vec{p}_e' \cdot \vec{p}_e' = 0 + 2\vec{p}_e \cdot \vec{p}_\gamma - m_e^2 = -m_e^2
\]

so that \( \vec{p}_e \cdot \vec{p}_\gamma = 0 \). But in the frame where \( \vec{p}_e = (m, 0) \) and \( \vec{p}_\gamma = (E, \vec{p}) \), this says that the photon energy must be zero, i.e., there is no photon.