PITCH CORRECTION IN $(g-2)$ EXPERIMENTS

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The pitch correction to the $(g-2)$ precession frequency of the electron or muon, arising from the axial oscillations of the particle rotating in an almost uniform magnetic field, is computed in a new way and extended to the case of axial focusing by electric fields. The main results are confirmed by a direct physical argument.

In $(g-2)$ experiments on the electron and muon the particle is constrained to turn in a magnetic field. It is well known that a small correction to the $(g-2)$ frequency is necessary if the orbit is not exactly perpendicular to the field. Granger and Ford [1] have recently reexamined the effect of spiral motion on the $(g-2)$ frequency ω_a, particularly when the pitch angle is varying due to axial focusing forces. They show that the pitch correction depends on the ratio of ω_a to the frequency ω_p of the axial oscillations, and compute significant corrections to the last two series of $(g-2)$ measurements for the electron [2]. In view of the importance of this question it may be useful to derive their result in a simpler way, extending the analysis to the case of axial focusing by electric fields currently being used in $(g-2)$ experiments for both the electron [3] and the muon [4], and to offer a more intuitive physical basis for understanding the equations. Such are the aims of this note.

Consider a particle spiralling in a main magnetic field B_z at a small angle ψ (pitch angle) with respect to the xy-plane perpendicular to the z-axis (referred to as the "axial" direction). Suppose further that the pitch angle varies harmonically according to

$$\psi = \psi_0 \sin \omega_p t \tag{1}$$

due to axial focusing forces (radial component of magnetic field or axial electric field). Choose a right-handed Cartesian coordinate frame rotating about the z-axis at frequency

$$\omega = \gamma^{-1}(e/mc)B_z\ , \tag{2}$$

such that the momentum vector lies always in the xz-plane making an angle ψ to the x-axis. The spin motion will be calculated relative to this frame, and from this the frequency of spin precession relative to the momentum vector, the so-called $(g-2)$ frequency, follows immediately. (Note that B_z is not necessarily constant. If it varies with particle position our analysis is still valid but the usual time average of B_z must be inserted in the final equations. B_x is zero on average over any closed path as no current flows through the orbit.)

We distinguish two components of the spin motion i) due to the main field B_z and ii) due to the axial focusing forces. For i) the angular velocity of the spin relative to our rotating frame is [5]

$$\omega_a = \gamma^{-1}\, a(e/mc)\, B^*\ , \tag{3}$$

where $a \equiv \frac{1}{2}(g-2)$ and B^* is the magnetic field in the rest frame of the particle. Resolving B_z parallel and perpendicular to the momentum, transforming to the rest frame and recombining, one finds the x- and z-components of ω_a,

$$\omega_z = \omega_0 \left\{1 - \frac{\gamma-1}{\gamma}\,\psi^2\right\} \tag{4}$$

$$\omega_x = -\,\omega_0\, \frac{\gamma-1}{\gamma}\psi\ , \tag{5}$$

where $\omega_0 = a(e/mc)B_z$ is the $(g-2)$ frequency for $\psi = 0$, and we use the approximation $\sin\psi = \psi$ for small ψ.

For the axial focusing forces we remark that when the momentum vector is deflected through angle ψ in the xz-plane the spin will rotate about the y-axis through angle $f\psi$ where [6]

$$\left.\begin{array}{ll} f = (1+\gamma a) & \text{for magnetic focusing,} \\ \text{and} \quad f = (1+\beta^2\gamma a-\gamma^{-1}) & \text{for electric focusing} \end{array}\right\}. \tag{6}$$

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Thus the harmonic pitch oscillation eq. (1) assumed above will induce an instantaneous angular velocity of the spin about the y-axis

$$\omega_y = -f\omega_p \psi_0 \cos\omega_p t\,. \tag{7}$$

We wish to follow the spin motion under the combined action of the three angular velocities (4), (5) and (7).

Transforming to spherical coordinates defined by

$$x = \cos\theta\,\cos\phi\ ,$$
$$y = \cos\theta\,\sin\phi\ ,$$
$$z = \sin\theta\ ,$$

the spin direction is defined by (θ, ϕ) and

$$\begin{aligned}\dot\theta &= \omega_x \sin\phi - \omega_y \cos\phi \\ &= -\omega_0\psi_0\gamma^{-1}(\gamma-1)\sin\phi\sin\omega_p t \\ &\quad + \omega_p\psi_0 f\cos\phi\cos\omega_p t\,,\end{aligned} \tag{8}$$

$$\begin{aligned}\dot\phi &= \omega_z - \omega_x\tan\theta\cos\phi - \omega_y\tan\theta\sin\phi \\ &= \omega_0\{1-\gamma^{-1}(\gamma-1)\psi_0^2\sin^2\omega_p t\} \\ &\quad + \omega_0\psi_0\gamma^{-1}(\gamma-1)\tan\theta\cos\phi\sin\omega_p t \\ &\quad + \omega_p\psi_0 f\tan\theta\sin\phi\cos\omega_p t\,.\end{aligned} \tag{9}$$

These equations are readily solved if one substitutes for ϕ in the right-hand side the close approximation $\phi = (\omega_0 t + \xi)$, where ξ is an arbitrary phase. We then find from eq. (8),

$$\begin{aligned}\theta &= \frac{A_1}{\omega_0+\omega_p}\sin\{(\omega_0+\omega_p)t+\xi\} \\ &\quad - \frac{A_2}{\omega_0-\omega_p}\sin\{(\omega_0-\omega_p)t+\xi\}\,,\end{aligned} \tag{10}$$

where

$$A_1 = \tfrac{1}{2}\psi_0\{\omega_0\gamma^{-1}(\gamma-1)+f\omega_p\}\,, \tag{11}$$

and A_2 is the same with negative sign before the last term.

Note that in general θ oscillates about zero with amplitude $\sim\psi_0$, the spin remaining on average in the xy-plane. However, the amplitude becomes large if $\omega_p \sim \omega_0$, and if $\omega_p = \omega_0$, $\theta = -A_2 t\cos\xi$, so the spin turns continously out of the plane. This is an example of the familiar depolarization resonance [7]. It follows from eq. (10) that large values of θ occur only if $(\omega_0-\omega_p)/\omega_0 \lesssim \psi_0/\gamma \lesssim 10^{-2}$ in a typical experiment.

If we are not too close to the resonance θ remains small and we may substitute θ as given by eq. (10) for $\tan\theta$ in eq. (9). This gives

$$\begin{aligned}\dot\phi &= \omega_0\left\{1-\frac{\gamma-1}{2\gamma}\,\omega_0^2\right\}\frac{A_1^2}{2(\omega_0+\omega_p)}+\frac{A_2^2}{2(\omega_0-\omega_p)} \\ &\quad + \text{terms oscillating at frequences } \omega_p, \omega_0, \\ &\quad (\omega_0-\omega_p)\,,\ \text{etc.}\end{aligned} \tag{12}$$

So, the observed $(g-2)$ frequency $\omega_a = \bar{\dot\phi} = \omega_0(1-C)$, where the correction factor C is found using eqs. (12) and (11) to be

$$\begin{aligned}C &= \tfrac{1}{4}\psi_0^2\{1-\omega_0^2/\gamma^2(\omega_0^2-\omega_p^2) \\ &\quad -\omega_p^2(f-1)(f-1+2/\gamma)/(\omega_0^2-\omega_p^2)\}\,.\end{aligned} \tag{13}$$

Applying this result to three cases of experimental interest we find:

a) When the axial focusing forces are magnetic, $f-1=\gamma a$, and

$$C = \tfrac{1}{4}\psi_0^2\{1-(\omega_0^2+2a\gamma^2\omega_p^2)/\gamma^2(\omega_0^2-\omega_p^2)\} \tag{14}$$

in agreement with Granger and Ford [1]. (Except at very high energies the term in a will be negligible in view of the uncertainties in determining ψ_0.) As ω_p is varied C changes according to a dispersion curve. When $\omega_p \ll \omega_0$ (the limit of slow pitch changes)‡, $C = \frac{1}{4}\beta^2\psi_0^2$. As ω_p increases C falls, eventually becoming negative, but after the resonance at $\omega_p = \omega_0$ C becomes large and positive, finally approching $\frac{1}{4}\psi_0^2(1+2a)$ when $\omega_p \gg \omega_0$ (the limit of rapid pitch changes). Near resonance our approximation $\theta \sim \tan\theta$ is not valid: a more detailed analysis shows that the infinitely large correction suggested by eq. (13) does not in fact occur.

b) With axial focusing by electric fields, $f-1 = \beta^2\gamma a - \gamma^{-1}$, so

$$C = \tfrac{1}{4}\psi_0^2\{\beta^2 - a^2\beta^4\gamma^2\omega_p^2/(\omega_0^2-\omega_p^2)\}\,. \tag{15}$$

‡ This agrees with the result of ref. [8] for a particle spiralling in a uniform magnetic field; also obtained by ref. [5].

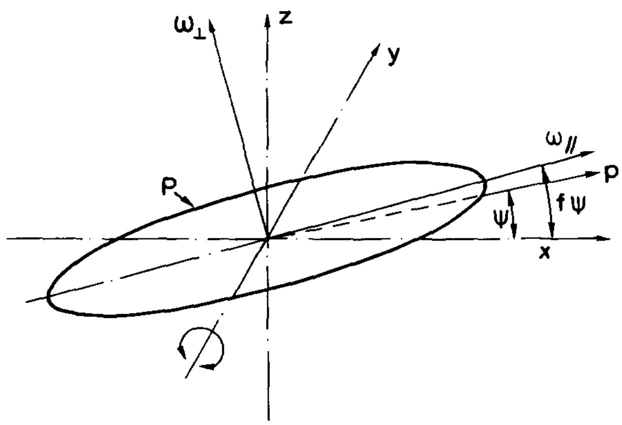

Fig. 1. Axes xyz rotating about z so that the momentum vector p lies always in the xz-plane. Owing to pitch oscillations the pitch angle ψ between p and the x-axis is varying. The spin angle is determined in the plane P (inclined at angle $f\psi$ to the x-axis) because a spin in this plane moves into the xy-plane when ψ passes through zero.

At low energy the second term is negligible and the corrections is virtually independent of the pitch frequency.

c) With electric focusing, using the "magic" particle energy given by $\beta^2\gamma^2 = a^{-1}$, so that electric fields do not affect the $(g-2)$ precession [4], then $f = 1$, and the final term in eq. (13) is zero: the pitch correction now shows the resonant behaviour discussed above, but contains no terms in a. Far from the resonance, $C = \frac{1}{4}\psi_0^2$ at both limits.

These results may be derived directly by means of simple physical argument based on eqs. (4) and (5). We are interested in the average rotation of the spin about the z-axis, i.e. the projection of the spin on the xy-plane. Let us consider the spin direction in this plane at instants when the pitch angle is zero, thus neglecting for the average motion the small non-cumulative mutations which occur during the pitch oscillation. The essential step is to recognize that when the pitch angle is ψ we must determine the progress of the spin in a plane making angle $f\psi$ with the x-axis, indicated in fig. 1 as the plane P. This is because any spin direction lying in this plane will be turned into the xy-plane when the pitch again becomes zero.

We therefore resolve the angular velocities (or magnetic fields) parallel ($\omega_\parallel$) or perpendicular ($\omega_\perp$) to the plane P. From eqs. (4) and (5)

$$\omega_\perp = \omega_z \cos(f\psi) - \omega_x \sin(f\psi)$$

$$= \omega_0 \{1 - \tfrac{1}{2}\psi^2[1 + (f-1)(f-1+2/\gamma)]\} \,. \qquad (16)$$

When the pitch oscillation is rapid ($\omega_p \gg \omega_0$), then $\omega_\parallel$ is rapidly changing in sign and contributes nothing to the net spin precession. So, in this case, the observed spin motion will be determined by $\omega_\perp$ alone. Recalling that $\overline{\psi}^2 = \frac{1}{2}\psi_0^2$ we see that eq. (16) and eq. (13) agree exactly in the limit $\omega_p \gg \omega_0$.

In the case of slow pitching however ($\omega_p \ll \omega_0$), $\omega_\parallel$ remains almost constant over many cycles of $(g-2)$ precession, and then $\omega_\perp$ and $\omega_\parallel$ must be *added vectorially* to give the resultant angular velocity. This is of course equal to the resultant of ω_x and ω_z, so from eqs. (4) and (5)

$$\omega_a = \omega_0 \{1 - \tfrac{1}{2}\beta^2\psi^2\} \qquad (17)$$

agreeing with eq. (13) in the limit $\omega_p \to 0$.

Thus the fast- and slow-pitching limits of eq. (13) are confirmed by an independent argument.

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