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Overview of Essential Beam Physics and Transport

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Fermilab

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Outline

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- Part I
  - Basics of Particle Transport
  - Courant-Snyder Parameters and Beam Emittance
- Part II
  - Analytical Solution of Betatron Motion
  - Weak Focusing Synchrotron/Betatron
  - Connections to Matrix Approach
- Part III
  - The Stability Criterion and the Discovery of Strong Focusing
  - Periodic Optics and Tune Calculations
  - Momentum Dispersion

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Part I

Basics of Particle Transport

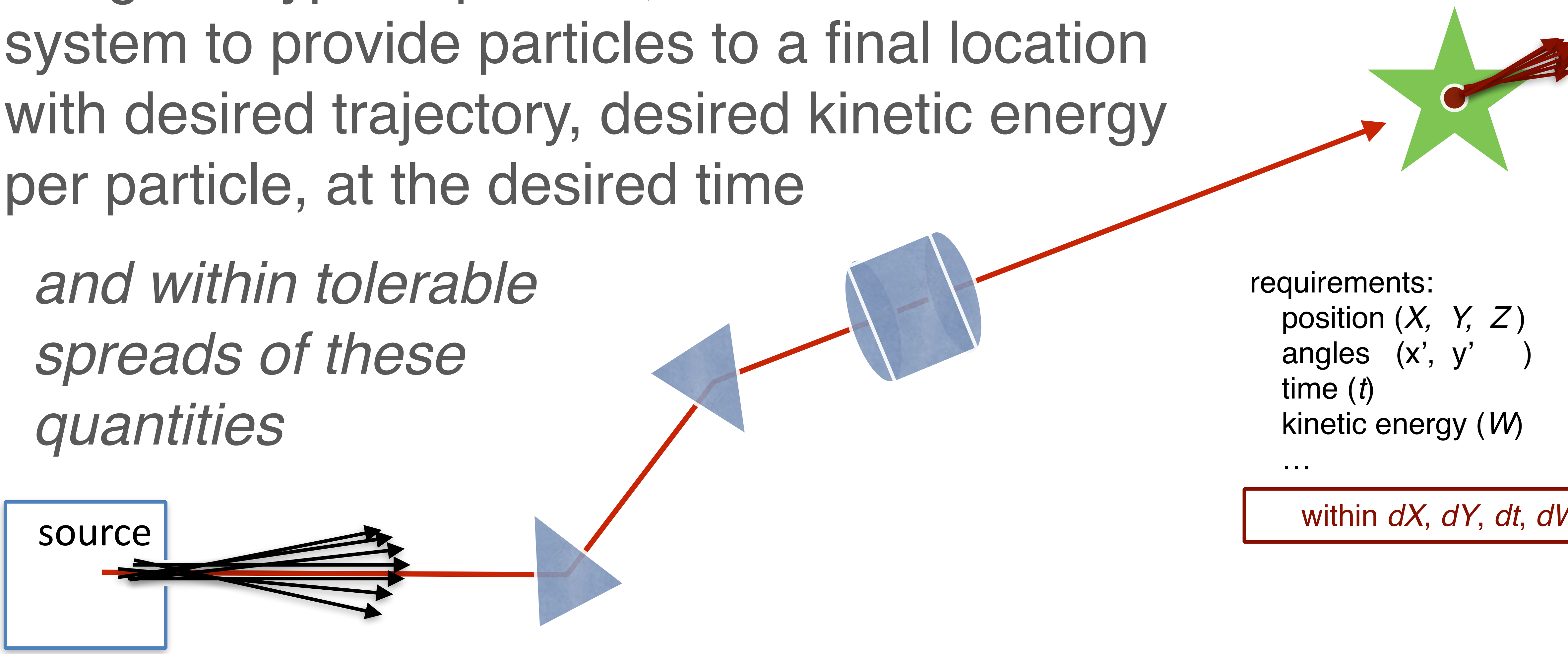
Courant-Snyder Parameters and Beam Emittance

The Problem

- 1927: Lord Rutherford requested a "copious supply" of projectiles "more energetic than natural alpha and beta particles"

- For given type of particle, create an ideal system to provide particles to a final location with desired trajectory, desired kinetic energy per particle, at the desired time

  and within tolerable spreads of these quantities



requirements:
position (X, Y, Z)
angles (x', y')
time (t)
kinetic energy (W)
…

within dX, dY, dt, dW, …

Single-Pass vs. Repetitive Systems

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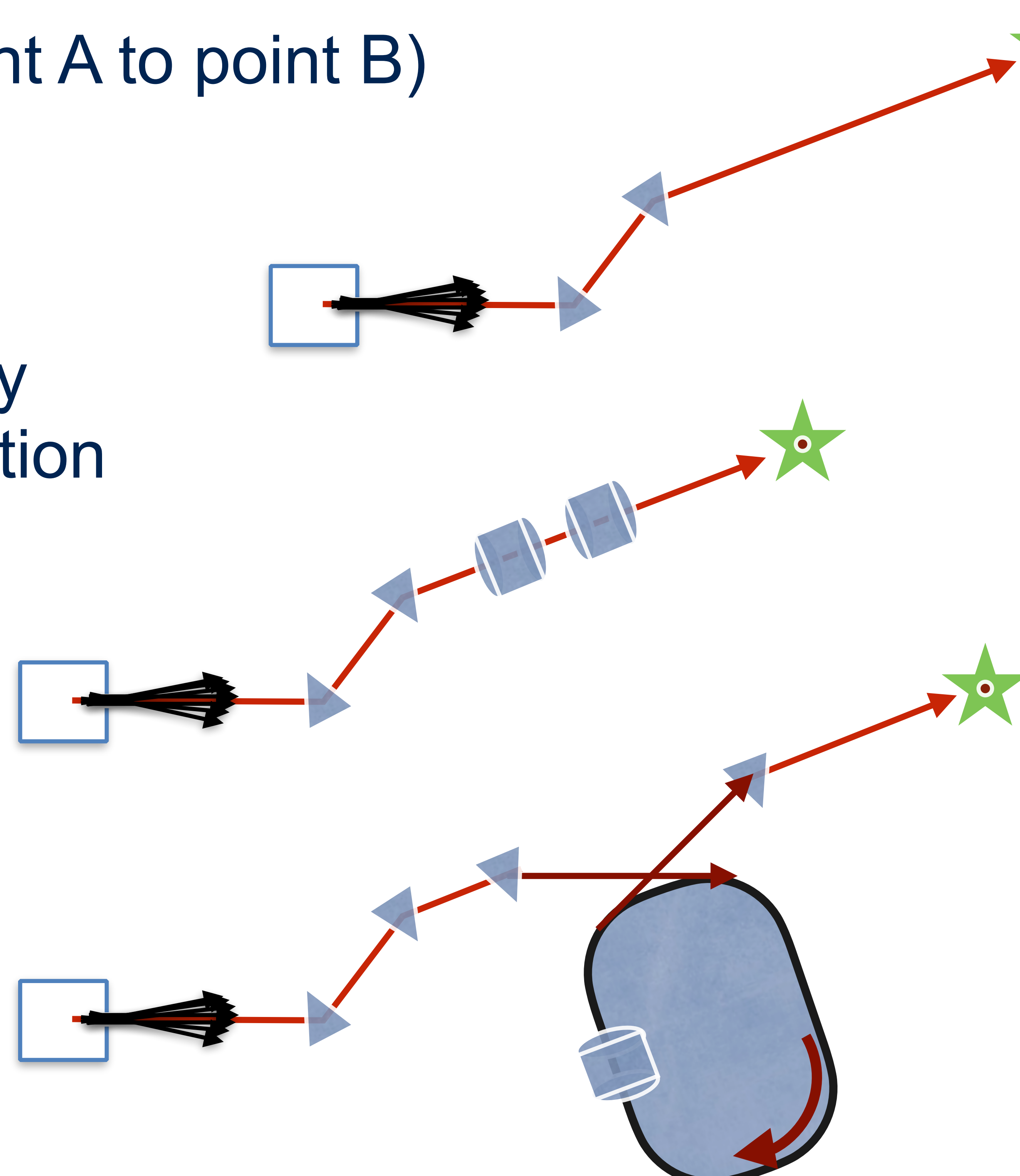
- Beam Transport (from point A to point B)
- Acceleration along the way
  - single-pass with acceleration
  - multi-pass acceleration



may need motion in such a system to be stable for many (millions or more?) revolutions

Stability of Motion Near the Ideal

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- Not all particles (any??) begin "on" the design trajectory with *exactly* the ideal energy/momentum
- We wish to have a system that will keep particles near the ideal conditions as they are transported (and possibly accelerated) through the system
- Particles emerge from their "source" with a slight divergence and will need to be guided back toward the ideal trajectory
- Also, particles with different energies/momenta will travel at different speeds, and hence may not arrive at cavities, experiments, etc., at the ideal time

Reduction of the Problem

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- Will treat transverse motion of particles through the accelerator as independent of the longitudinal motion, and study these two cases separately. Must show along the way that this is viable approach.
- Certainly not always be the case...
  - electric fields used for focusing at low energies can also accelerate the particles as well;
  - fields in the gaps of cavities will have focusing effects; etc.
- However, much of the "cross talk" can be minimized, and for much of the particle's journey, especially at higher energies, the major transverse focusing can be performed by magnetic fields -- particle's energy not changed
- Look at "linear" fields, *i.e.* linear restoring forces

Equations of Motion

- Lorentz Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

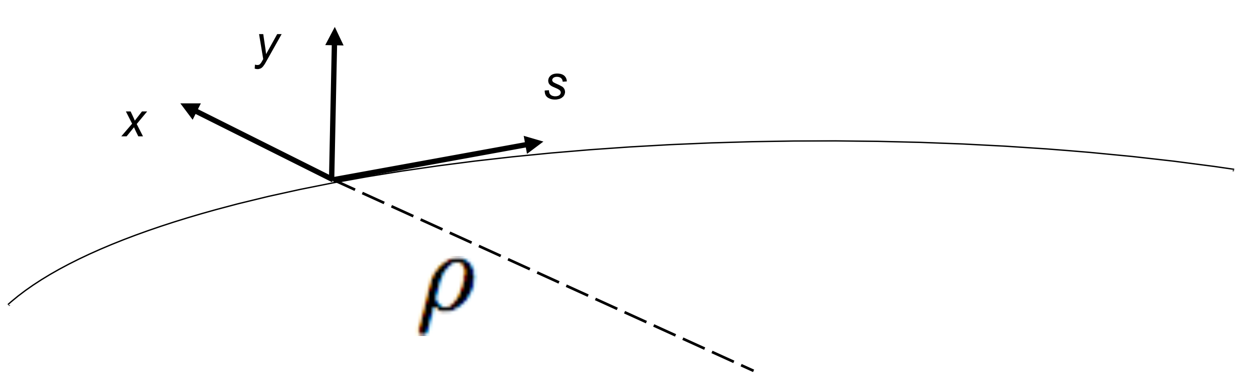
- Magnetic Rigidity
  - particle of unit charge, q = e:

$$B\rho \equiv \frac{p}{q} = \frac{p}{e}$$

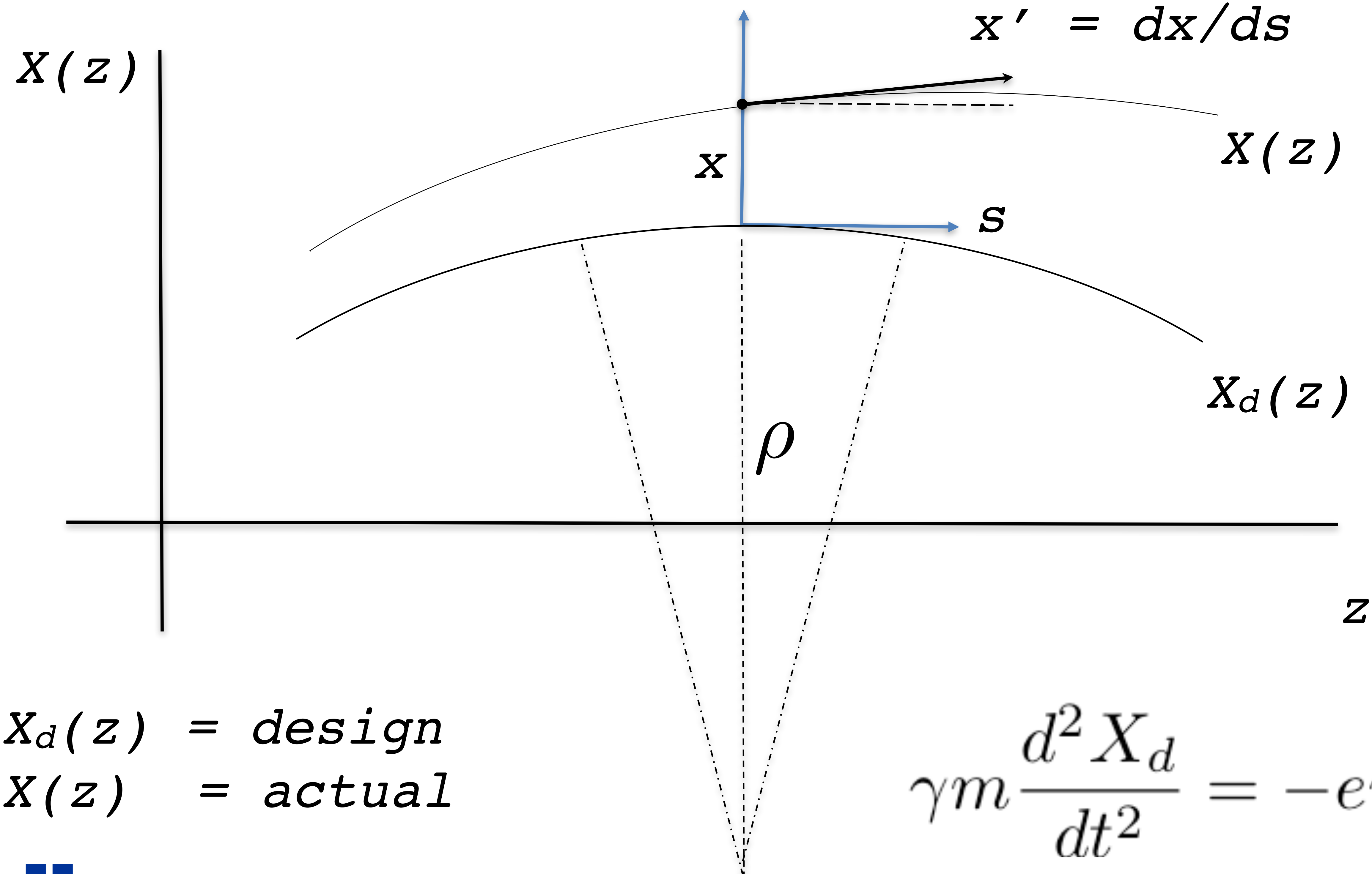
  - ion w/ mass A (atomic units, u), charge Q:

$$B\rho = \frac{A}{Q}\left(\frac{1}{300}\frac{\mathrm{T\cdot m}}{\mathrm{MeV/c/u}}\right) p_u$$

- Reference Trajectory
  - Local Coordinate System



Linear Magnetic Fields for Guiding & Focusing



$X_d(z)$ = design

$X(z)$ = actual

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Transverse Fields ($B_z = 0$)

- Drift Space: $B_x = 0$ $B_y = 0$
-
- Bending Region: $B_x = 0$ $B_y = B_0$
  - *(dipole magnet)*
- Focusing Region:
  - *(quadrupole magnet)* $B_x = B'y$ $B_y = B'x$
  - *(electrostatic quadrupole)* $E_y = -E'y$ $E_x = E'x$
- Combined Function Region: $B_x = B'y$ $B_y = B_0 + B'x$
  - *(uniform magnet + ES quad):* $B_x = 0, E_y = -E'y$ $B_y = B_0, E_x = E'x$
- Accelerating Device: $B_x = 0, B_y = 0;$ $E_z = V/g$
  - *(cavity)*

Linear Restoring Forces

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- Assume linear guide fields: --

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

- Look at radial motion:

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x' v_s$$

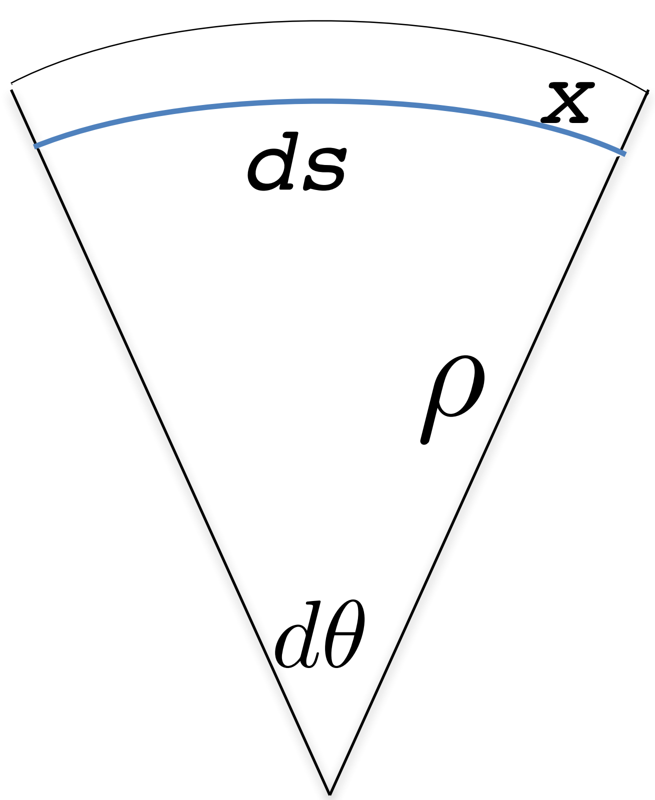
$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}$$

$$\ddot{r} - r\dot{\theta}^2 = -\frac{e v_s^2 B_y}{p}$$

$$v_s^2 \left(\frac{\rho}{r}\right)^2 x'' - (\rho + x)\left(\frac{v_s}{r}\right)^2 = -\frac{e v_s^2 B_y}{p}$$

$$dt = \frac{\rho + x}{\rho}\frac{ds}{v_s}$$

$$\frac{ds}{dt} = v_s \frac{\rho}{r}$$



$$x'' - \frac{\rho + x}{\rho^2} = -\frac{e B_y}{p}\left(\frac{r}{\rho}\right)^2$$

linearize…

$$x'' - \frac{1}{\rho} - \frac{x}{\rho^2} = -\frac{B_0 + B'x}{B\rho}\left(1 + \frac{2x}{\rho} + \frac{x^2}{\rho^2}\right)$$

$$x'' + \left(\frac{1}{\rho^2} + \frac{B'}{B\rho}\right) x = 0$$

Hill's Equation

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- Now, for vertical motion:

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

$$y'' = \frac{eB_x}{p}\left(\frac{r}{\rho}\right)^2$$

$$y'' - \frac{eB_x}{p}\left(1 + \frac{x}{\rho}\right)^2 = 0$$

- So we have,
  - to lowest order,

$$x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right)x = 0$$
$$y'' - \left(\frac{B'}{B\rho}\right)y = 0$$

linearize…

$$y'' - \frac{eB'y}{p} = 0$$

Hill's Equation

General Form: → $x'' + K(s)x = 0$

Piecewise Method of Solution

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- Hill's Equation: $$x'' + K(s)x = 0$$

- Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: drift $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$

- *K*

- *K*

Here, x refers to horizontal or vertical motion, with relevant value of K

Piecewise Method -- Matrix Formalism

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- Write solution to each piece in matrix form
  - for each, assume K = const. from s=0 to s=L

- $K = 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K > 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K < 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Determining *K* — *Examples*

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- Quadrupole Magnets

$$K_x = \frac{B'}{B\rho}$$
$$K_y = -\frac{B'}{B\rho}$$

- Sector Bend Dipole Magnets

$$K_x = \frac{1}{\rho^2}$$
$$K_y = 0$$

$$x'' + K_x\, x = 0$$
$$y'' + K_y\, y = 0$$

- Sector Bends with Electrostatic Focusing

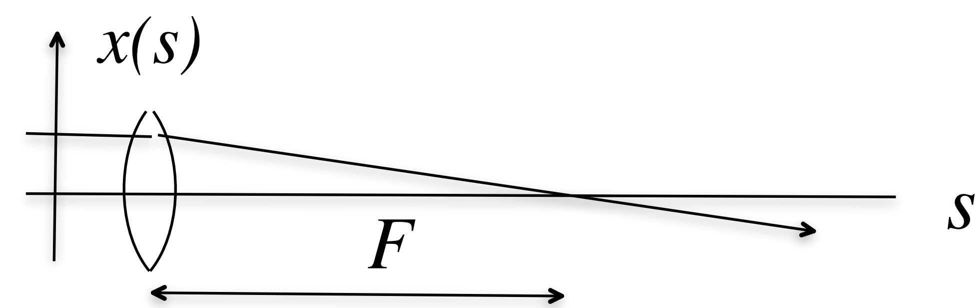
$$K_x = \frac{1}{\rho^2} - \frac{E'}{v(B\rho)}$$
$$K_y = \frac{E'}{v(B\rho)}$$

g-2 arrangement

- Other Considerations
  - Combined function magnet
  - Rectangular Bend
  - Bend with arbitrary Edge Angles

"Thin Lens" Quadrupole

- If quadrupole magnet is short enough, particle's offset through the quad does not change by much, but the slope of the trajectory does -- acts like a "thin lens" in geometrical optics



- Take limit as *L* --> 0, while *KL* remains finite
  - (similarly, for defocusing quadrupole)

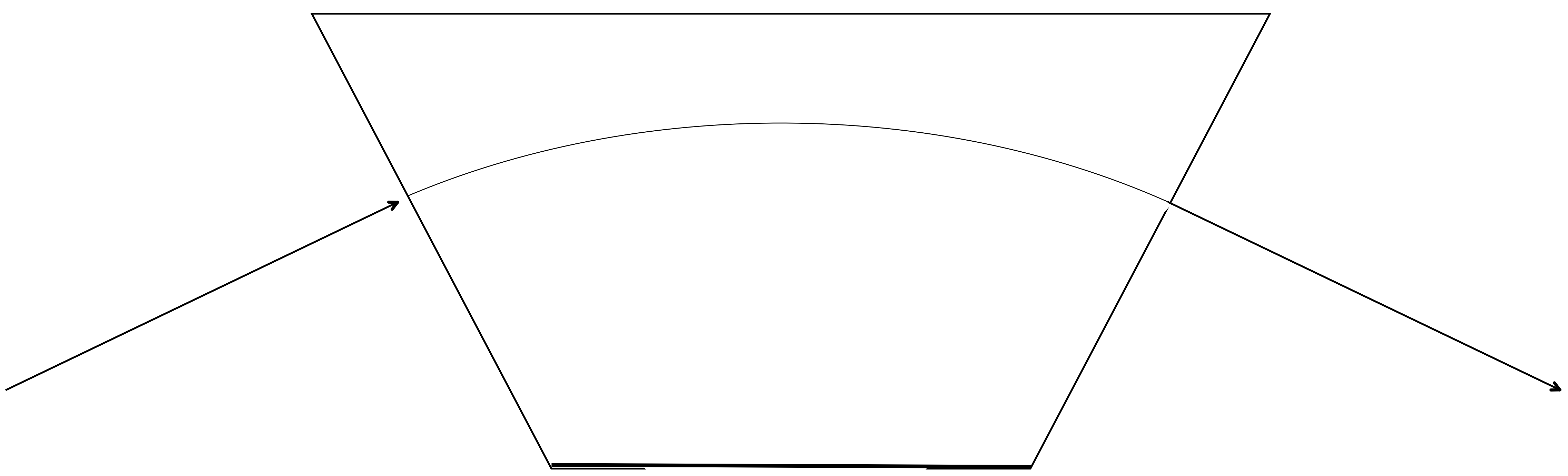
$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

- Valid approx., if *F* >> L

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

Sector Magnets

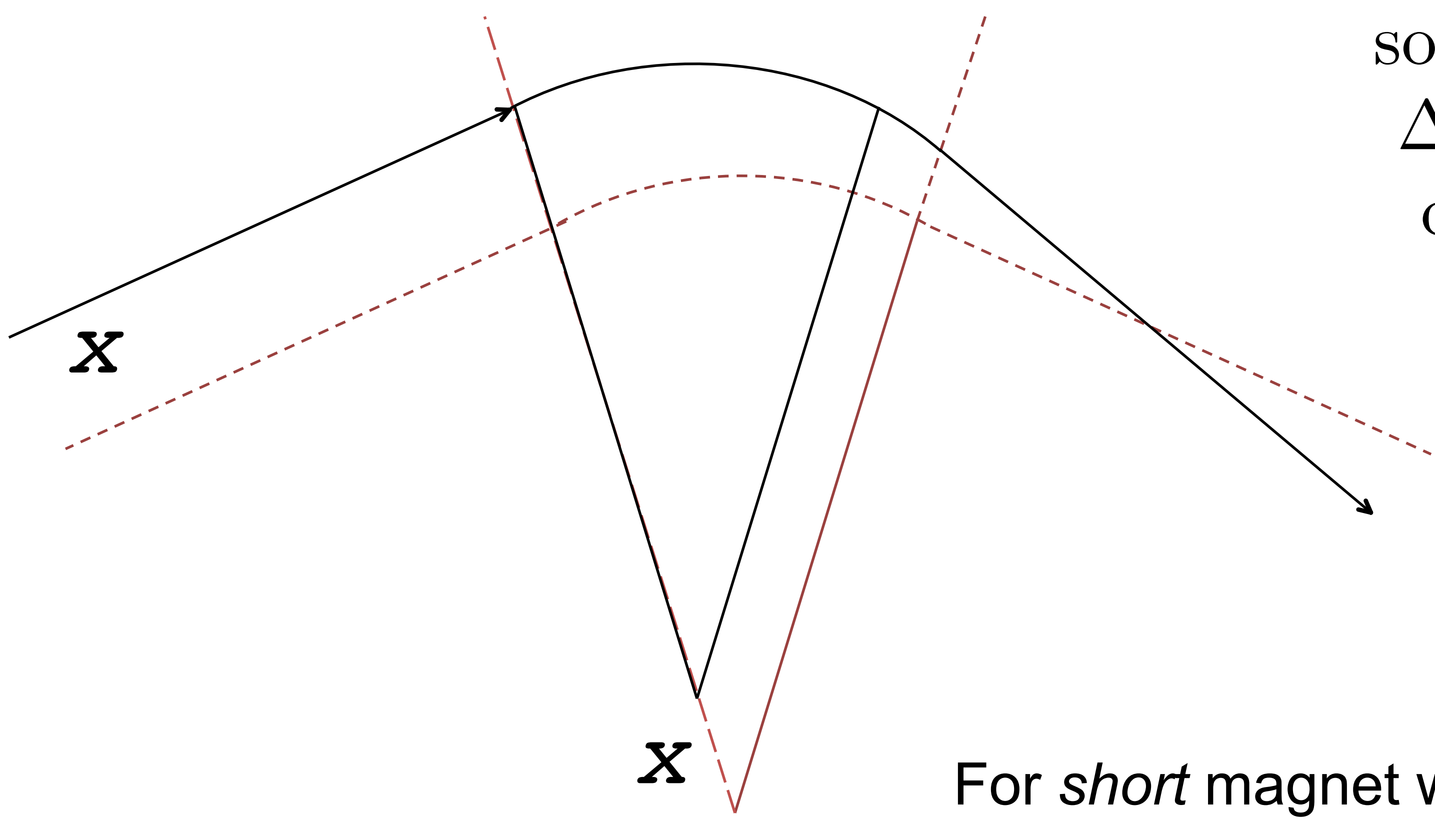
- Sector Dipole Magnet: "edge" of magnetic field is perpendicular to incoming/outgoing design trajectory:



Field points "*out of the page*"

Sector Magnets & Sector Focusing

- Incoming ray displaced from ideal trajectory will experience more/less bending field, thus is "focused" toward axis in the bend plane:



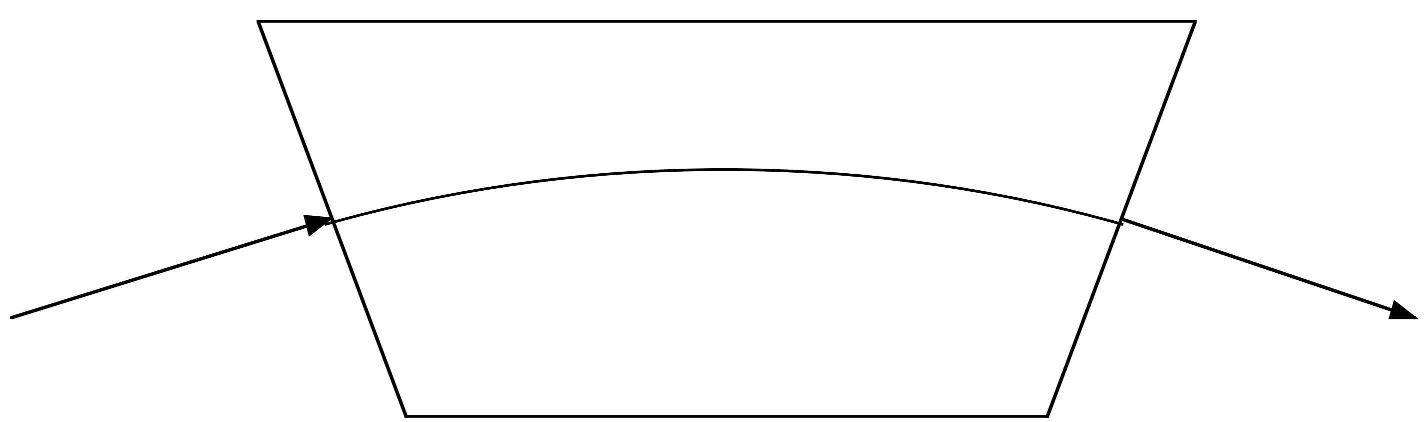
$$\text{Extra path length} = \Delta s = x\ \theta$$
$$\text{so extra bend angle} = \Delta x' = -\Delta s/\rho$$
$$\Delta x' = -(\theta/\rho)x = -(\ell/\rho^2)x$$
$$\text{or, } x'' = dx'/ds = -(1/\rho^2)x$$

$$\text{Thus, } K_x = 1/\rho^2,\ K_y = 0.$$

For *short* magnet with small bend angle, will also acts like lens in the bend plane ⇨ $$\frac{1}{f_x} = \frac{\theta}{\rho}$$

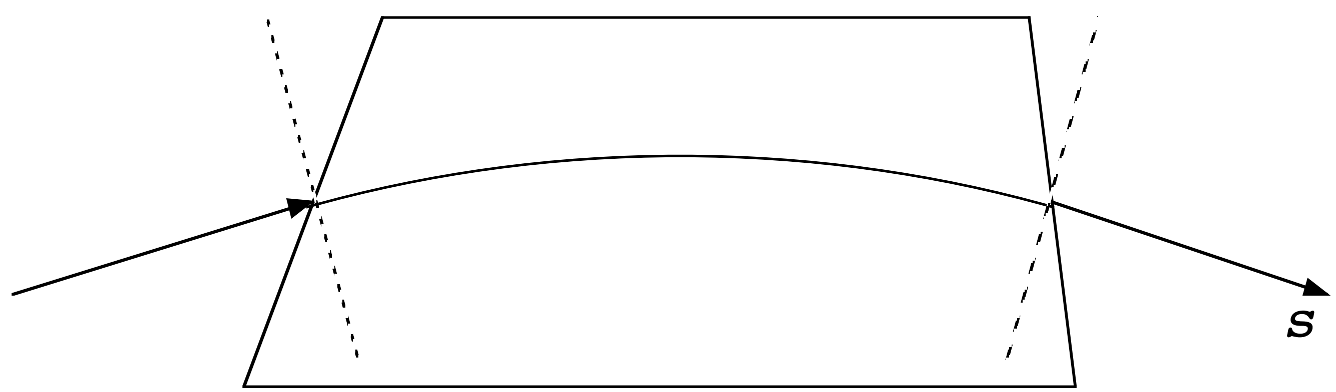
Edge Focusing

- In an ideal *sector magnet*, the magnetic field begins/ends exactly at $s = 0$, L independent of transverse coordinates x, y relative to the design trajectory.
  - *i.e.*, the face of the magnet is perpendicular to the design trajectory at entrance/exit



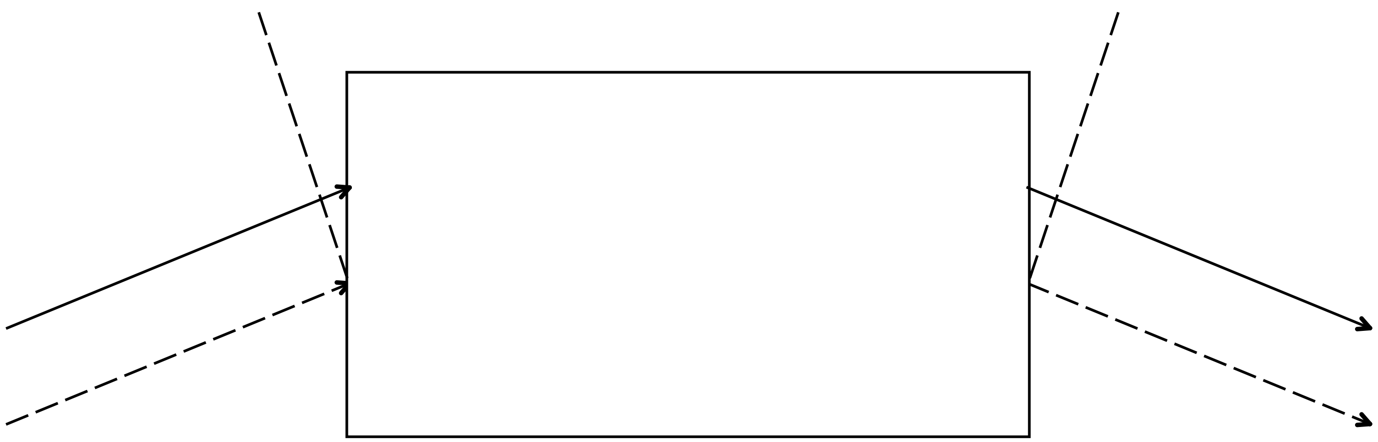
Edge Focusing

- However, could (and often do) have the faces at angles w.r.t. the design trajectory -- provides “edge focusing”



- Since our transverse coordinate x is everywhere perpendicular to s, then a particle entering with an offset will see more/less bending at the interface...

Rectangular Bending Magnet



In the bending plane, each edge acts as a defocusing lens with focal length:

$$\frac{1}{f} = -\frac{\theta/2}{\rho} = -\frac{\theta}{2\rho}$$

For Pure Sector Magnet,

hor: $$\frac{1}{f_x} \approx \frac{\theta}{\rho}$$

ver: $$\frac{1}{f_y} \approx 0$$

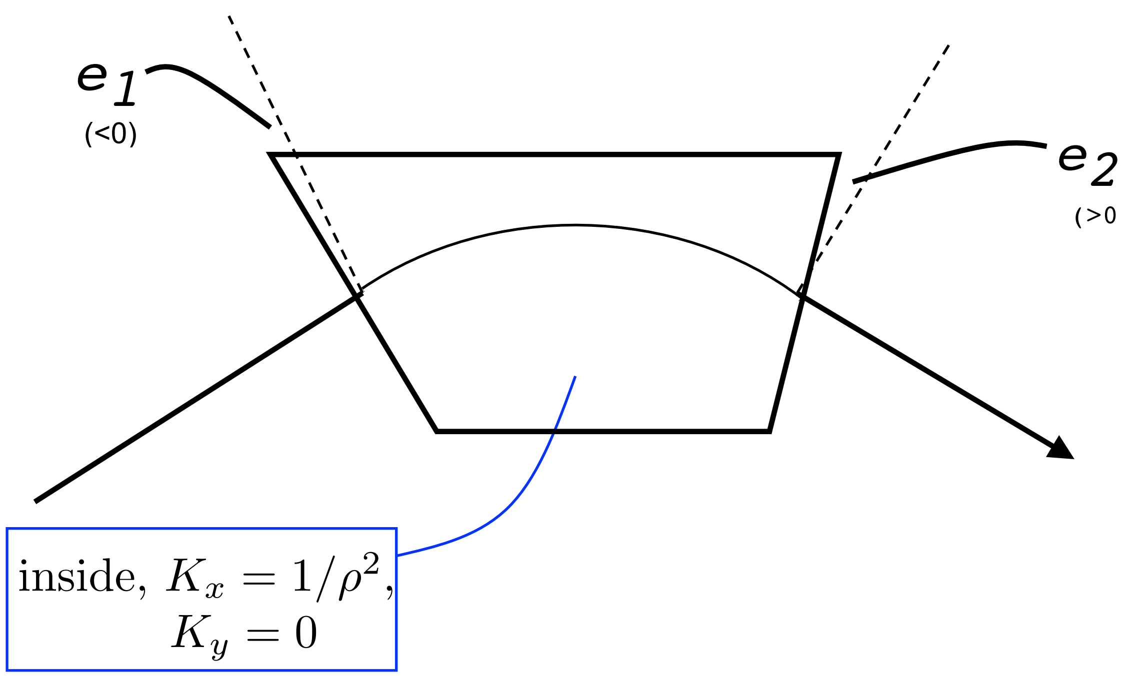
For Rectangular Magnet,

hor: $$\frac{1}{f_x} \approx -\frac{\theta}{2\rho} + \frac{\theta}{\rho} - \frac{\theta}{2\rho} = 0$$

ver: $$\frac{1}{f_y} \approx \frac{\theta}{2\rho} + \frac{\theta}{2\rho} = \frac{\theta}{\rho}$$

Transport through a General Bending Magnet

- Put all the pieces together…



- $M_{total} = M_{e2}\, M_{body}\, M_{e1}$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan e_2}{\rho} & 1 \end{pmatrix} \begin{pmatrix} \cos(\ell/\rho) & \frac{1}{\rho}\sin(\ell/\rho) \\ -\frac{1}{\rho}\sin(\ell/\rho) & \cos(\ell/\rho) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\tan e_1}{\rho} & 1 \end{pmatrix}$$

$$M_y = \begin{pmatrix} 1 & 0 \\ -\frac{\tan e_2}{\rho} & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\tan e_1}{\rho} & 1 \end{pmatrix}$$

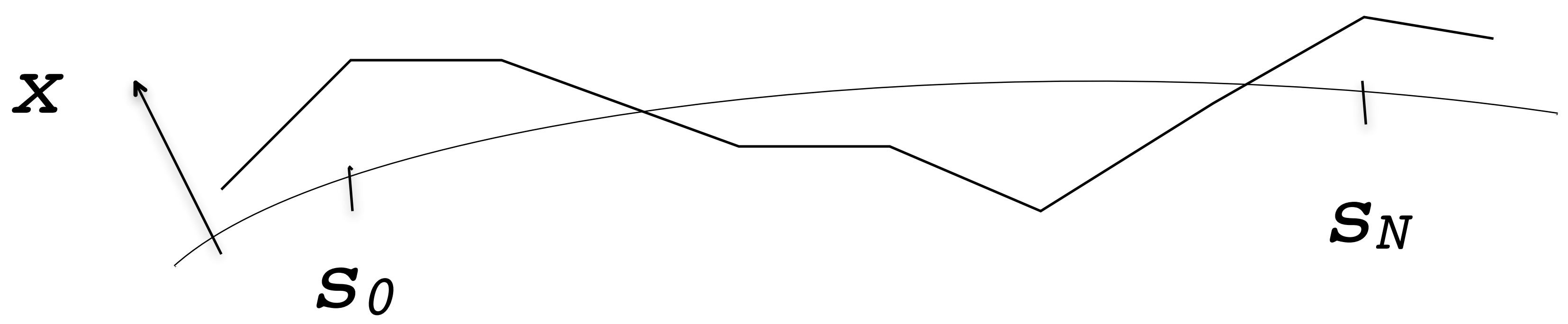
Comments

- Very often, especially for highly-relativistic particles, the bend radii with bending magnets can be large, and hence the sector focusing can be a small effect. However, in accelerators with dozens, hundreds, or thousands of elements, it can certainly add up.
- Same can be said for edge effects in many circumstances.
- One must always seek to understand the particular situation and determine what assumptions can be made for the level of detail one is studying.

Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Example: A Beam Line Calculation

- Will consider two particle trajectories, starting with
  - (x,x') = (0, 0.5 mrad), and (x,x') = (5 mm, 0)
- A distance 6 m later, the trajectories enter a thin lens quadrupole of focal length F = 3 m. This is followed by a second quadrupole of focal length -F, a distance 1 m later.

  - Find the trajectories (x,x') for each case at the exit of the second quad

Example: A Beam Line Calculation

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- A distance 6 m later, the trajectories enter a thin lens quadrupole of focal length F = 3 m. This is followed by a second quadrupole of focal length -F, a distance 1 m later.

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 1\ m \\ \frac{1}{3\ m} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ -\frac{1}{3\ m} & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

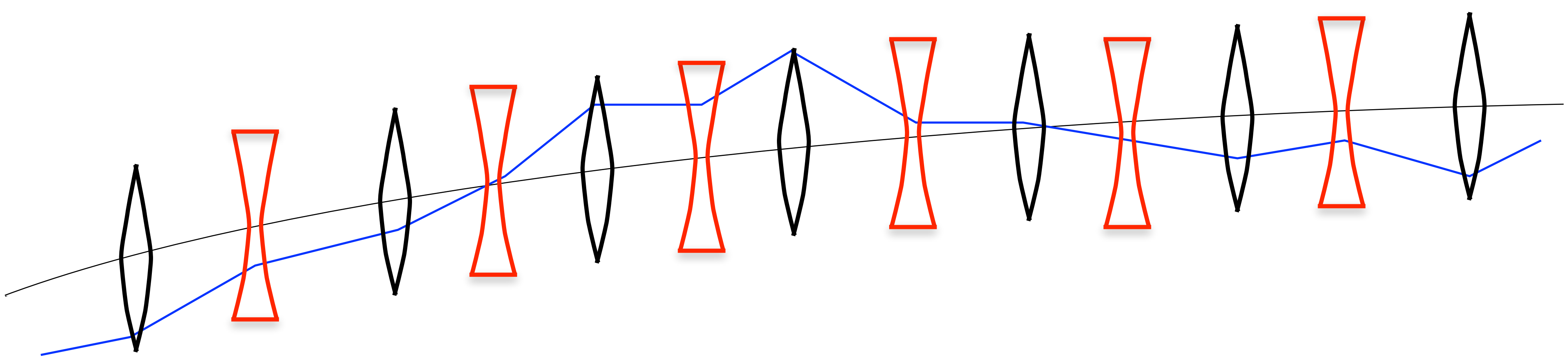
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 5\ m \\ -\frac{1}{9\ m} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

x_0 = 0 mm, x'_0 = 0.5 mr —> x = 2.5 mm, x' = 0.33 mr

x_0 = 5 mm, x'_0 = 0.0 mr —> x = 3.3 mm, x' = -0.6 mr

Can now make LARGE accelerators!

- Since the lens spacing can be made arbitrarily short, with corresponding focusing field strengths, then in principal can make beam transport systems (and linacs and synchrotrons, for instance) of arbitrary size

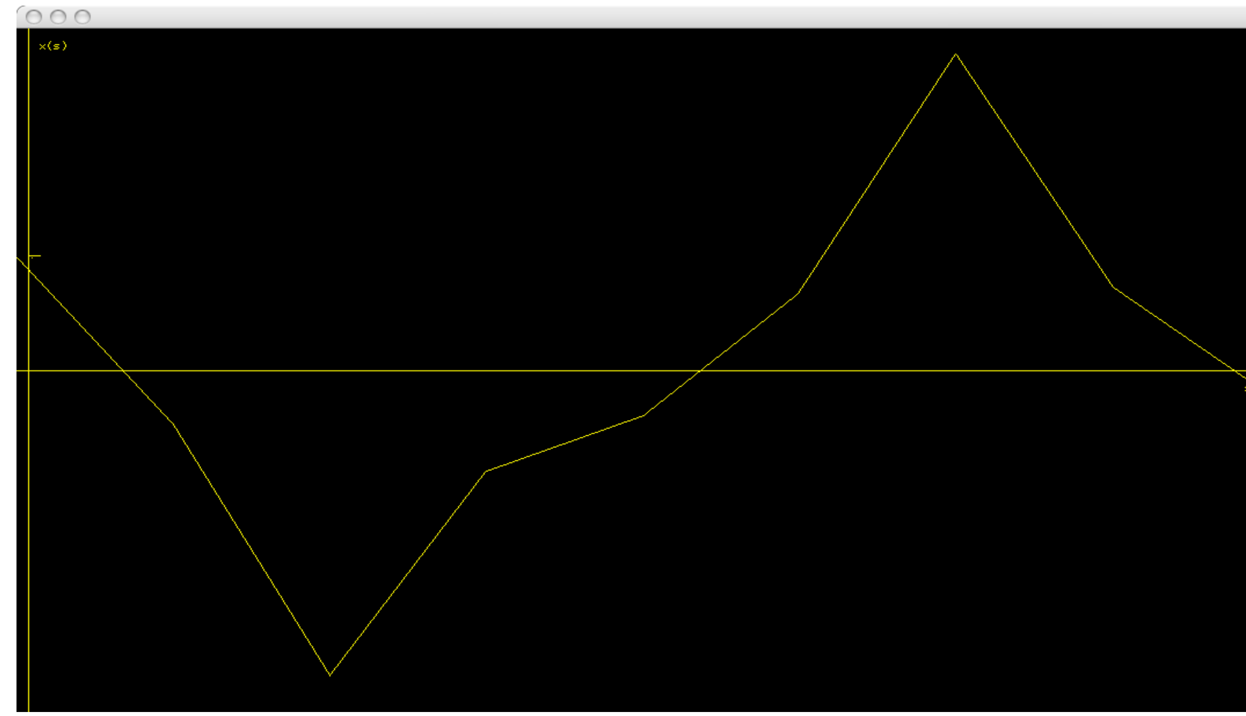


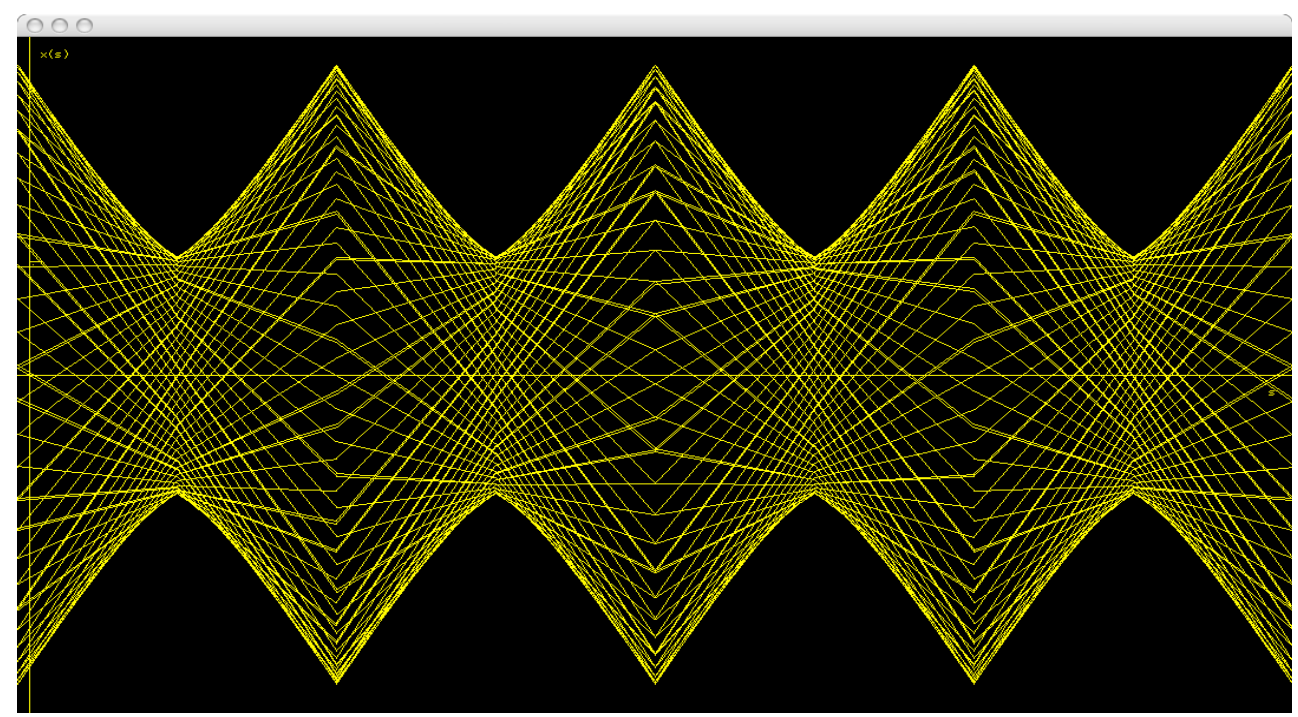
- Instrumental in paving the way for very large accelerators, both linacs and especially synchrotrons, where the bending and focusing functions can be separated into distinct magnet types

The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line



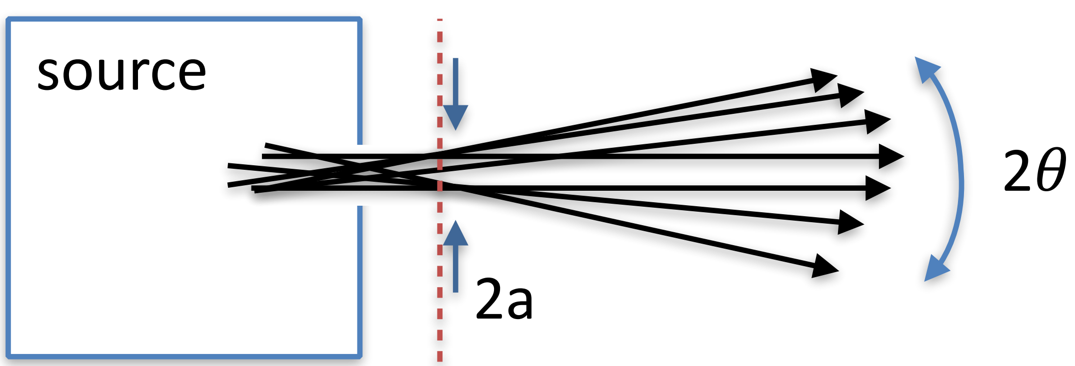


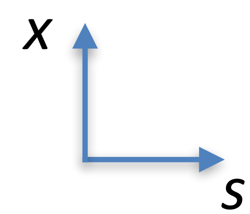
Can we describe the maximum amplitude of particle excursions in analytical form?

of course! *coming up soon …*

Particle Beams and Phase Space

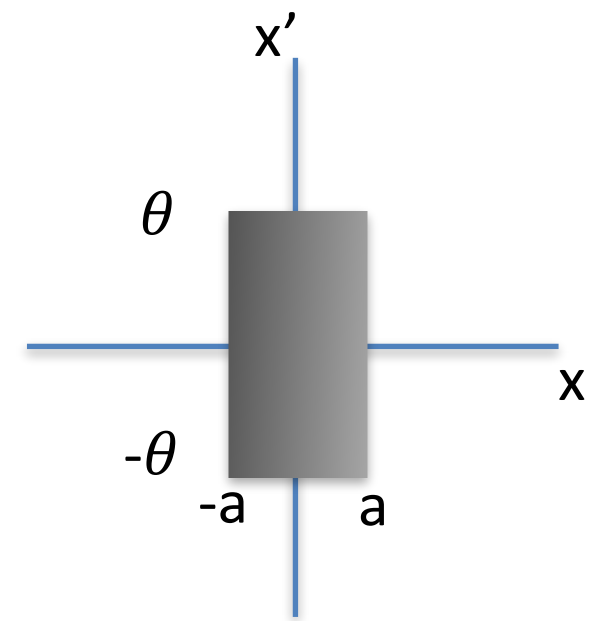
Transverse coordinates:



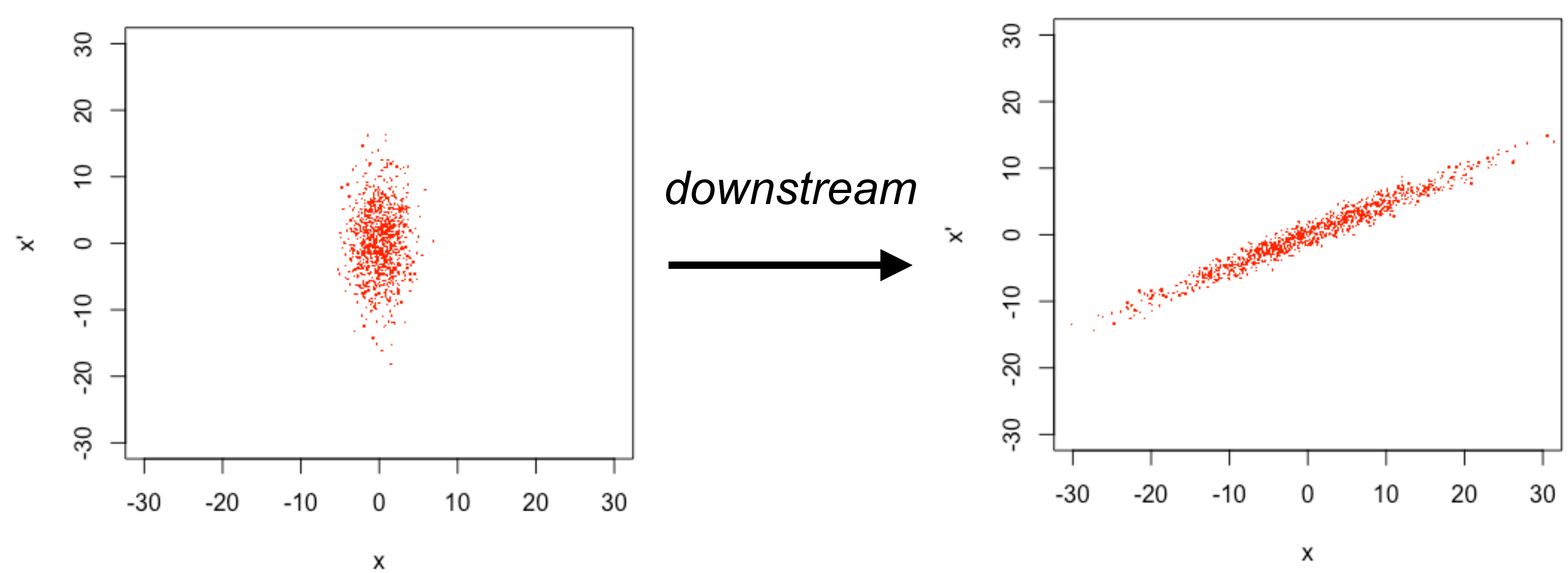


angle: $x' = dx/ds$

Phase Space:



Shape, orientation of distribution in "phase space" will change as particles progress downstream, but effective "area" of distribution will remain constant (*Liouville*); correlations will naturally develop



Emittance in Terms of Moments

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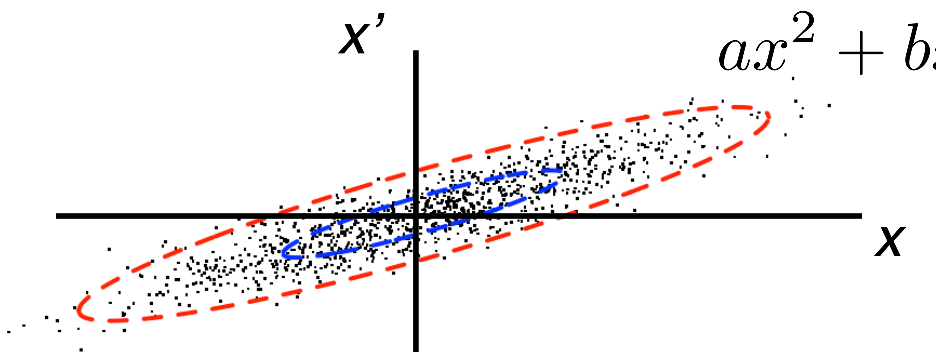
- Considering the general equation of an ellipse, the area enclosed by the ellipse is related to its coefficients by:



$$ax^2 + bxx' + cx'^2 = 1$$

area of ellipse:

$$\mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

- Can define quantities scaled by an area, ϵ, of our elliptical distribution:

$$\alpha \equiv -\frac{\langle xx' \rangle}{\epsilon/\pi} \qquad \beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \qquad \gamma \equiv \frac{\langle x'^2 \rangle}{\epsilon/\pi}$$

$$\epsilon = \pi\sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

the "rms emittance"

α, β, γ collectively are called the ***Courant-Snyder* parameters**, or *Twiss* parameters

So, equation of the **blue** curve above:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon/\pi$$

The ellipse (red curve above) that contains ~95% has area ~6ϵ

(*for Gaussian distribution*)

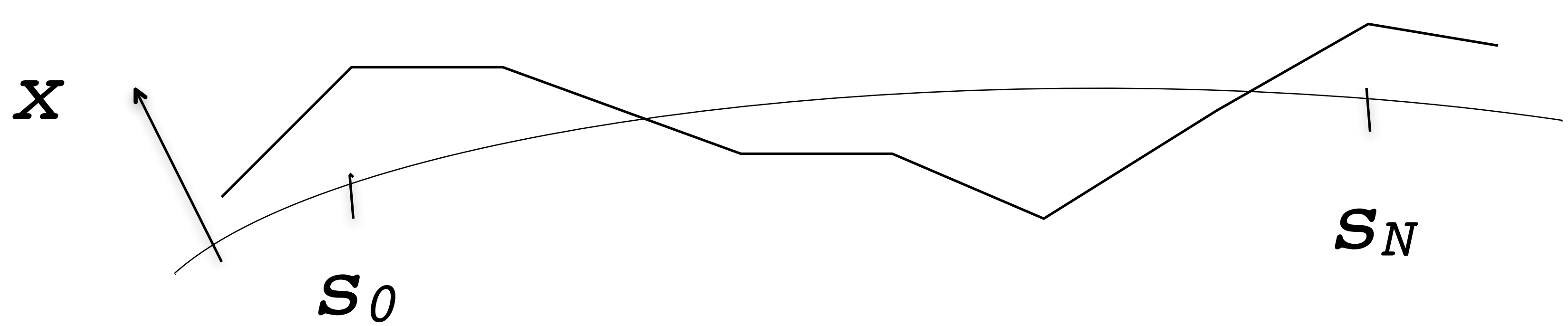
Linear Optics

- Let *x* be the transverse (horizontal, say) displacement of a particle from the ideal beam trajectory. Let the angle it makes to the ideal trajectory be $x' = dx/ds$, where *s* is the distance along the ideal trajectory. Transport through a magnetic element is then described by a matrix *M*, such that

$$\vec{X} = M\vec{X}_0 \qquad \vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

- An arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication for elements all along the beam line…

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



TRANSPORT of Beam Moments

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- Transport of particle state vector downstream from position 0

$$\vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix} \qquad \vec{X} = M\vec{X}_0$$

- Create a "covariance matrix" of the resulting vector…

$$\vec{X}\vec{X}^T = \begin{pmatrix} x^2 & xx' \\ x'x & x'^2 \end{pmatrix} = M\vec{X}_0(M\vec{X}_0)^T = M\vec{X}_0\vec{X}_0^T M^T$$

- … then, by averaging over all the particles in the distribution,

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

we get: $$\Sigma = M\Sigma_0 M^T$$

TRANSPORT of Beam Moments

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- So, since

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} \sim \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = \epsilon \cdot K$$

- where

$$K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\boxed{K = M\ K_0\ M^T}$$

- If know matrices *M*, then can "transport" beam parameters from one point to any point downstream, which determines beam distribution along the way.

$$\beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \quad \longrightarrow \quad x_{rms}(s) = \sqrt{\epsilon\beta(s)/\pi}$$

Conservation of Emittance

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- Note that from

$$\Sigma = M\Sigma_0 M^T$$

$$\Sigma = \epsilon \cdot K \qquad K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\det\Sigma = \det M \ \det\Sigma_0 \ \det M^T = \det\Sigma_0$$

- and

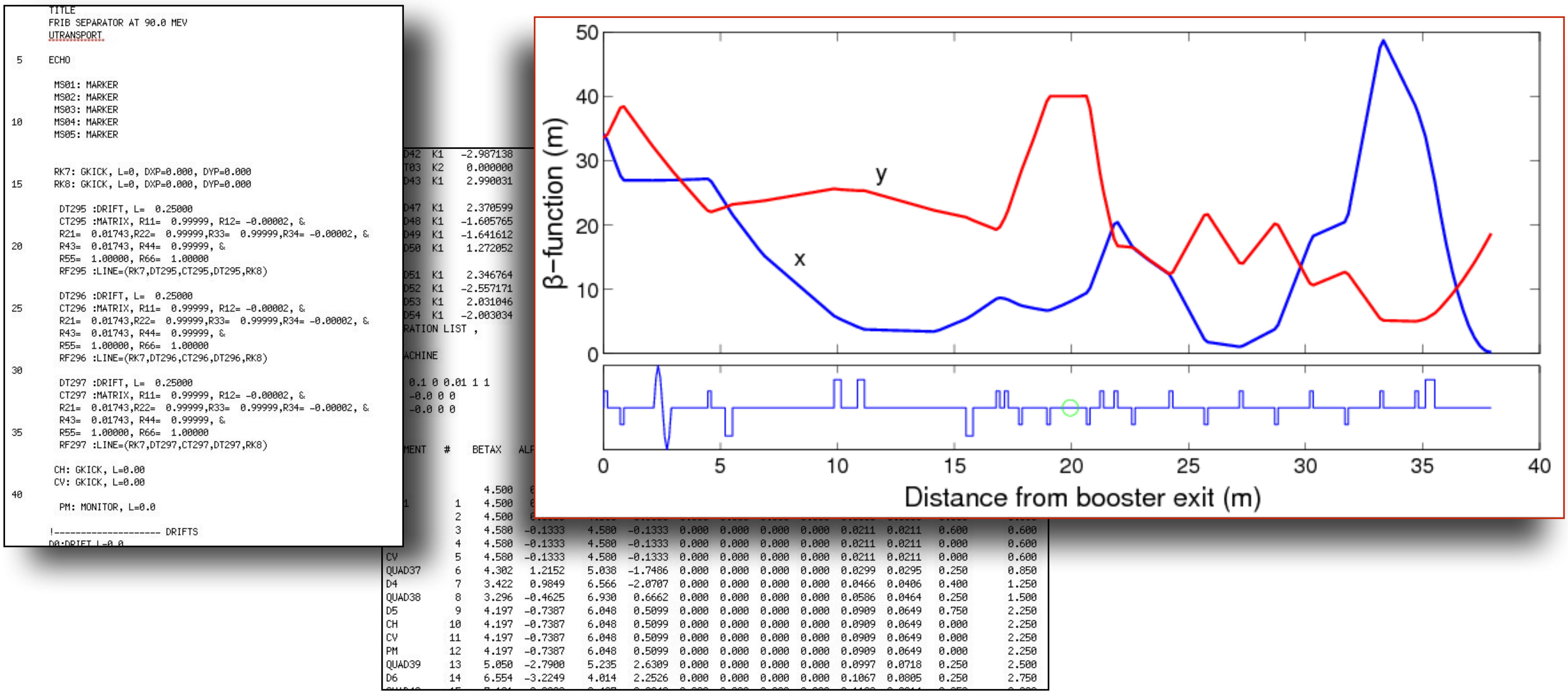
note: $\det M = 1$

$$\det\Sigma = \epsilon^2 \det K = \epsilon^2(\beta\gamma - \alpha^2) = \epsilon^2$$

- Thus, the emittance is conserved upon transport through the system

Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT, MAD, DIMAD, TRACE, TRACE3D, COSY, SYNCH, CHEF, many more ...



Let's Think About the Numbers & Units…

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$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

- If <x^2> ~ mm^2, and <x'^2> ~ mrad^2, then the emittance can have units of mm-mrad (*also* = μm)
- Courant-Snyder parameters

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon}$$

mm^2/(mm-mrad) ~ mm/mrad = m

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

(mm-mrad)/(mm-mrad) = dimensionless

$$\gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

mrad^2/(mm-mrad) ~ 1/m

The "π" comes from our definition of emittance as an area in phase space; emittance is often expressed in units of "π mm-mrad"

Summary

- Given an initial particle distribution in phase space at the input to a beam transport system, can describe that distribution (sometimes not all that well, but we try…) using Courant-Snyder parameters:

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon} \qquad \gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

- The C-S parameters can then be computed downstream, using

$$\Sigma = M\Sigma_0 M^T$$

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Part II

Analytical Solution of Betatron Motion

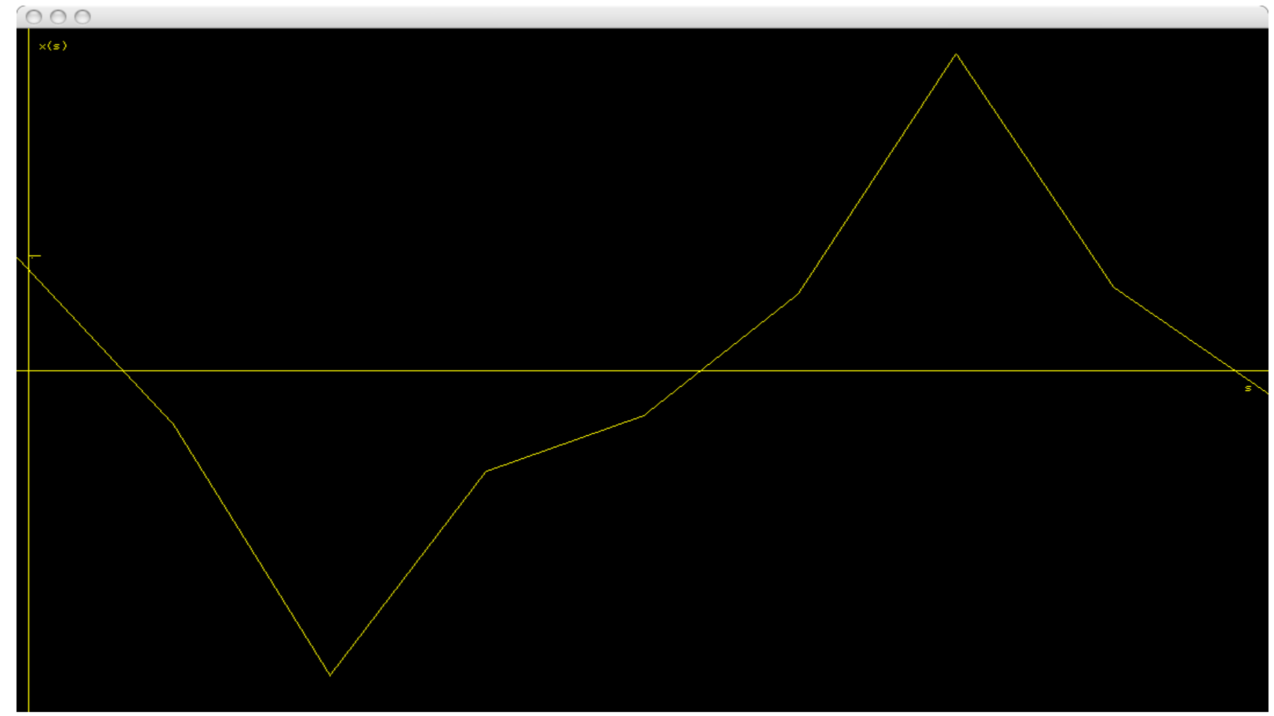
Weak Focusing Synchrotron/Betatron

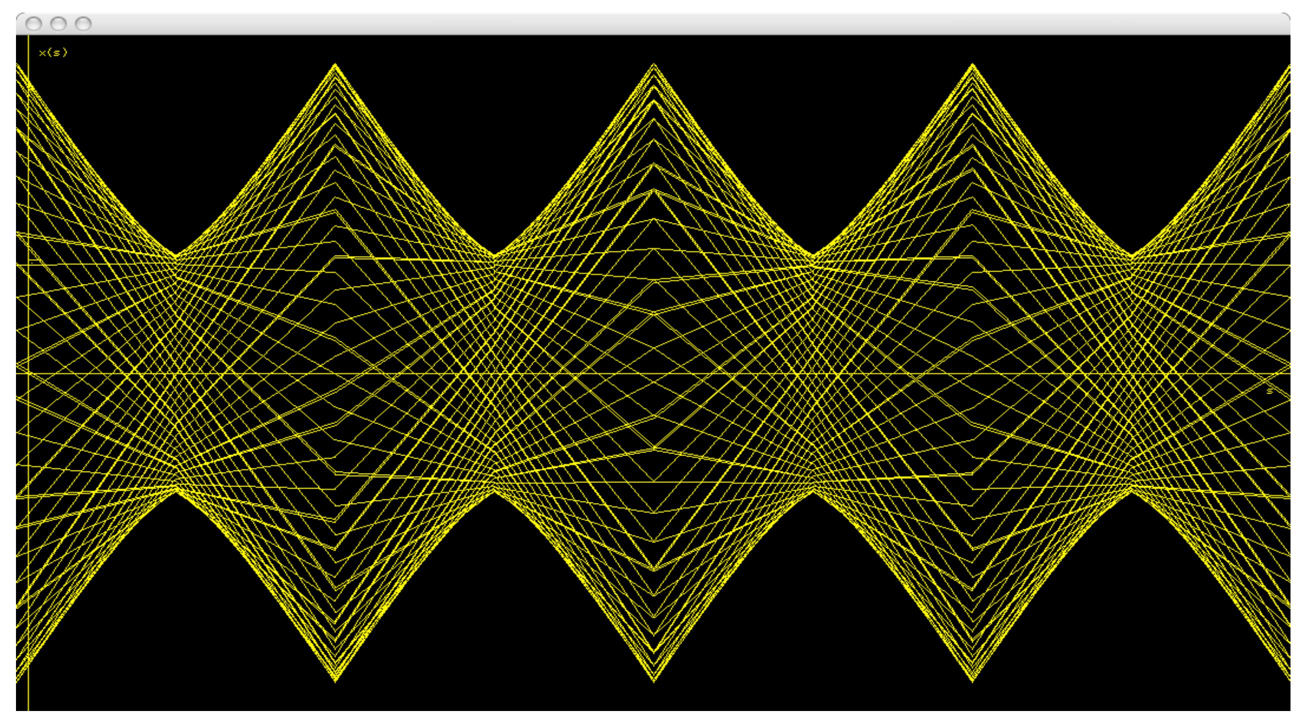
Courant-Snyder Invariant

The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line





Can we describe the maximum amplitude of particle excursions in analytical form?

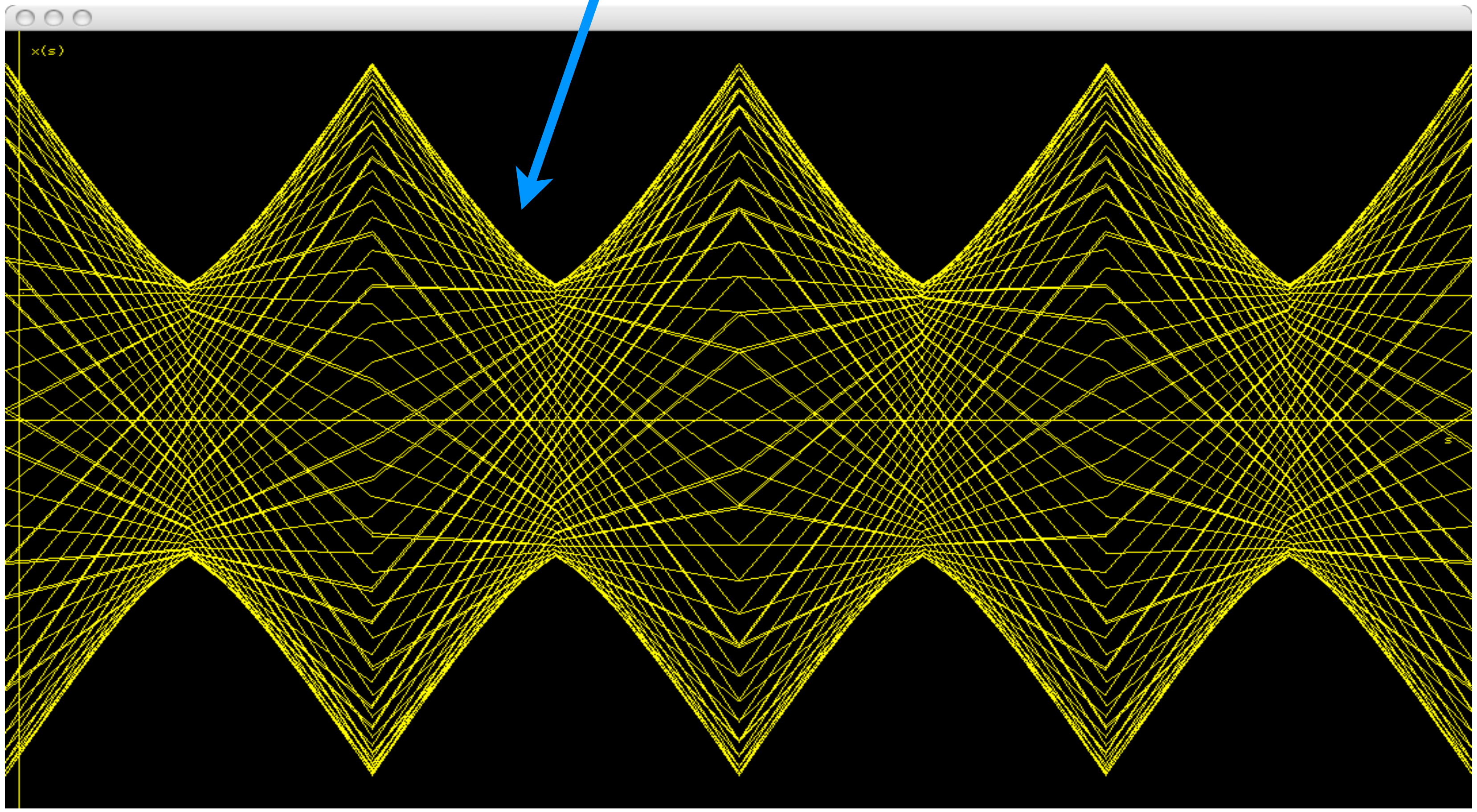
of course!   *coming up next …*

Pushing the "Envelope"

- Wish to look for a functional form of the outer envelope of particle motion, and the rate at which the phase of the oscillatory motion develops within that envelope

- This will enable us to decouple the motion of individual particle from intrinsic properties of the accelerator design

Envelope described by an "amplitude function"



Hill's Equation — Analytical Solution

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- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, *s*.
- So, assume solution is sinusoidal, with a phase which advances as a function of location *s*; also assume amplitude is modulated by a function which also depends upon *s*:

- Then, plug into Hill's Equation ...

$$\boxed{x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]}$$

Analytical Solution (cont'd)

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$$\begin{aligned} x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\ x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\ x'' &= \ldots \end{aligned}$$

Plugging into Hill's Equation, and collecting terms...

$$\begin{aligned} x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\ &\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0 \end{aligned}$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, must have contents of each [ ] = 0 simultaneously for *sum* = 0.

Analytical Solution (cont'd)

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- Thus, we must have ...

thus, we need

$$\psi'' + \frac{\beta'}{\beta}\psi' = 0$$

and

$$-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0$$

Some Comments

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But the amplitude function is also a local wavelength of the motion.
- This seems strange at first, but ...
  - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
  - Thus, the spacing and/or strengths (i.e., *K(s)*) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes must be tied together.

Equation of Motion of Amplitude Function

From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\boxed{\beta'' + 4K\beta = const.}$$

is the general equation of motion for the amplitude function, β.

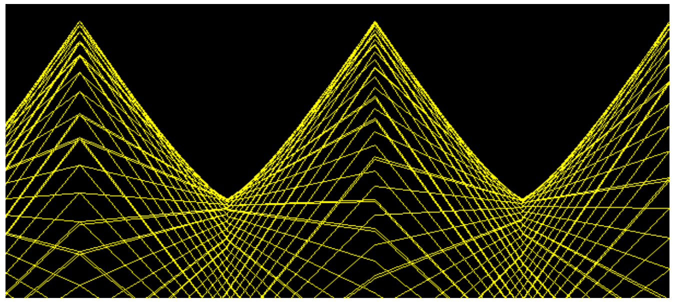
(in regions where K is either zero or constant)

Piecewise Solutions

- $K = 0$: $\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$ Parabola!
  - since $\beta > 0$, then from original diff. eq. …

$$2\beta\beta'' - (\beta')^2 = 4 \qquad \beta'' = \frac{4 + (\beta')^2}{2\beta} > 0$$

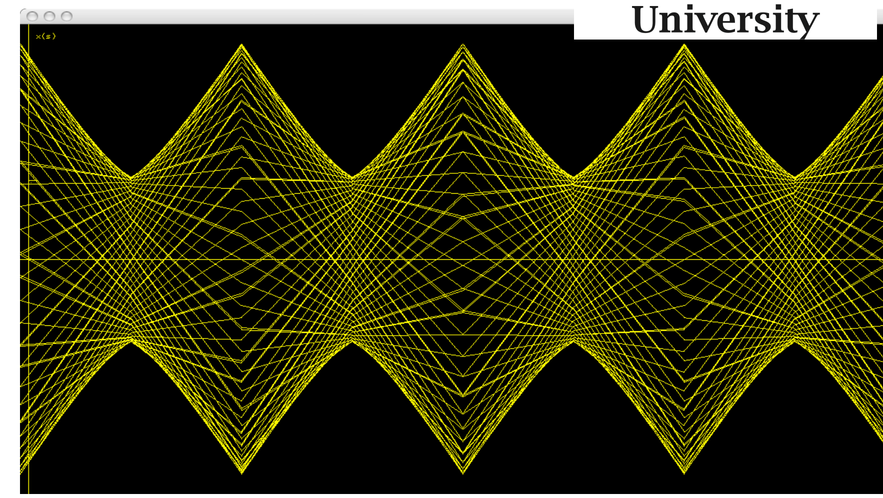
  - Therefore, parabola is ***always*** concave up



- $K > 0$, $K < 0$: sinusoidal + *constant*

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$$

Summary



$$x'' + K(s)x = 0$$ Hill's Equation

trial solution: $x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$

requires:

$$\psi' = 1/\beta \quad \longrightarrow \quad \psi(s) = \int \frac{ds}{\beta(s)}$$

and

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \quad \longrightarrow \quad \beta'' + 4K\beta = const.$$

(for *K'* = 0)

for *K* = 0 : $\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$

for *K* > 0 : $\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$

Courant-Snyder Parameters, & Connection to Matrix Approach

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- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Previously have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,

$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}$$

- Collectively, β, α, γ are called the *Courant-Snyder Parameters* (sometimes called "Twiss parameters" or "lattice parameters")

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \qquad \textit{becomes} \qquad K\beta = \gamma + \alpha'$$

Courant-Snyder Parameters, & Connection to Matrix Approach

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- Define two new variabl…
- Collectively, β, α … ed the *Courant-Snyder Parameters* (sometimes called …ss parameters" or "lattice parameters")

YES! They ARE the same
α, β, γ seen earlier!

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \qquad \textit{becomes} \qquad K\beta = \gamma + \alpha'$$

Solutions using Courant-Snyder Parameters

- Our previous results become

  - drift space:

$$\beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$

$$\text{-->} \quad \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

  - gradient field:

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$$

$$\text{-->} \quad \beta(s) = \frac{\beta_0}{2}[1 + \cos(2\sqrt{K}s)] - \frac{\alpha_0}{\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\gamma_0}{2K}[1 - \cos(2\sqrt{K}s)]$$

The Transport Matrix

- We can always write: $x(s) = a\sqrt{\beta}\sin\Delta\psi + b\sqrt{\beta}\cos\Delta\psi$

- Solve for *a* and *b* in terms of initial conditions and write in matrix form
  - we get:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

So, can write any of our transport matrices in terms of values of C-S parameters at the two end points, and the phase advance between them.

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

Tracking β, α, γ ...

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- Saw earlier that if given values of the Courant-Snyder parameters at one location in the beam line, and if know the matrix of the linear motion between that location and another location downstream, then can compute the values at the second location *via*:

$$K = M\ K_0\ M^T$$

where $K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$

- Have not explicitly proven that the ellipse coefficients found earlier are the SAME as the parameters above, but they are — and, we will.

Evolution of the Phase Advance

- Also, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\rightarrow 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan\Delta\psi_{1\rightarrow 2}$$

- So, from knowledge of matrices, can "transport" phase *and* the Courant-Snyder parameters along a beam line from one point to another

Simple Examples

- Propagation through a Drift:

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = \tan^{-1}\left(\frac{L}{\beta_1 - L\alpha_1}\right)$$

$$\beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha = \alpha_0 - \gamma_0 L$$

$$\gamma = \gamma_0$$

- Propagation through a Thin Lens:

$$M = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = 0$$

$$\beta = \beta_0$$

$$\alpha = \alpha_0 + \beta_0/F$$

$$\gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2$$

$$K = M\ K_0\ M^T$$

- Given *α, β* at one point, can calculate *α, β* at all downstream points

Another Summary

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- So, with knowledge of the layout of (linear) magnetic (and electrostatic) fields from which matrices describing the horizontal and vertical motion can be derived, and with an initial set of Courant-Snyder parameters describing the beam distribution, can *transport* the Courant-Snyder parameters along the beam line
  - Hence, can design a first-order focusing system without having to track particles. Within such a system the beam size will be determined by the value of the emittance used.
- These same C-S parameters describing the beam ellipse in phase space are found to be the same parameters found in the analytical solution to Hill's Equation if we identify

$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta} \qquad \Delta\psi = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$$

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

The Weak Focusing Synchrotron/Betatron

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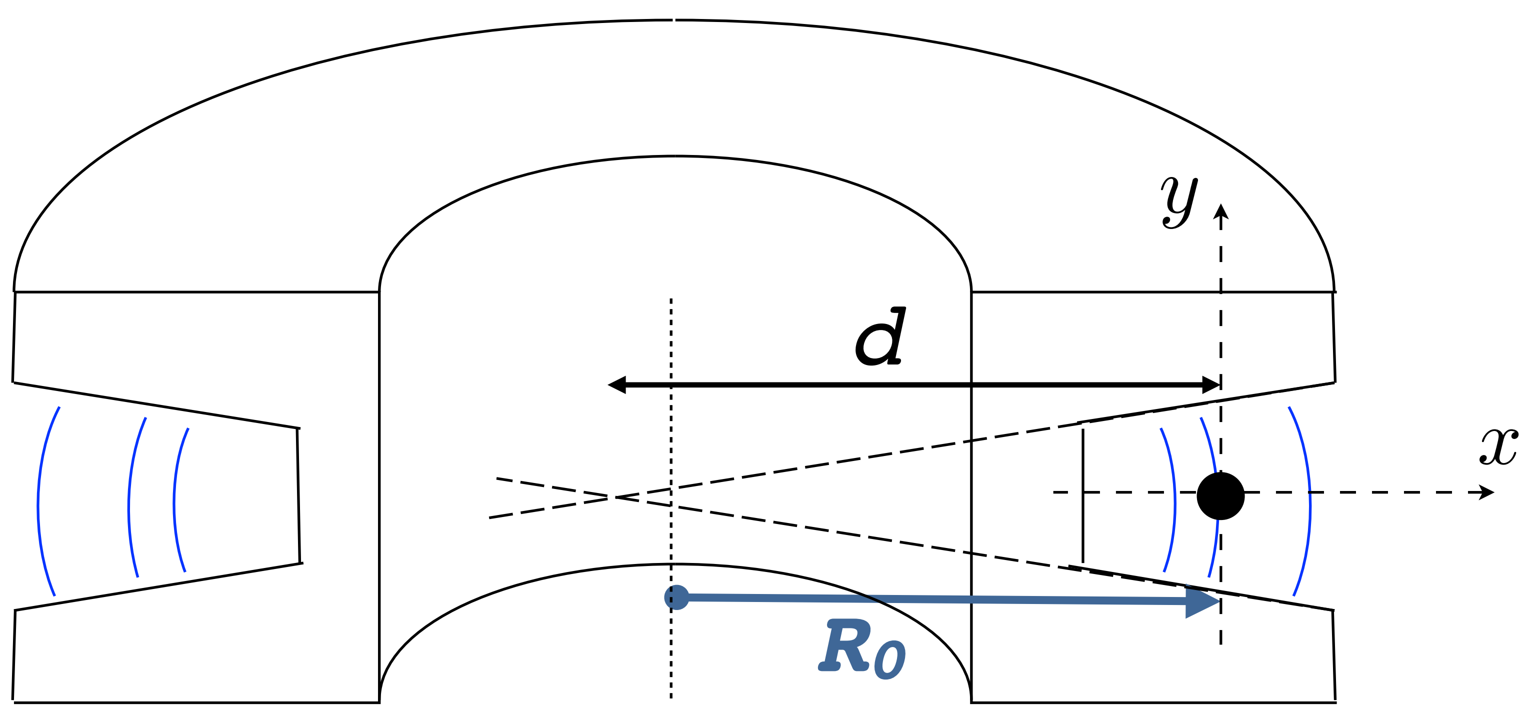
- Early accelerators (betatrons in particular, and early synchrotrons) employed what is now called "weak focusing"



$$B = B_0 \left(\frac{R_0}{r} \right)^n$$

n is determined by adjusting the opening angle between the poles

n = "field index"

$$n \approx \frac{R_0}{d}$$

Let's look at the stability
of transverse motion in this system…

Equation of Motion

- In rotating coordinate system,

$$r = R_0 + x$$

$$\frac{d^2x}{ds^2} - \frac{R_0 + x}{R_0^2} = -\frac{eB_y}{p}\left(1 + \frac{x}{R_0}\right)^2$$

- Hence,

$$B = B_y(y=0) = B_0\left(\frac{R_0}{r}\right)^n$$

$$B = B_0\left(\frac{1}{1 + x/R_0}\right)^n \approx B_0\left(1 - \frac{n}{R_0}x\right)$$

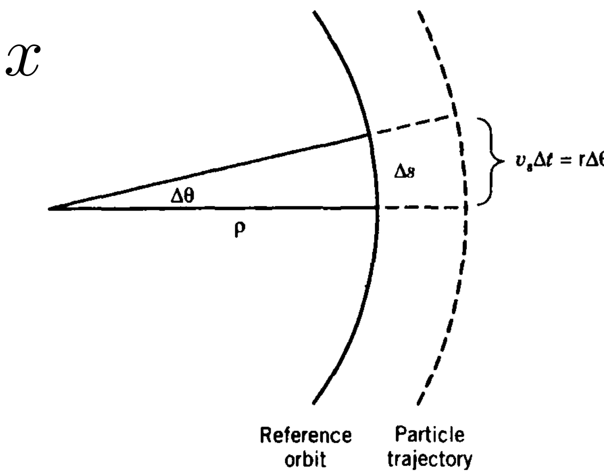


Figure 3.9. Comparison of reference orbit path length ds and particle path length $v_s\,dt$.

Since $v_x \ll v_s$ and $v_y \ll v_s$, to a very good approximation the total momentum p of the particle is $\gamma m v_s$. So

$$\ddot{r} - r\dot{\theta}^2 = -\frac{ev_s^2 B_y}{p}. \qquad (3.38)$$

Now, change to s as the independent variable. Then

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds}, \qquad (3.39)$$

and from Figure 3.9 we see that

$$ds = \rho\, d\theta = v_s\, dt \frac{\rho}{r}. \qquad (3.40)$$

Hence, assuming $d^2s/dt^2 = 0$,

$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}. \qquad (3.41)$$

Replacing r with $\rho + x$, the equation of motion becomes

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2, \qquad (3.42)$$

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Stability within a Weak Focusing System

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- Thus: $$B_y = B_0 - \frac{nB_0}{R_0} x \qquad B_x = -\frac{nB_0}{R_0} y \qquad (\nabla \times \vec{B} = 0)$$

- So we get,

$$x'' + K_x x = x'' + \left(-\frac{nB_0/R_0}{B_0 R_0} + \frac{1}{R_0^2} \right) x = 0$$

$$y'' + K_y\, y = y'' + \frac{n}{R_0^2}\, y = 0$$

- or,

$$x'' + \frac{1-n}{R_0^2} x = 0$$

$$y'' + \frac{n}{R_0^2}\, y = 0$$

must have $0 \leq n \leq 1$ for stability

Aperture of Weak Focusing System

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- The solutions of the equations of motion are:

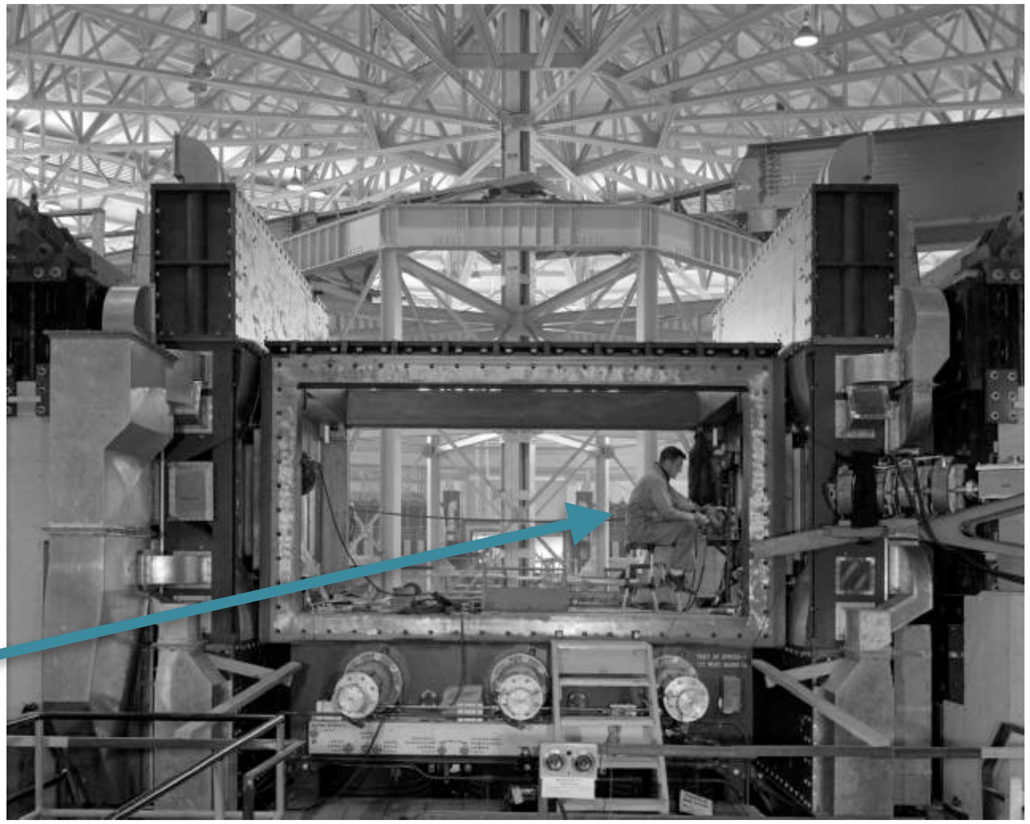
$$x'' + \frac{1-n}{R_0^2}x = 0 \quad \longrightarrow \quad x = x_0 \cos(\frac{\sqrt{1-n}}{R_0}\, s) + x_0' \frac{R_0}{\sqrt{1-n}} \sin(\frac{\sqrt{1-n}}{R_0}\, s)$$

$$y'' + \frac{n}{R_0^2}\, y = 0 \qquad y = y_0 \cos(\frac{\sqrt{n}}{R_0}\, s) + y_0' \frac{R_0}{\sqrt{n}} \sin(\frac{\sqrt{n}}{R_0}\, s)$$

SO, maxima in *x, y* grow with the RADIUS of the accelerator, for a given set of initial beam conditions

Higher energies required larger radii (for ~ constant *B),* and hence the *apertures* had to grow as well

sitting inside the ***beam chamber*** of the Bevatron (LBNL)



Betatron Oscillation Amplitude

- Transverse oscillations in a synchrotron (or beam line) are called Betatron Oscillations (first observed/analyzed in a "betatron" accelerator)
- Write *x,x'* in terms of initial conditions x_0, x'_0 :

$$x(s) = a\sqrt{\beta}\cos\Delta\psi + b\sqrt{\beta}\sin\Delta\psi$$

$$x' = \frac{1}{\sqrt{\beta}}([b - a\alpha]\cos\Delta\psi - [a + b\alpha]\sin\Delta\psi)$$

$$\downarrow$$

$$a = \frac{x_0}{\sqrt{\beta_0}}, \qquad b = \frac{\alpha_0 x_0 + \beta_0 x'_0}{\sqrt{\beta_0}}$$

$$\Longrightarrow x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\left[x_0\cos\Delta\psi + (\alpha_0 x_0 + \beta_0 x'_0)\sin\Delta\psi\right]$$

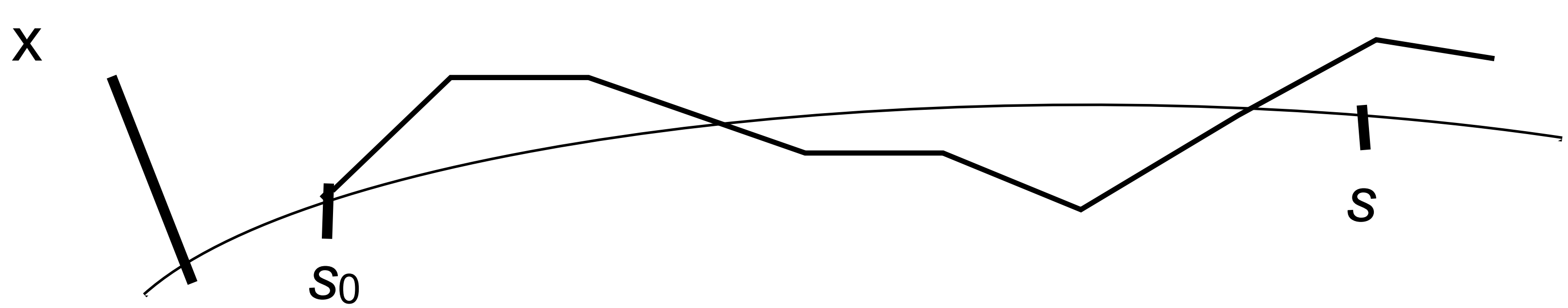
$$\text{amplitude:}\ \ A = \sqrt{\frac{x_0^2 + (\alpha_0 x_0 + \beta_0 x'_0)^2}{\beta_0}}$$

Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle
- Then, downstream, we have

$$\Delta x' = x'_0 = \Delta\theta$$

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\,\sin[\psi(s) - \psi_0]$$



Example:
Suppose $\Delta\theta = 0.4$ mrad, $\beta_0 = 4.0$ m, $\beta(s) = 6.4$ m, and $\Delta\psi = n \times 2\pi + 30^\circ$. Then $x(s) = 1$ mm.

Courant-Snyder Invariant

- In general,

$$\begin{aligned} x &= A\sqrt{\beta}\sin\psi \\ x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\ \beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\ &= A\sqrt{\beta}\cos\psi - \alpha x \\ \beta x' + \alpha x &= A\sqrt{\beta}\cos\psi \end{aligned}$$

$$\begin{aligned} x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\ A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\ &= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta} \end{aligned}$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

While C-S parameters evolve along the beam line, the combination above remains constant.

Properties of the Phase Space Ellipse

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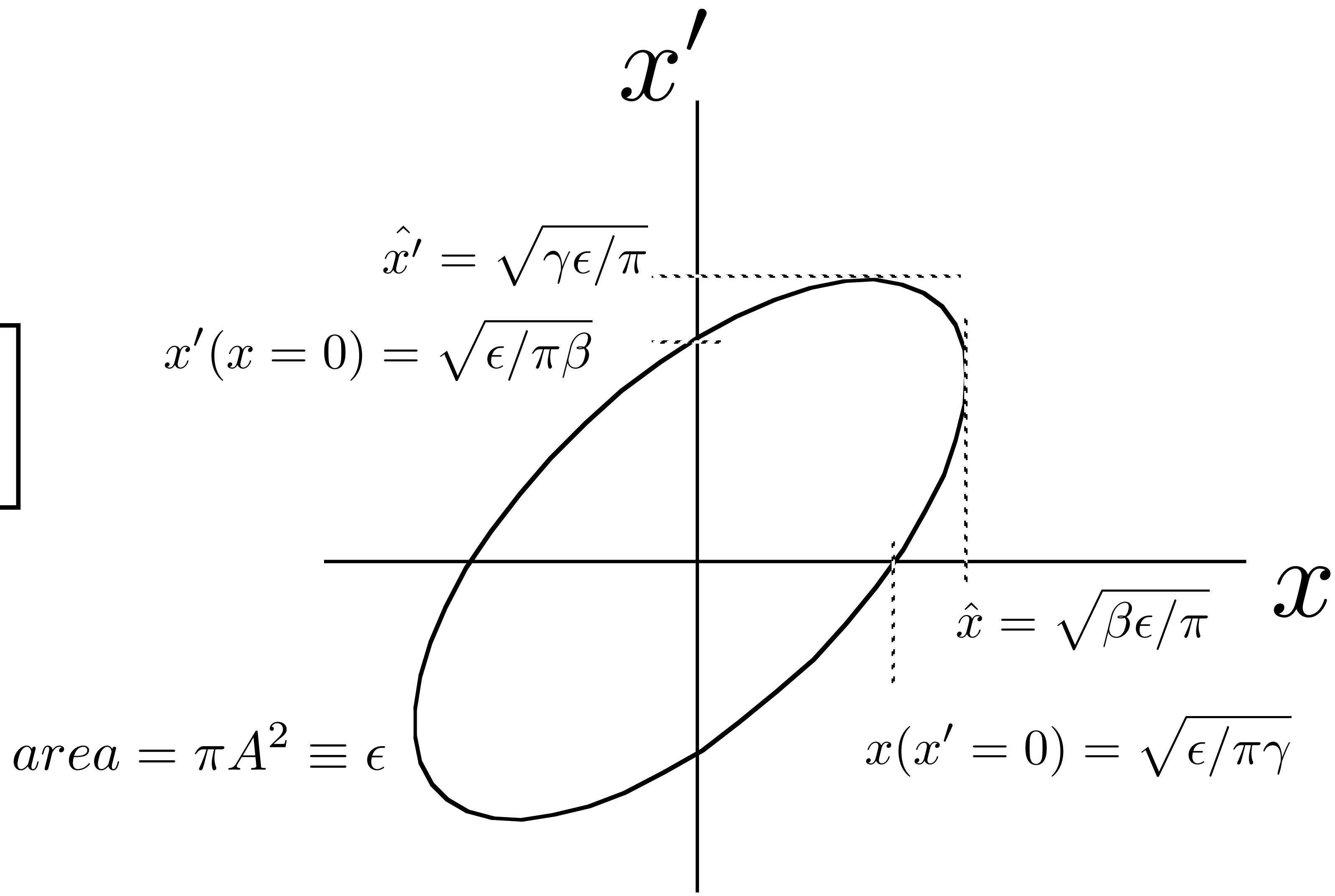
- The initial conditions of a freely-oscillating particle in the beam optics system determine its C-S invariant and hence the particle's phase space ellipse

$$area = \pi A^2$$

while the ellipse changes shape along the beam line, its area remains constant

Emittance = area within a phase space trajectory

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2$$



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Part III

The Stability Criterion

Discovery of Strong Focusing

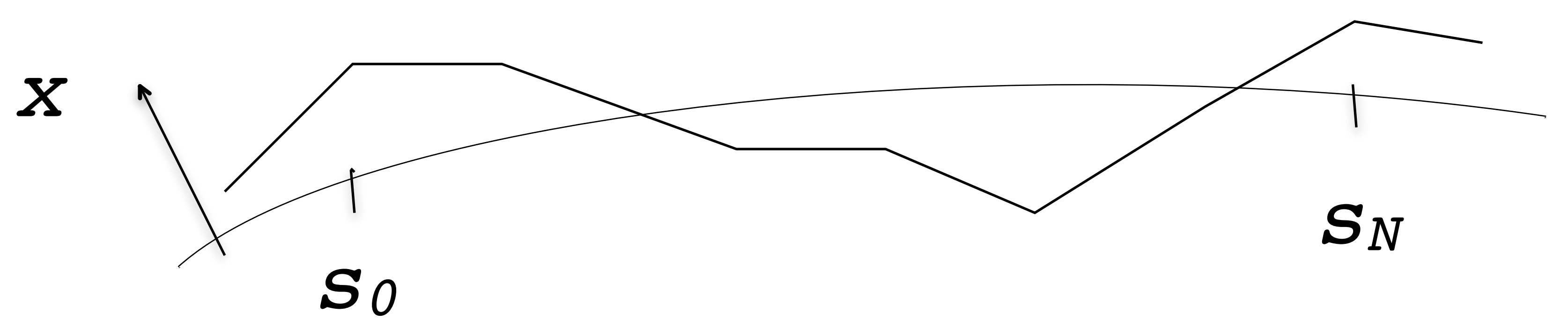
Periodic Optics and Tune Calculations

Momentum Dispersion

Arbitrary Focusing System

- What if the focusing is not *continuous* but rather varies with location *s*?
- Generate a single-turn matrix of the linear motion, made from matrices of individual elements (Note: each with unit determinant)
- Look at matrix describing motion for one passage through a repetitive period:

$$M = M_N M_{N-1} \cdots M_2 M_1$$



- Now suppose repeat this operation k times. We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \to \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

The Stability Criterion

- From the discipline of “linear algebra”, we know that any vector within a vector space (i.e., that is operated on, say, by a matrix *M*) can be written in terms of the eigenvectors of the matrix *M*

  - Eigenvector: V $MV = \lambda V$
    - »
    - » *where* λ is an eigenvalue of *M* *(real or imaginary)*

- A 2x2 matrix *M* will have two eigenvalues, λ_1 and λ_2 and two corresponding eigenvectors, V_1 and V_2; so any vector that *M* operates on can be written as

$$X = c_1 V_1 + c_2 V_2$$

The Stability Criterion

- So, if the matrix *M* is applied to vector X_0 a number of times, the resulting vector X_k after the *k*-th iteration will be

$$X_k = M^k X_0 = M^k(c_1 V_1 + c_2 V_2) = c_1 \lambda_1^k V_1 + c_2 \lambda_2^k V_2$$

V = eigenvector
λ = eigenvalue

- Now, also from linear algebra, the determinant of the matrix *M* will be the product of the eigenvalues. So,

$$\det M = 1 = \lambda_1 \lambda_2 \rightarrow \lambda_2 = 1/\lambda_1 \rightarrow \lambda = e^{\pm i\mu}$$

The Stability Criterion

- Since for our case the eigenvalues are reciprocals of each other, and since we can write $\lambda = e^{\pm i\mu}$, then

$$X_k = M^k X_0 = c_1 \lambda_1^k V_1 + c_2 \lambda_2^k V_2 = c_1 e^{ik\mu} V_1 + c_2 e^{-ik\mu} V_2$$

If μ is imaginary, then repeated application of *M* gives exponential growth; if μ is real, gives oscillatory solutions…

- To find the eigenvalues, we solve the "characteristic equation": from $MV = \lambda V$,

$$\text{characteristic equation: } \det(M - \lambda I) = 0$$

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } (a - \lambda)(d - \lambda) - bc = 0$$

The Stability Criterion

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- Solving for the eigenvalues,

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{then } \det(M - I\lambda) = \left| \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \right| = (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$
$$\lambda^2 - trM\lambda + 1 = 0$$
$$\lambda + 1/\lambda = trM$$
$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = trM$$

$$ad - bc = \det M = 1$$

$$a + d = trM = \text{``}trace\text{''}\ of\ M$$

$$\text{So, } \mu \text{ real (stability)}$$
$$\rightarrow |trM| < 2$$

The Stability Criterion

Check: The Weak Focusing Synchrotron

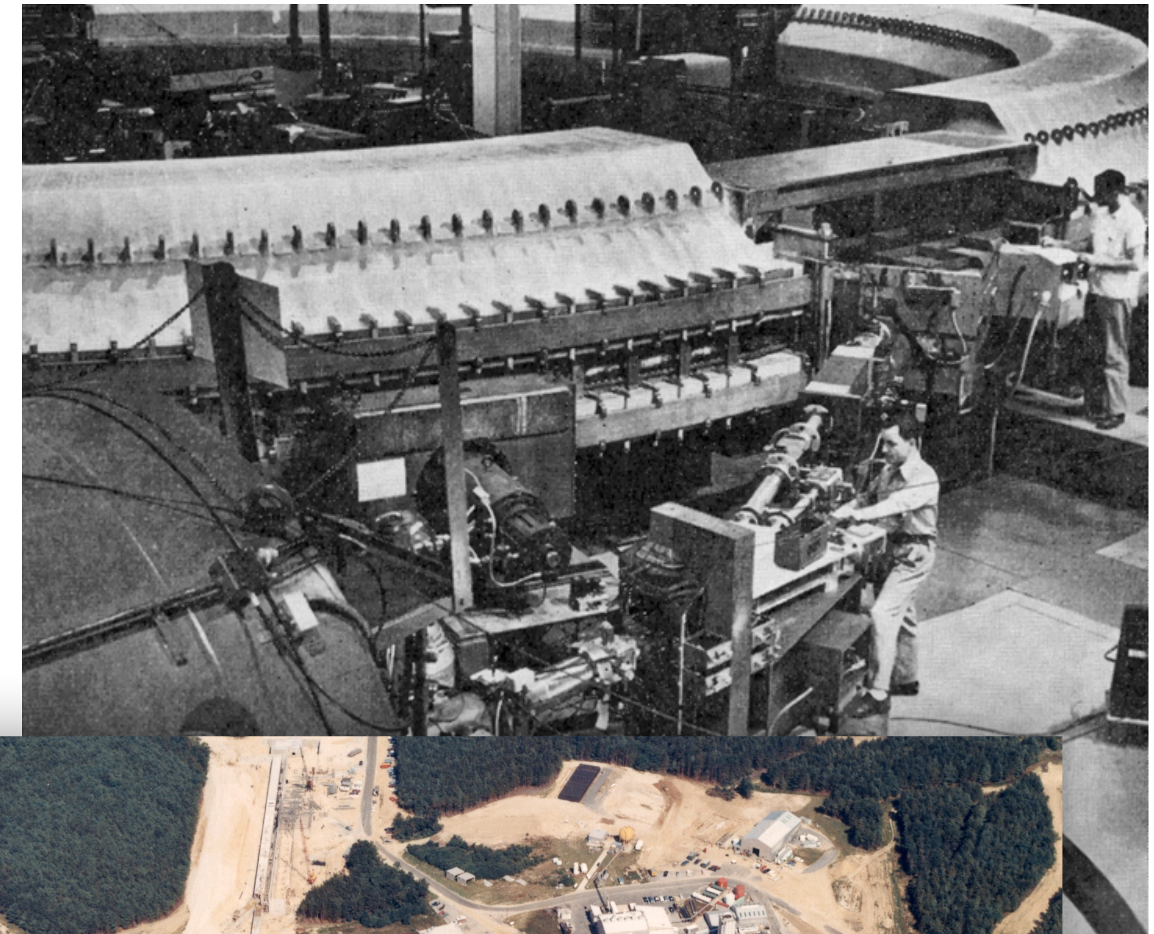
- We had, for example: $$y'' + \frac{n}{R_0^2}\, y = 0$$
  - » from which follows:
  $$y = y_0 \cos(\frac{\sqrt{n}}{R_0}\, s) + y_0' \frac{R_0}{\sqrt{n}} \sin(\frac{\sqrt{n}}{R_0}\, s)$$
  $$y' = -y_0 \frac{\sqrt{n}}{R_0} \sin(\frac{\sqrt{n}}{R_0} s) + y_0' \cos(\frac{\sqrt{n}}{R_0} s)$$
  - » or, in matrix form:
  $$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\frac{\sqrt{n}}{R_0} s) & \frac{R_0}{\sqrt{n}} \sin(\frac{R_0}{\sqrt{n}} s) \\ -\frac{\sqrt{n}}{R_0} \sin(\frac{R_0}{\sqrt{n}} s) & \cos(\frac{\sqrt{n}}{R_0} s) \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}_0$$
- For 1 revolution, $s = 2\pi R_0$ and the trace of *M* is …

$$|trM| = |2\cos(2\pi\sqrt{n})| \leq 2 \qquad (|2\cos(2\pi\sqrt{1-n})| \leq 2, \text{ for horizontal})$$

$$0 \leq n \leq 1$$
for stability

Discovery of Strong Focusing

- The Cosmotron (BNL)
  - (weak focusing)



- Through looking at upgrade options, strong focusing was discovered and the decision was made to go for a new, much larger synchrotron
  - The Alternating Gradient Synchrotron (AGS)

Discovery of Strong Focusing*

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- Consider the "weak-focusing" magnet system just discussed. Suppose the ring is made up of $2N$ identical magnets, each with field index n

- Take every other magnet and have the magnet *open* to the inside, instead of the outside: $n \rightarrow -n$

  - All have the same central field value, B_0, but the field "gradients" will alternate $n, -n, n, -n, \ldots$

- Analyze the resulting system by using a matrix approach and applying the stability criterion

*Courant, Livingston, and Snyder, 1952.
Christofolis, c. 1950

Discovery of Strong Focusing [2]

- Consider one degree of freedom, say the vertical
  - for one of the N cells, the matrix would be…

(B’>0) (B’<0) $K = |B'|/B\rho$

$$M_c = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\sqrt{K}L)\cosh(\sqrt{K}L) + \sin(\sqrt{K}L)\sinh(\sqrt{K}L) & \cdots \\ \cdots & \cos(\sqrt{K}L)\cosh(\sqrt{K}L) - \sin(\sqrt{K}L)\sinh(\sqrt{K}L) \end{pmatrix}$$

from which

$$tr M = 2\cos(\sqrt{K}L)\cosh(\sqrt{K}L)$$

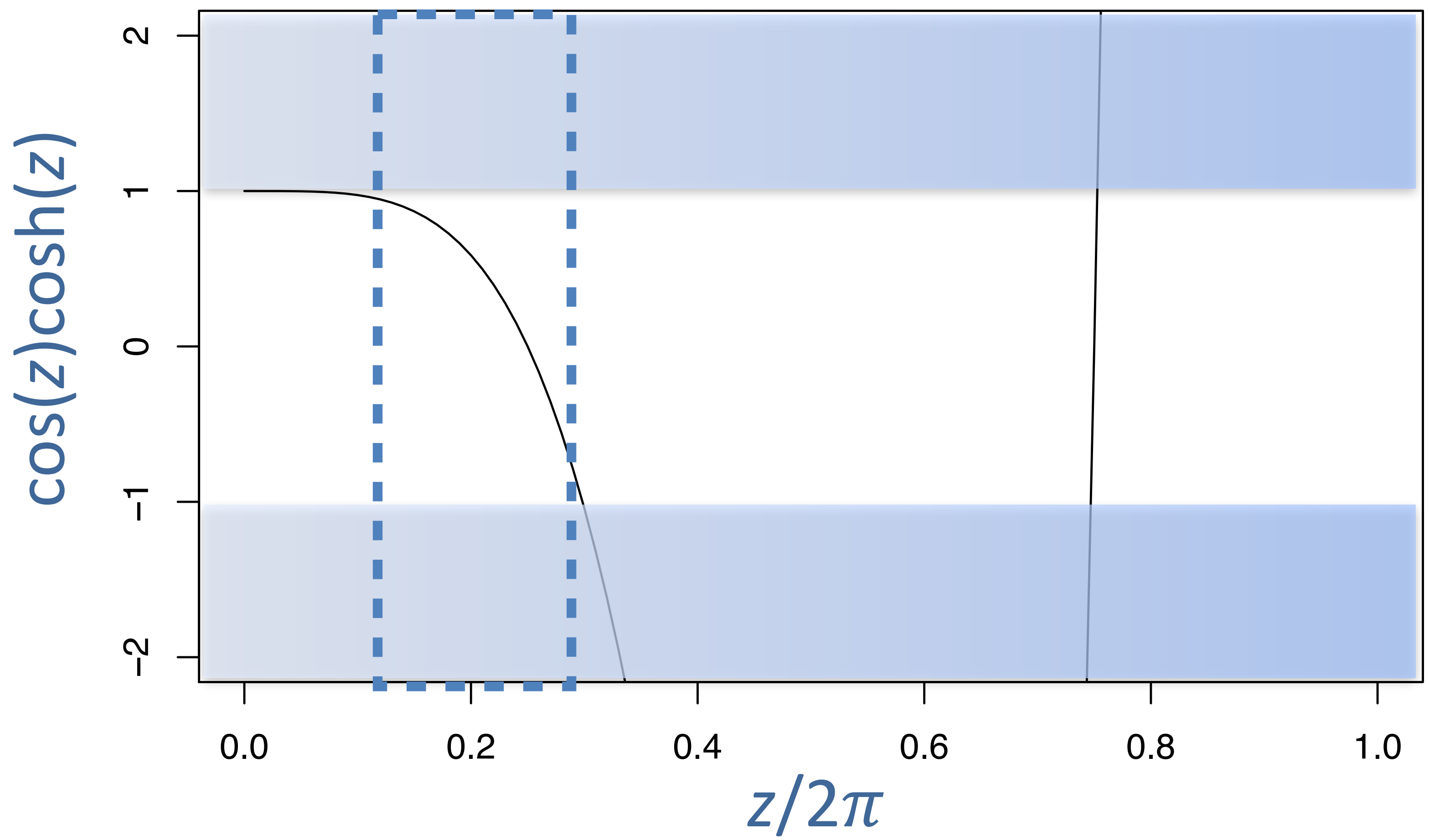
- So, for stability, we would need:

$$|\cos(\sqrt{K}L)\cdot\cosh(\sqrt{K}L)| < 1$$

*Courant, Livingston, and Snyder, 1952.
Christofolis, c. 1950

Discovery of Strong Focusing [3]

- We see a range in which the system would be stable

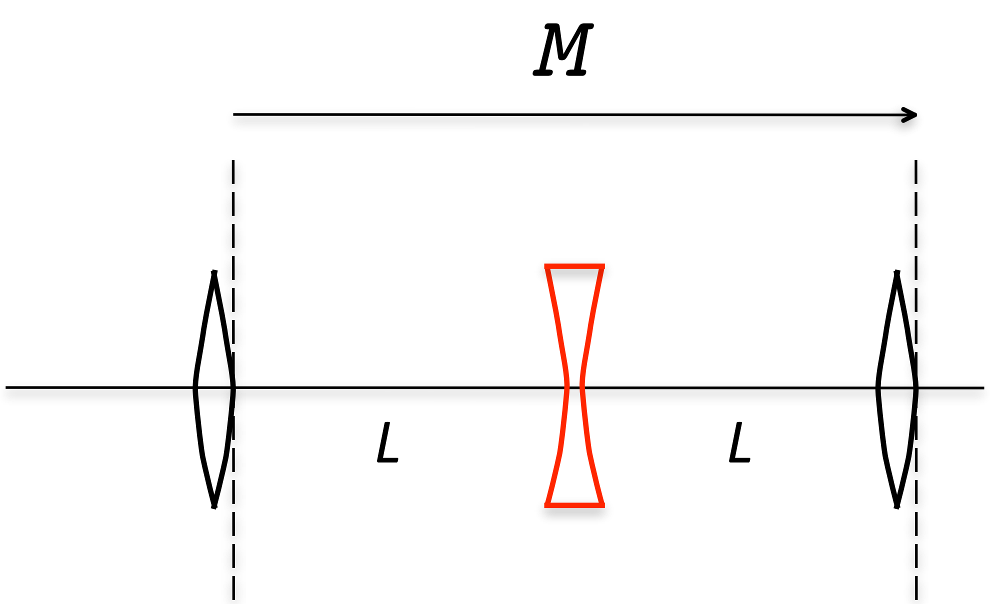


- Choose $z = \sqrt{K}\, L$
  $K = (z/L)^2$
- Also, $L = 2\pi R_0 / 2N$ $\quad K = (zN/\pi R_0)^2$
- In the weak focusing case, $K_0 = n/R_0^2$. So,… $$\frac{K}{K_0} = \left(\frac{z}{\pi}\right)^2 \frac{N^2}{n}$$
- Let's pick $z/\pi = 0.4$, $n = 0.5$, and $N = 25$:

$$K/K_0 = 200!$$

Another Example: FODO system

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$



and repeat…

So, $tr\, M = 2 - L^2/F^2$ and thus, for stability,

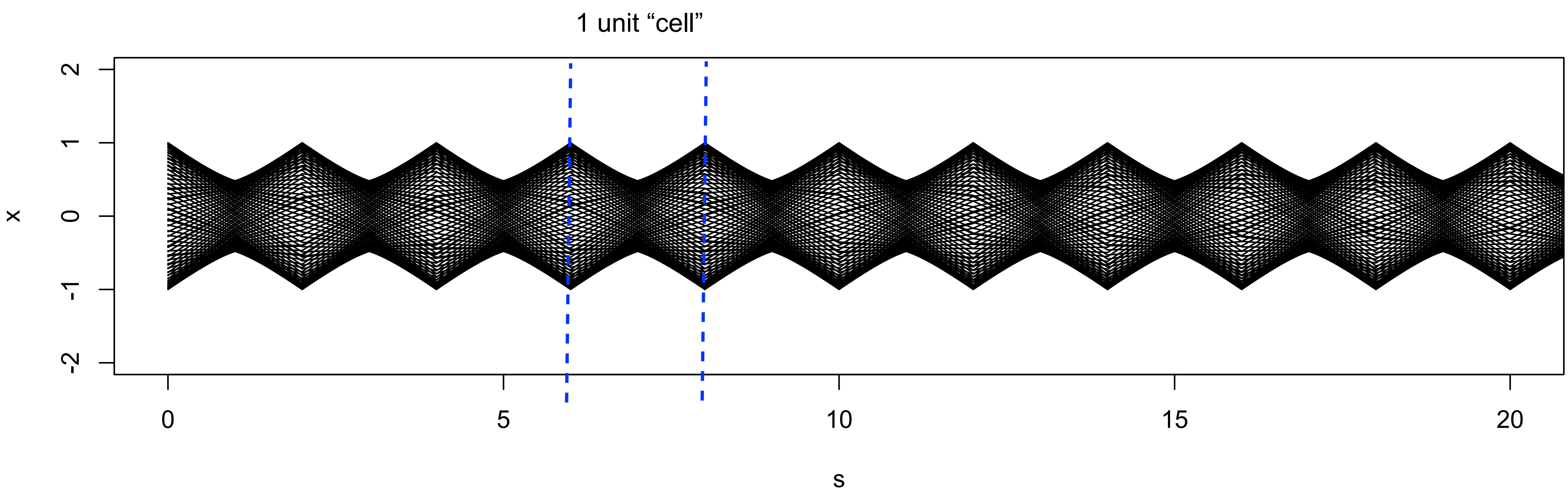
$$-2 < 2 - L^2/F^2 < 2$$

$$-4 < -L^2/F^2 < 0$$

$$F > L/2$$

Particle Trajectories in a Periodic Lattice

Northern Illinois University



$$\frac{dx'}{ds} = \frac{d^2x}{ds^2} = -\frac{eB'(s)}{p}x$$

$$x'' + K(s)x = 0$$

(Hill's Equation)

$$K(s) = \frac{e}{p}\frac{\partial B_y}{\partial x}(s)$$

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

The Periodic Amplitude Function

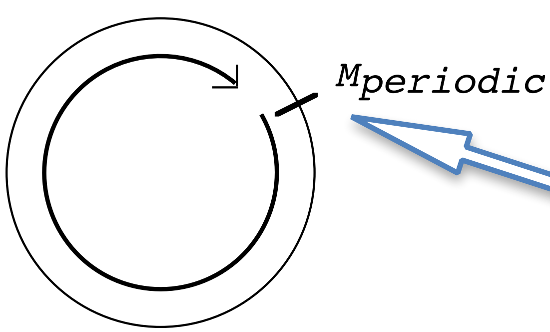
- Previously, …
  - Transport matrix, in terms of amplitude function at end points, and phase advance between:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

$$M_{periodic} = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$

sometimes "μ" is used to denote the **periodic** phase advance



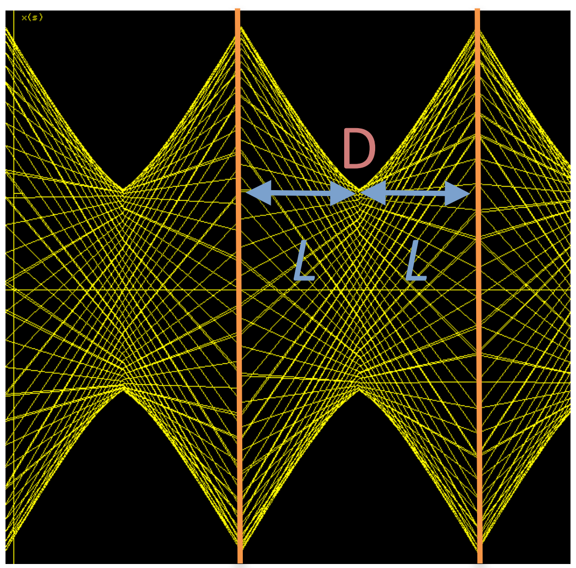
Natural choice in a circular accelerator, when values of β, α above correspond to one particular point in the ring

Choice of Initial Conditions

- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a "ring," then natural to choose the **periodic** solutions for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- In a system that takes a distribution from a source or off of a target, wish to "match" to desired initial conditions at the input to the downstream beam line system by using an arrangement of tunable focusing elements

Computation of Courant-Snyder Parameters

- As an example, consider again the FODO system



F

-F

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$

- Let's, use above matrix of the periodic section to compute functions at exit of the F quad..

FODO Cell Courant-Snyder Parameters

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- From the matrix: $$M_{periodic} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4 numbers

$$traceM = a + d = 2 - L^2/F^2 = 2\cos\mu \quad \Rightarrow \quad \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1 + L/2F}{1 - L/2F}} \qquad \alpha = \frac{a - d}{2\sin\mu} = \sqrt{\frac{1 + L/2F}{1 - L/2F}}$$

- If go from D quad to D quad, simply replace *F* --> *-F* in matrix *M* above
  - So, at exit of the D quad:

$$\beta = 2F\sqrt{\frac{1 - L/2F}{1 + L/2F}} \qquad \alpha = -\sqrt{\frac{1 - L/2F}{1 + L/2F}}$$

for completeness,

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

Periodic FODO Cell Functions

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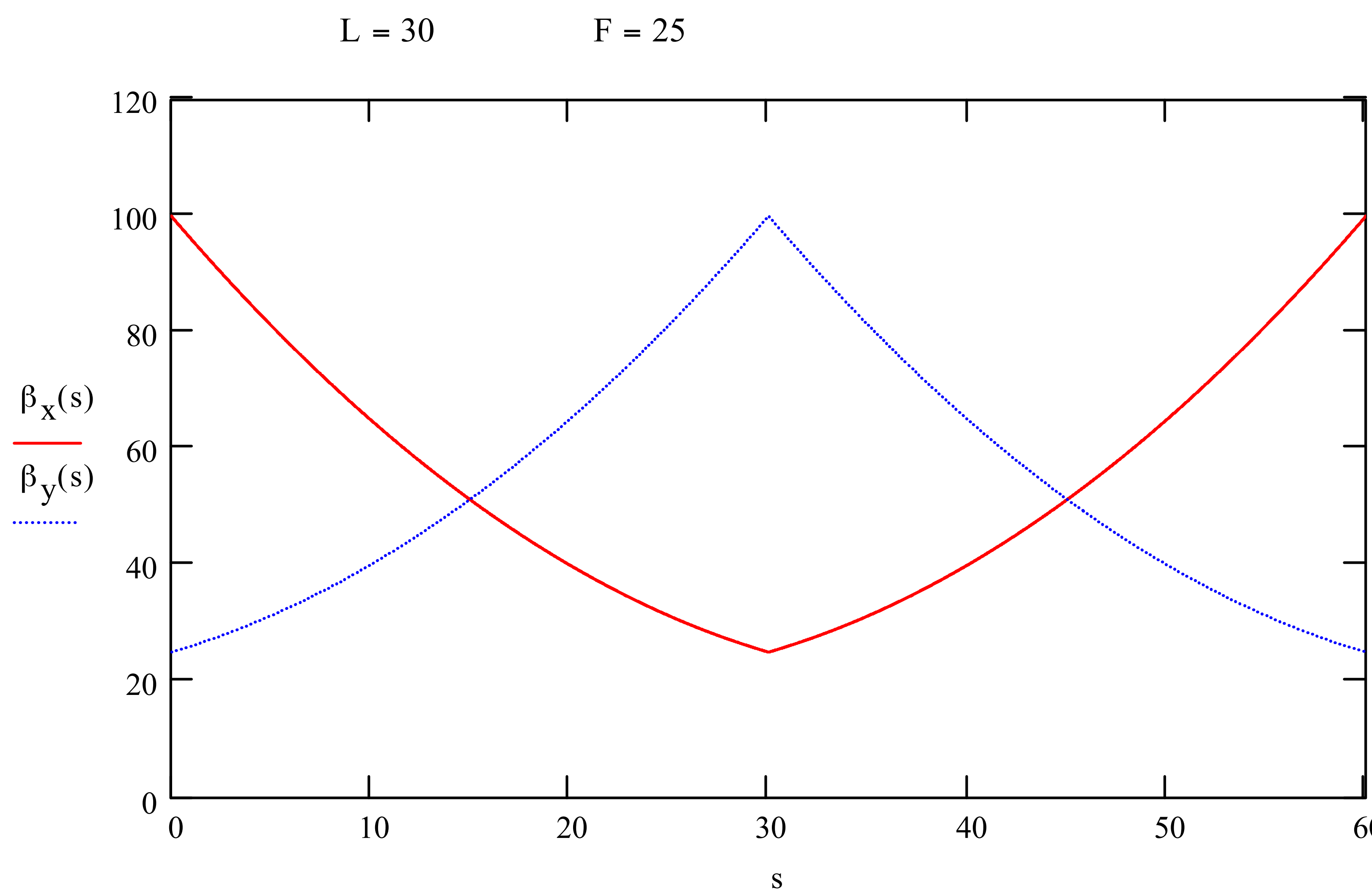
- Numerical Example: Standard FODO Cell of the old Fermilab Tevatron (~100 of these made up the ring)

$$\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ)$$
$$\beta_{max} = 2(25\text{ m})\sqrt{1.6/0.4} = 100\text{ m}$$
$$\beta_{min} = 2(25\text{ m})\sqrt{0.4/1.6} = 25\text{ m}$$
$$\nu \approx 100 \times 1.2/2\pi \sim 20$$

Note: this thin lens example is actually accurate to a few percent!

$$(F/\ell = 25/2 >> 1)$$



Propagation of Periodic Courant-Snyder Parameters

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- We can write the matrix of a periodic section as:

$$
\begin{aligned}
M_0 &= \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\Delta\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin\Delta\psi \\
&= I\cos\Delta\psi + J\sin\Delta\psi \qquad = e^{J\Delta\psi}
\end{aligned}
$$

- where

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$det J = 1, \quad trace(J) = 0; \quad J^2 = -I$$

α, β are values at the beginning/end of the periodic section described by matrix M

Tracking β, α, γ ...

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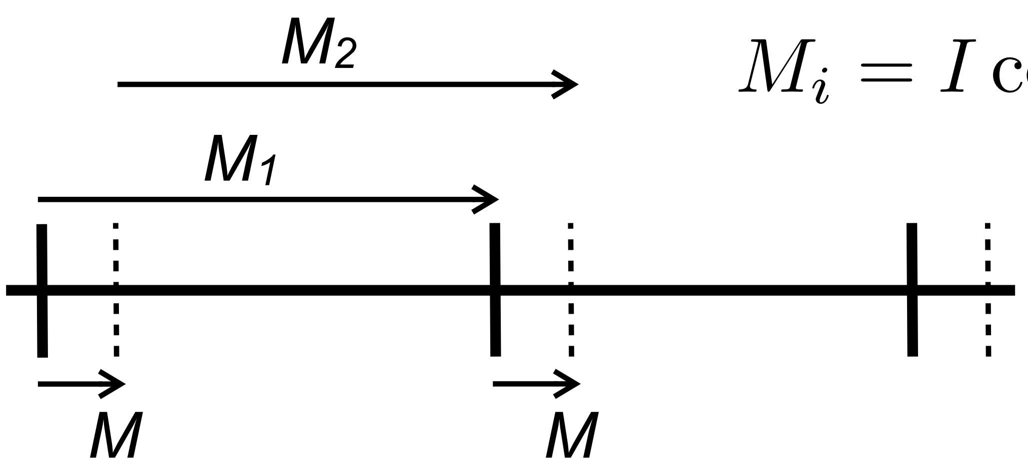
- Let M_1 and M_2 be the "periodic" matrices as calculated at two points, and M propagates the motion between them. Then,



$$M_i = I\cos\mu + J_i \sin\mu \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$M_2 = M\ M_1\ M^{-1}$$

- Or, equivalently,
  - if know C-S parameters (i.e., J) at one point, can find them at another point downstream if given the matrix for motion in between:

$$J_2 = M\ J_1\ M^{-1}$$

- Doesn't have to be part of a periodic section; valid between any two points of an arbitrary arrangement of elements

For comparison, remember $K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, and $K = M\ K_0\ M^T$; these are equivalent

Evolution of the Phase Advance

- Again, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase and the Courant-Snyder parameters along a beam line from one point to another

The Betatron Tune

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- In a cyclic accelerator (synchrotron), the particles will oscillate (betatron oscillations) with a certain oscillation frequency — the betatron frequency.
- The betatron frequency is determined by the total phase advance once around the ring:

$$\Delta\psi_{total} = \oint \frac{ds}{\beta(s)}$$

$$\nu \equiv \Delta\psi_{total}/2\pi$$

Betatron Tune: # of oscillations per revolution

$$\mathrm{tr}M = 2\cos(2\pi\nu)$$

$$f_{betatron} = \nu f_{rev}$$

Ex: Tune of a FODO synchrotron

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- Suppose a ring is made up of *N* FODO cells
- Each cell has phase advance given by the lens spacing and lens focal length:

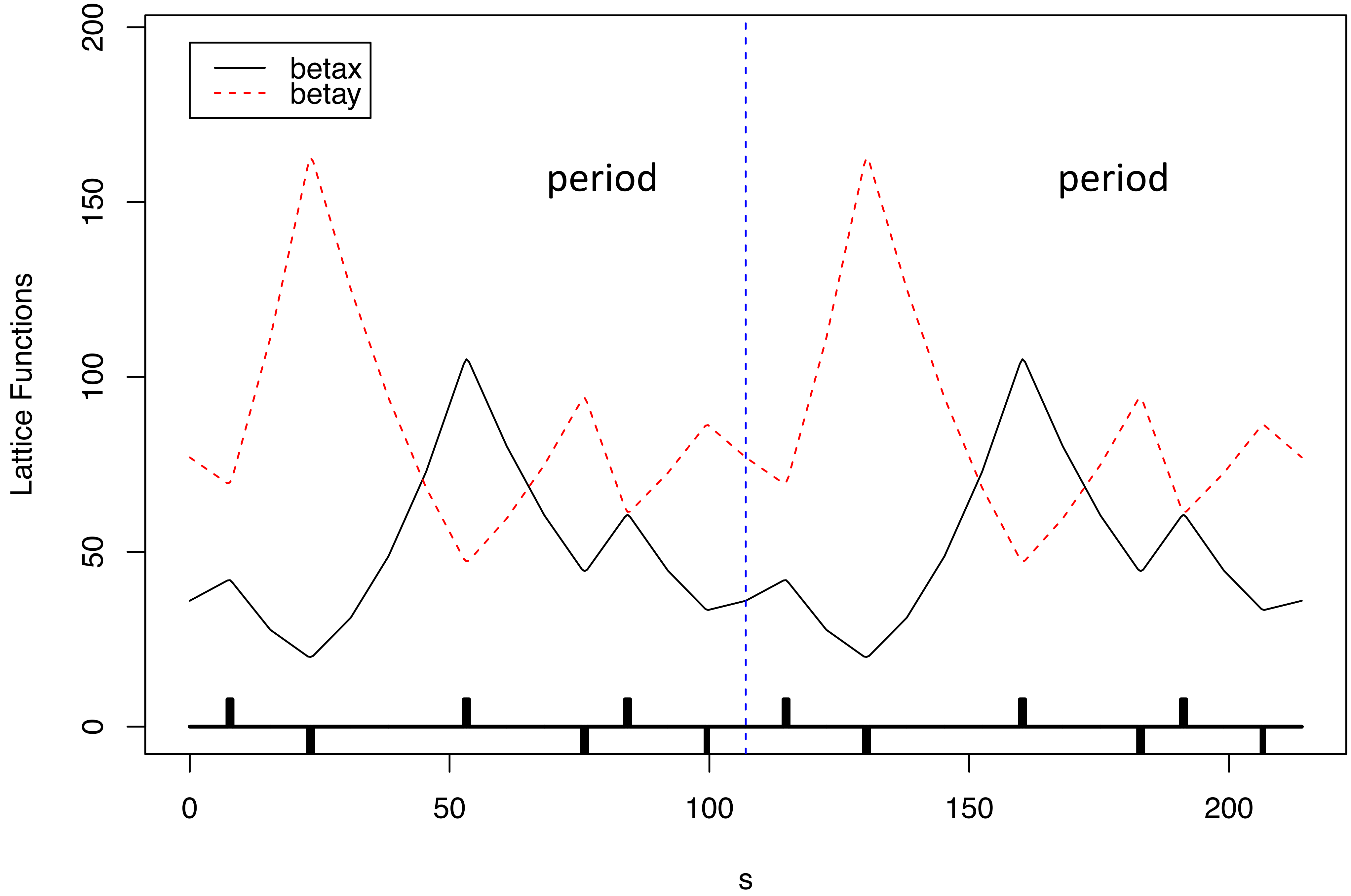
$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

- So, the tune of this simple synchrotron would be:

$$\nu = N\mu/2\pi \approx N\frac{L}{2\pi F} = \frac{2LN}{4\pi F} = \frac{C}{2\pi}\frac{1}{2F} = \frac{R}{2F}$$

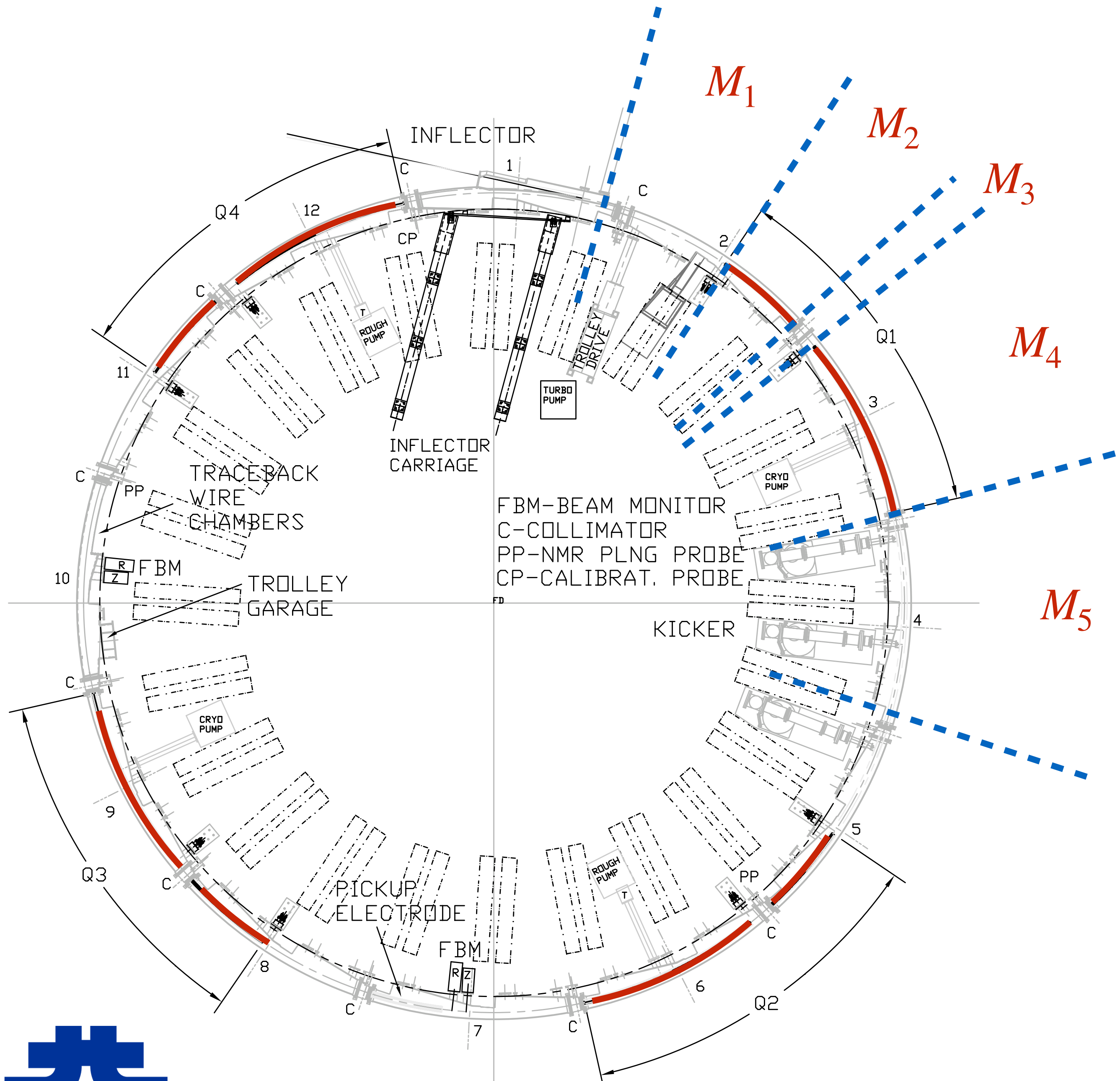
- Ex: Main Injector at Fermilab: *R* ~ 500 m; *F* ~ 13 m
  - so, ν ~ 20
  - thus, if initiate a betatron oscillation in this synchrotron it will oscillate ~20 times per revolution around the ring

Arbitrary Distribution of Quadrupoles



The Storage Ring

- Matrix description of discrete focusing regions



$$x'' + K_x(s)x = 0$$
$$y'' + K_y(s)y = 0$$

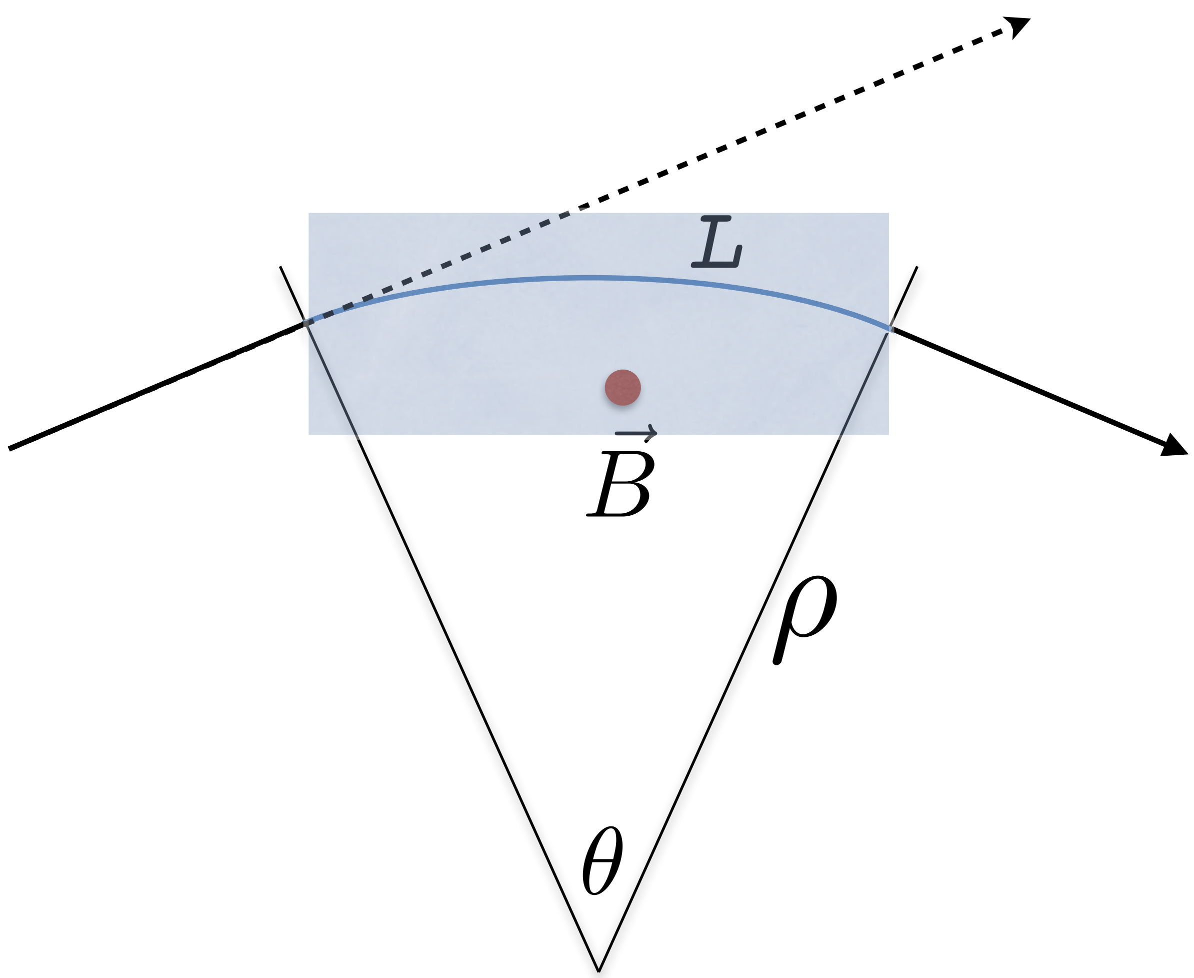
where

$$K_x(s) = \frac{1}{R_0^2} - \frac{E'(s)}{vB_0R_0}$$
$$K_y(s) = + \frac{E'(s)}{vB_0R_0}$$

$$M_{1/4} = M_5M_4M_3M_2M_1$$

$$M_{tot} = (M_{1/4})^4$$

Bending through Dipole Field



$$\theta = \frac{L}{\rho} = \frac{B \cdot L}{(B\rho)}$$

$$= \frac{q \cdot B \cdot L}{p}$$

Dispersion

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$$B\rho = \frac{p}{q} \qquad \theta = \frac{qB\cdot\ell}{p}$$

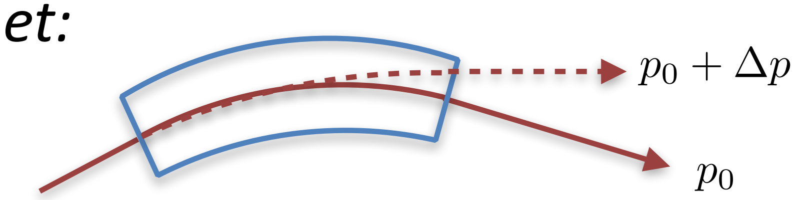
The bend angle (and/or focusing strength) depends upon momentum

Similar to index of refraction depending upon frequency

dipole steering "error" due to a different momentum —> "dispersion"

focusing "error" due to a different momentum —> "chromatic aberration"

dipole magnet:



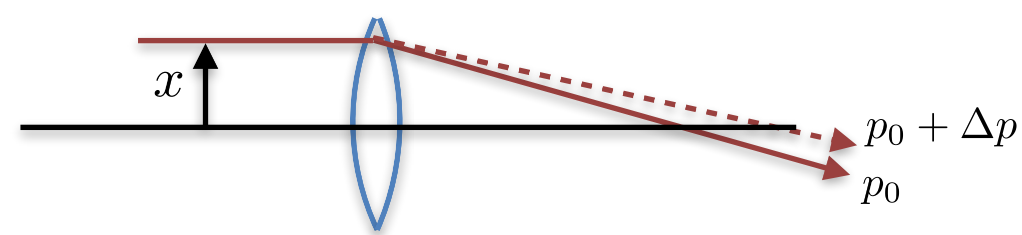
$$\frac{\Delta\theta}{\theta_0} = -\frac{\Delta p}{p}$$

[i.e., in "opposite" direction of bend]

at exit, to lowest order, $\Delta x' = \theta_0 \frac{\Delta p}{p}$

$$\Delta x \approx \frac{1}{2}\ell\theta_0 \frac{\Delta p}{p}$$

likewise, for quadrupole:



$$f = f_0(1 + \frac{\Delta p}{p})$$

Trajectory differences due to momentum differences referred to as "dispersion"

and,

$$D(s, \Delta p/p) \approx D(s) \equiv \frac{\Delta x(s)}{\Delta p/p}$$

"dispersion function"

Dispersion [2]

(see E&S text for details…)

Equation of Motion:

$$x'' + (\frac{B'}{B\rho} + \frac{1}{\rho^2})x = 0 \quad \text{becomes} \quad x'' + \left\{ \left(\frac{1}{1+\Delta p/p}\right)\frac{B'}{B\rho} + \frac{p_0 - \Delta p}{p_0 + \Delta p} \cdot \frac{1}{\rho_0(s)^2} \right\} x = \frac{1}{\rho_0} \frac{\Delta p}{p_0 + \Delta p}$$

let $x = D\ \Delta p/p,$ particular solution

(must add the homogeneous solution, which we have found previously)

betatron oscillation

then,

$$D''\frac{\Delta p}{p_0} + \left\{ \left(\frac{1}{1+\Delta p/p}\right)\frac{B'}{B\rho} + \frac{p_0 - \Delta p}{p_0 + \Delta p} \cdot \frac{1}{\rho_0(s)^2} \right\} D\frac{\Delta p}{p_0} = \frac{1}{\rho_0} \frac{\Delta p}{p_0 + \Delta p}$$

keep only terms linear in the relative momentum deviation,

$$D''\frac{\Delta p}{p_0} + \left(\frac{B'}{B\rho} + \frac{1}{\rho_0^2}\right) D\frac{\Delta p}{p_0} = \frac{1}{\rho_0}\frac{\Delta p}{p_0}$$

a driven betatron oscillation, with a *constant* driving term. The "driver" is the dipole field within a bending magnet

$$D'' + K\ D = \frac{1}{\rho_0}$$

so, solutions are $\sin\sqrt{K}\ell$ & $\cos\sqrt{K}\ell$ plus *const.*

Dispersion [3]

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$$D'' + K\ D = \frac{1}{\rho_0}$$

In terms of matrices…

K = 0 :

$$D'' = \frac{1}{\rho}, \quad D' = \frac{s}{\rho} + D'_0$$

$$D = D_0 + D'_0 s + \frac{1}{2}\frac{s^2}{\rho}$$

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & s & \frac{1}{2}s^2/\rho \\ 0 & 1 & s/\rho \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

in the limit of short, or "thin" elements, a bending magnet primarily changes the slope of the dispersion function by an amount equal to the bend angle of the magnet

1/ρ = 0 :

same 2x2 as before

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} M & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

otherwise, the *D* transports roughly like a betatron oscillation

So, can use matrix methods (3x3 now; and 2x2 in "vertical" plane) to solve for:

$$\beta_x, \quad \alpha_x, \quad \psi_x$$

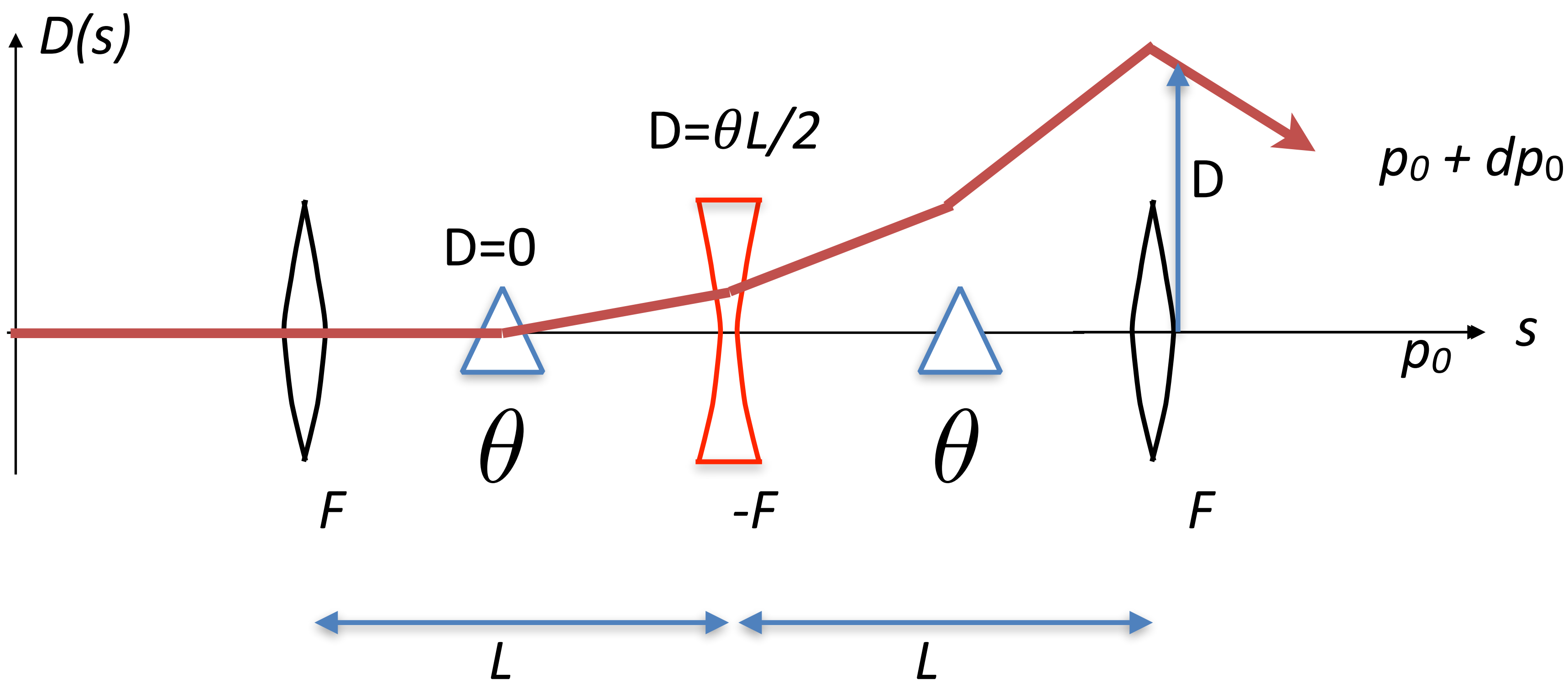
$$\beta_y, \quad \alpha_y, \quad \psi_y$$

$$D_x, \quad D'_x$$

(& D_y, D'_y , if also have vertical bending)

Generating Dispersion

- System of quadrupoles with alternating-sign gradients (F, D, F, …) separated by distance *L*, and with bending magnets present…



D=θL/2(1+L/2F)

Ex: D = 3 m, dp/p = 0.3%, then Δx = 9 mm

Beam Size Including Dispersion

- Total excursion due to "off momentum" plus betatron oscillation:

$$x = x_\beta + D\ \delta \qquad\qquad \delta \equiv \Delta p/p$$

$$x^2 = x_\beta^2 + 2x_\beta D\delta + D^2\delta^2$$

- Assuming no correlation between x_β and particle's momentum:

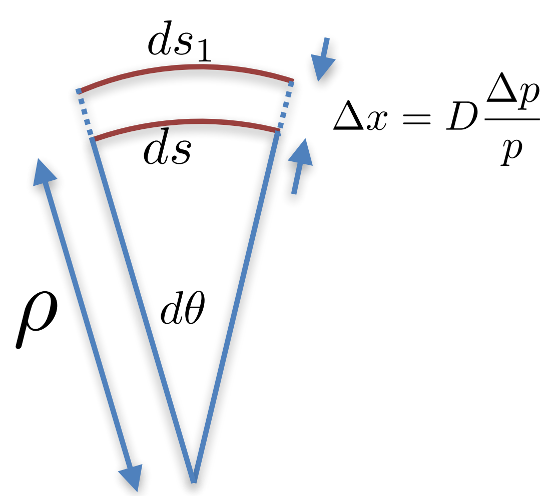
$$\langle x^2 \rangle = \langle x_\beta^2 \rangle + D^2 \langle \delta^2 \rangle$$

$$\langle x^2 \rangle = \epsilon\beta/\pi + D^2 \langle \delta^2 \rangle$$

Momentum Compaction Factor

- How does path length along the beam line depend upon momentum?
  - in straight sections, no difference; in bending regions, *can* be different

Look closely at an infinitesimal section along the ideal trajectory…



$$d\theta = \frac{ds}{\rho} = \frac{ds_1}{\rho + \Delta x}$$

$$ds_1 - ds = \left(\frac{\rho + \Delta x}{\rho} - 1 \right) ds$$

$$= \frac{\Delta x}{\rho} ds = \frac{D}{\rho} \frac{\Delta p}{p} ds$$

if L = path length along ideal trajectory between 2 points, then

$$\frac{\Delta L}{L} = \boxed{\frac{\int \frac{D(s)}{\rho(s)} ds}{\int ds}} \cdot \frac{\Delta p}{p}$$

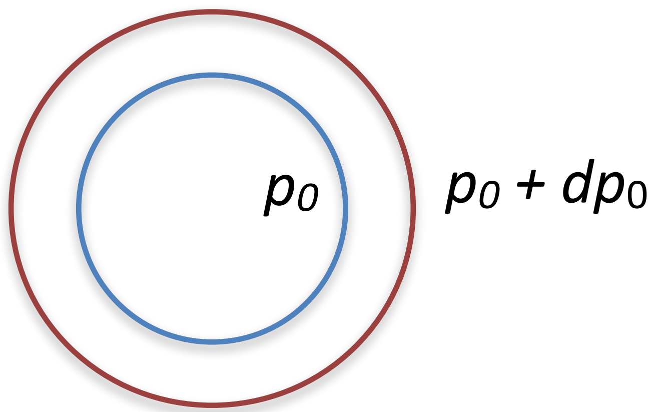
The relative change in path length, per relative change in momentum, is called the *momentum compaction factor,*

$\alpha_p = \langle D/\rho \rangle$ along the ideal path

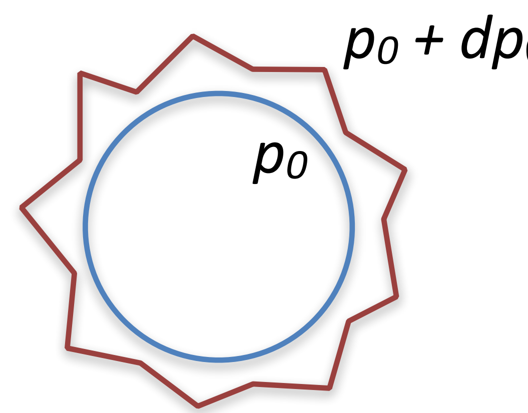
Periodic Dispersion Function

Northern Illinois University

uniform bend field:



add gradients…

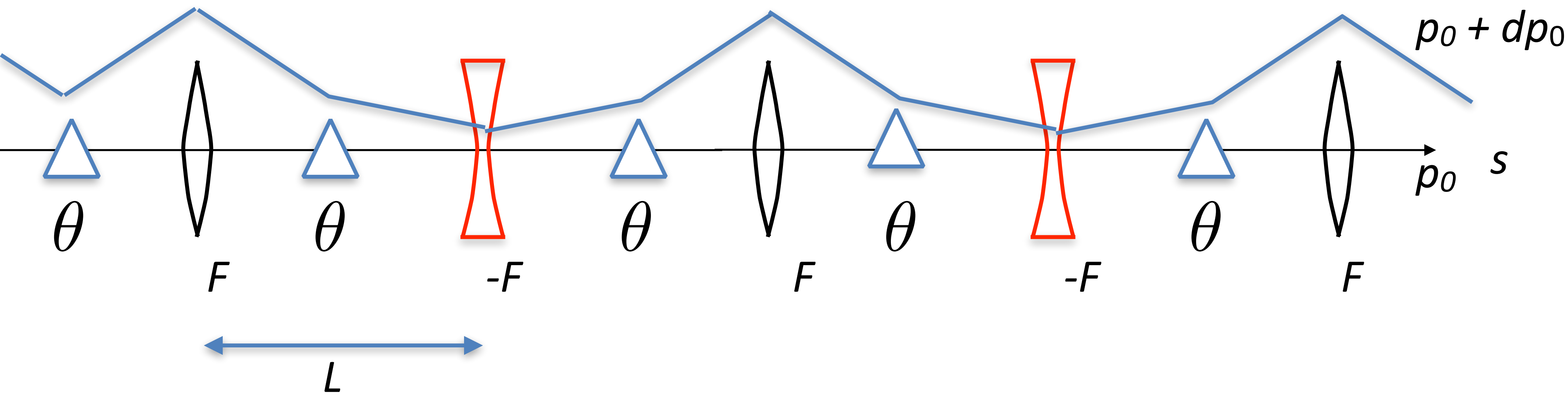


the trajectory "closed" orbit for momentum p+Δp

$$D(s,p) = \frac{\Delta x(s,p)}{\Delta p/p}$$

the orbit of an off-momentum particle which closes on itself is described by the *periodic* dispersion function

Ex: FODO Cells with Bending Magnets



$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

$$D_{max,min} = \frac{L\theta}{2\sin^2(\mu/2)}\left(1 \pm \frac{1}{2}\sin(\mu/2)\right)$$

Values of dispersion function are typically ~ few meters

Note: in a weak-focusing synchrotron, would have $D = R_0$!

Dispersion

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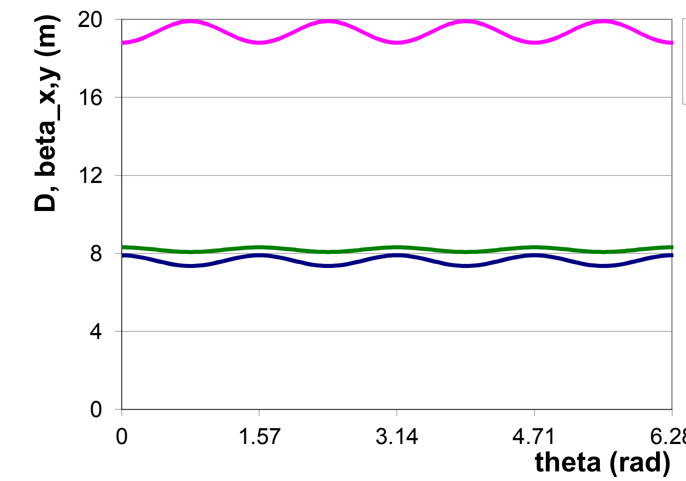
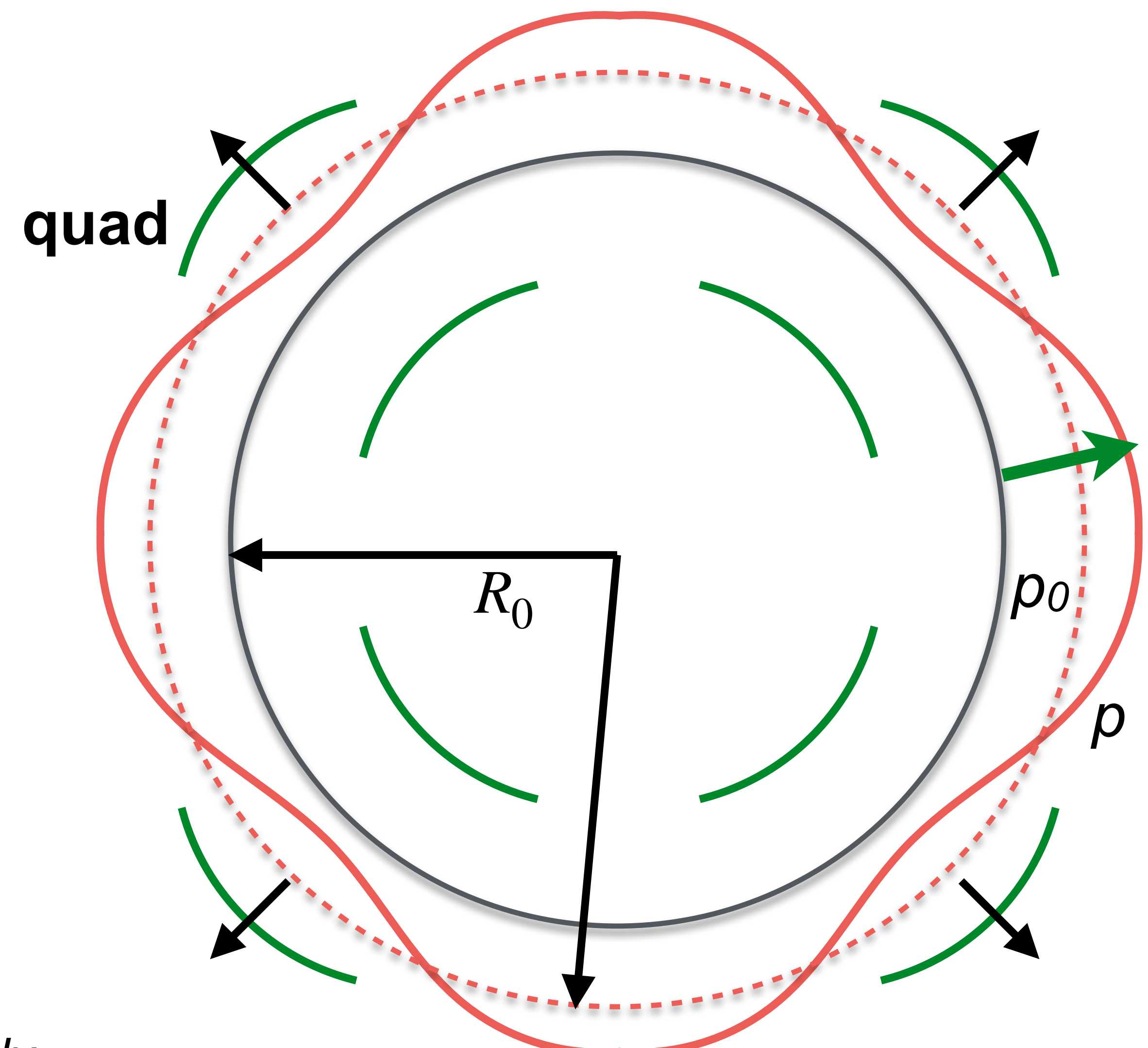
- Periodic Orbit of an Off-Momentum Particle





$$\Delta x_e(s) = D(s) \cdot \frac{\Delta p}{p_0}$$

schematic only,
***highly** exaggerated*

$$R = R_0(1 + \Delta p/p)$$

$D(s)$ will depend upon the quad voltage, hence upon the field index, n

R_0 = 7.112 m
D ≈ 8 m