Emittance Preservation

- Liouville's Theorem: the volume enclosed by surface in phase space is invariant under conservative forces
- Another theorem from classical dynamics: integration over a time period of the "action variables" is an adiabatic invariant

$$J = \int_0^T p_x(t) \frac{dx}{dt} dt$$

  - transverse: (x, p_x), (y, p_y) are action variables
  - longitudinal: ΔE and Δt are also action variables
  - "normalized" transverse phase space emittances,
    - $\epsilon_N = (\beta\gamma)\epsilon = (p/mc)\int x'\,dx = \int p_x\,dx\,/mc$

Protons vs. Electrons

- When dealing with a beam line or along a linac, the same issues affecting beam emittance exist for both electron and proton (or heavier ion) beams.
- In the case of circular accelerators, there is a distinct difference: charged particles radiate as they are accelerated, and electrons will radiate much more than protons and, as we have seen, the final emittance of electron beams in a ring will be defined by the optics of the ring.
- This is not true for a proton beam. If the emittance is increased due to errors or mismatches, the damage is done and cannot be undone without ***much*** effort

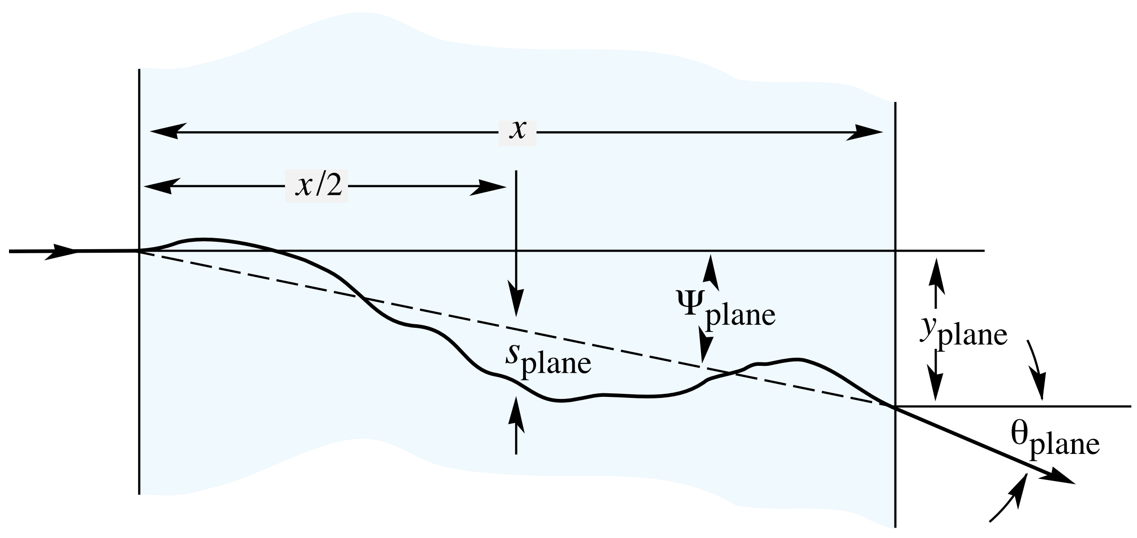
Sources of Emittance Growth

- Will discuss two classes of disruptive processes…
  - **Single non-adiabatic disturbance** of the distribution
    - *examples*: injection errors (steering, focusing); electrostatic "spark"; single pass through a vacuum window; a pinger/kicker excitation; intrusive diagnostic measurement; …
  - **Repetitive random disturbances** of individual particles, leading to diffusion
    - *examples*: RF noise; beam-gas scattering; power supply noise; mechanical vibrations; …

Non-adiabatic Disturbances

Example: single pass through a thin object (vacuum window)

The beam particles interact with the atoms in the material and scatter, primarily from Coulomb interactions. In either plane — x or y — the distribution of scattering angles emerging from the material is given by:



$$\theta_{rms} \approx \frac{13.6\ \text{MeV}}{\beta pc}\sqrt{\frac{\ell}{L_{rad}}}$$

where L_{rad} is the "radiation length" of the material:

$$\frac{1}{L_{rad}} \approx 2\alpha \frac{N_A}{A} \rho Z^2 r_e^2 \ln \frac{a}{R}$$

N_A = Avogadro's No., A = atomic mass, Z = charge state, r_e = "classical electron radius", a = radius of target atom, R = radius of target nucleus, α = fine structure constant

for more accurate estimates, see *Particle Data Booklet*, http://pdg.lbl.gov

Side Note: The Bethe Formula

H. Bethe und J. Ashkin in "Experimental Nuclear Physics, ed. E. Segré, J. Wiley, New York, 1953, p. 253

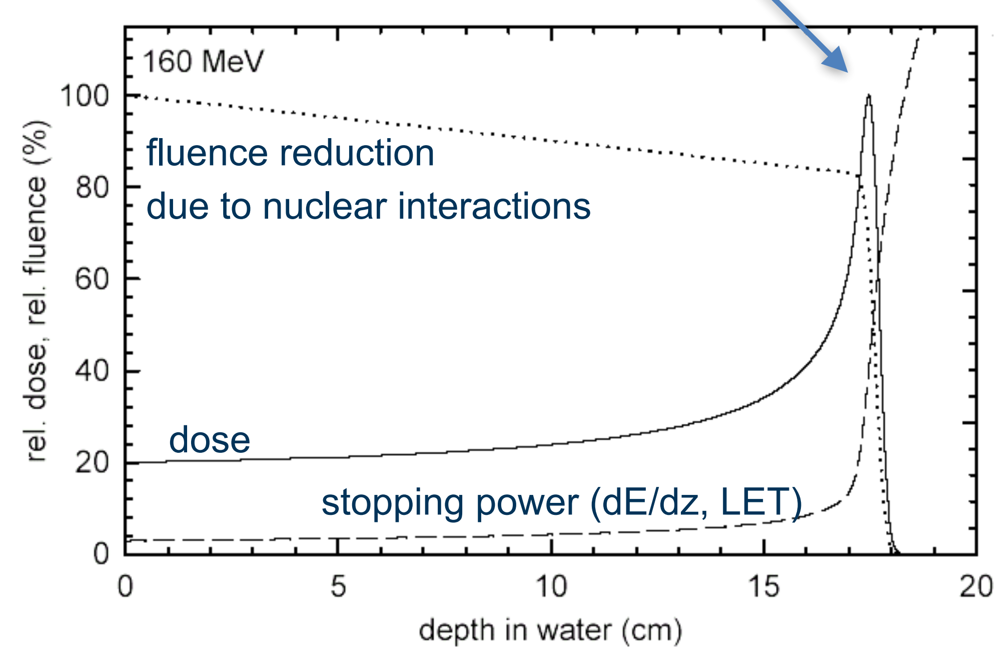
- Radiation Length is related to the stopping power of material as charged particles pass through
  - mean distance e$^-$ travels before losing all but 1/e of its energy $\frac{1}{L_{rad}} \approx 2\alpha \frac{N_A}{A} \rho Z^2 r_e^2 \ln \frac{a}{R}$
- The average energy loss rate is given by the Bethe formula:

$$-\left\langle \frac{dE}{dx} \right\rangle = \frac{4\pi}{m_e c^2} \cdot \frac{nz^2}{\beta^2} \cdot \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \cdot \left[\ln\left(\frac{2m_e c^2 \beta^2}{I \cdot (1-\beta^2)}\right) - \beta^2\right]$$

$$n = \frac{N_A \cdot Z \cdot \rho}{A \cdot M_u},$$

Used to determine depth of energy deposition for proton or ion therapy, for instance:

“Bragg Peak”



Non-adiabatic Disturbances

Example: single pass through a thin object (vacuum window)

- As we saw earlier, the emittance and Courant-Snyder parameters describing a distribution can be written as:

$$\epsilon_N = (\beta\gamma)\epsilon \qquad \epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon} \qquad \alpha = -\frac{\pi\langle xx'\rangle}{\epsilon} \qquad \gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

- From the scattering, the angular distribution will be altered:

$$x' = x'_0 + \Delta\theta$$

$\Delta\theta$ is random, with $\langle\Delta\theta^2\rangle \equiv \theta^2_{rms}$

- We then average over the distribution to see the effect on the CS parameters and emittance…

Non-adiabatic Disturbances

Example: single pass through a thin object (vacuum window)

$$
\begin{aligned}
x &= x_0 & \langle x^2\rangle &= \langle x_0^2\rangle \\
x' &= x_0' + \Delta\theta & \langle x'^2\rangle &= \langle (x_0' + \Delta\theta)^2\rangle \\
& & &= \langle x_0'^{\,2} + \Delta\theta^2 + 2x_0'\Delta\theta\rangle \\
& & &= \langle x_0'^{\,2}\rangle + \langle \Delta\theta^2\rangle \\
& & \langle xx'\rangle &= \langle x_0(x_0' + \Delta\theta)\rangle = \langle x_0 x_0'\rangle
\end{aligned}
$$

assuming that the scattering process is uncorrelated with the phase space variables. The new emittance after scattering is then given by,

$$(\epsilon/\pi)^2 = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = \langle x_0^2\rangle(\langle x_0'^{\,2}\rangle + \theta_{rms}^2) - \langle x_0 x_0'\rangle^2$$

or,

$$\epsilon = \epsilon_0\sqrt{1 + \frac{\langle x_0^2\rangle\theta_{rms}^2}{(\epsilon_0/\pi)^2}}$$

where $\theta_{rms} = \langle\Delta\theta^2\rangle^{1/2}$. Since $\beta_0 = \pi\langle x_0^2\rangle/\epsilon_0$, then we can write

$$\epsilon = \epsilon_0\sqrt{1 + \theta_{rms}^2\frac{\beta_0}{(\epsilon_0/\pi)}}.$$

And, using this new emittance, we can describe the resulting distribution after the scattering by new Courant-Snyder parameters

$$
\begin{aligned}
\beta &= \frac{\pi\langle x^2\rangle}{\epsilon} = \beta_0\,\frac{\epsilon_0}{\epsilon} \\
\alpha &= \alpha_0\,\frac{\epsilon_0}{\epsilon} \\
\gamma &= \left(\gamma_0 + \frac{\theta_{rms}^2}{(\epsilon_0/\pi)}\right)\frac{\epsilon_0}{\epsilon}
\end{aligned}
$$

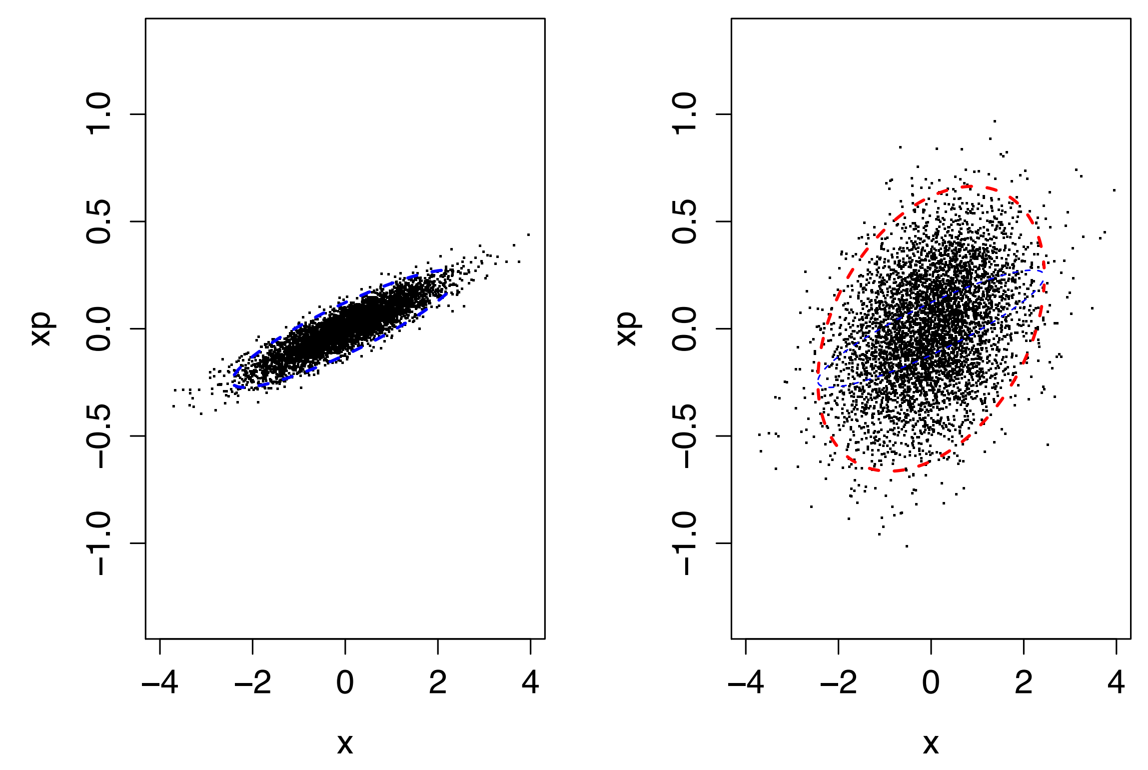


Figure 1: Phase space before/after (left/right) scattering. Blue/red lines indicate 95% emittances before/after scattering.

If the scattering is inherent in the system, then can imagine re-tuning the downstream optical system to match to the new conditions; but will still have an overall emittance growth

Non-adiabatic Disturbances

Example: single pass through a linac/beamline

$$\theta_{rms} \approx \frac{13.6 \text{ MeV}}{\beta pc}\sqrt{\frac{\ell}{L_{rad}}}$$

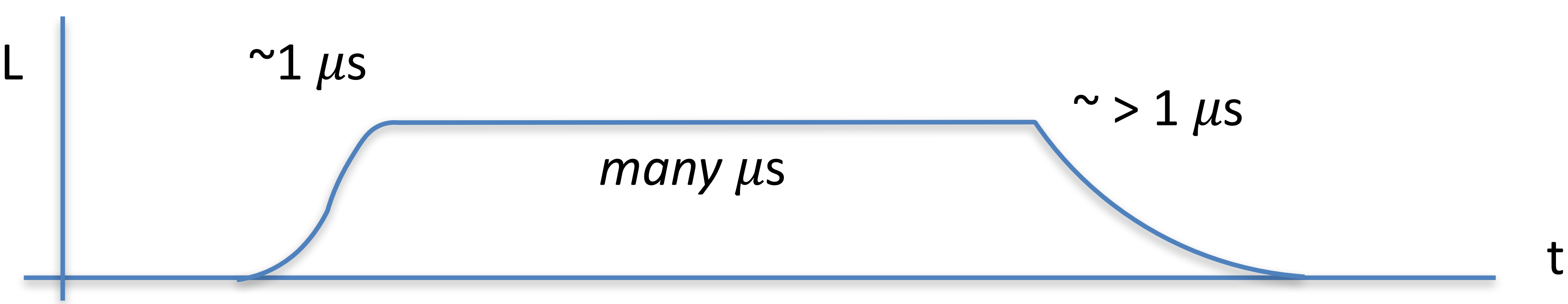
- Suppose we pass through a long "evacuated" tube of length *L*. As an example, consider a tube which started with air, and has been evacuated to an average pressure of 10^{-6} torr (760 torr = 1 atm)
- From the PDG report, find L_{rad} of air (dry; 1am):
  - density = 1.205 g/l , L_{rad} = 36.6 g/cm^2
  - so, L_{rad} = (36.6 g/cm^2)/(1.205 g/ml)(l/1000 cm^3)
  - = 30373 cm = 304 m
  - at 10^{-6} torr, through *PV=nRT*, L_{rad} = 231x10^9 m
- Estimate the rms scattering angle of a typical particle, just due to this effect:

$$\theta_{rms} \approx \frac{13.6 \text{ MeV}}{27 \text{ MeV}}\sqrt{\frac{230 \text{ m}}{23 \times 10^{10} \text{ m}}} \approx 16\ \mu\text{rad}$$

Side Note: Kicker Magnet

- Want to induce an angular deflection of a particle bunch, or bunch train, without affecting other particles outside of the bunch/train
- Need significant B fields that turn on/off on the scale of, say, μs   ex: bunches @ 1 MHz = 1 μs
  - Ex: discharge large current into an inductive load (magnet) with a resistance,   gives time constants on the scale of ~ L/R: μH/Ohm = μs

$B = B\rho\ \theta/L$



fast means low inductance, thus low fields and low turns, high currents in the magnets

Side Note: Kicker Magnet

- Kickers typically used to deflect beam into and out of beam lines and accelerators

Ex: injection into a synchrotron:

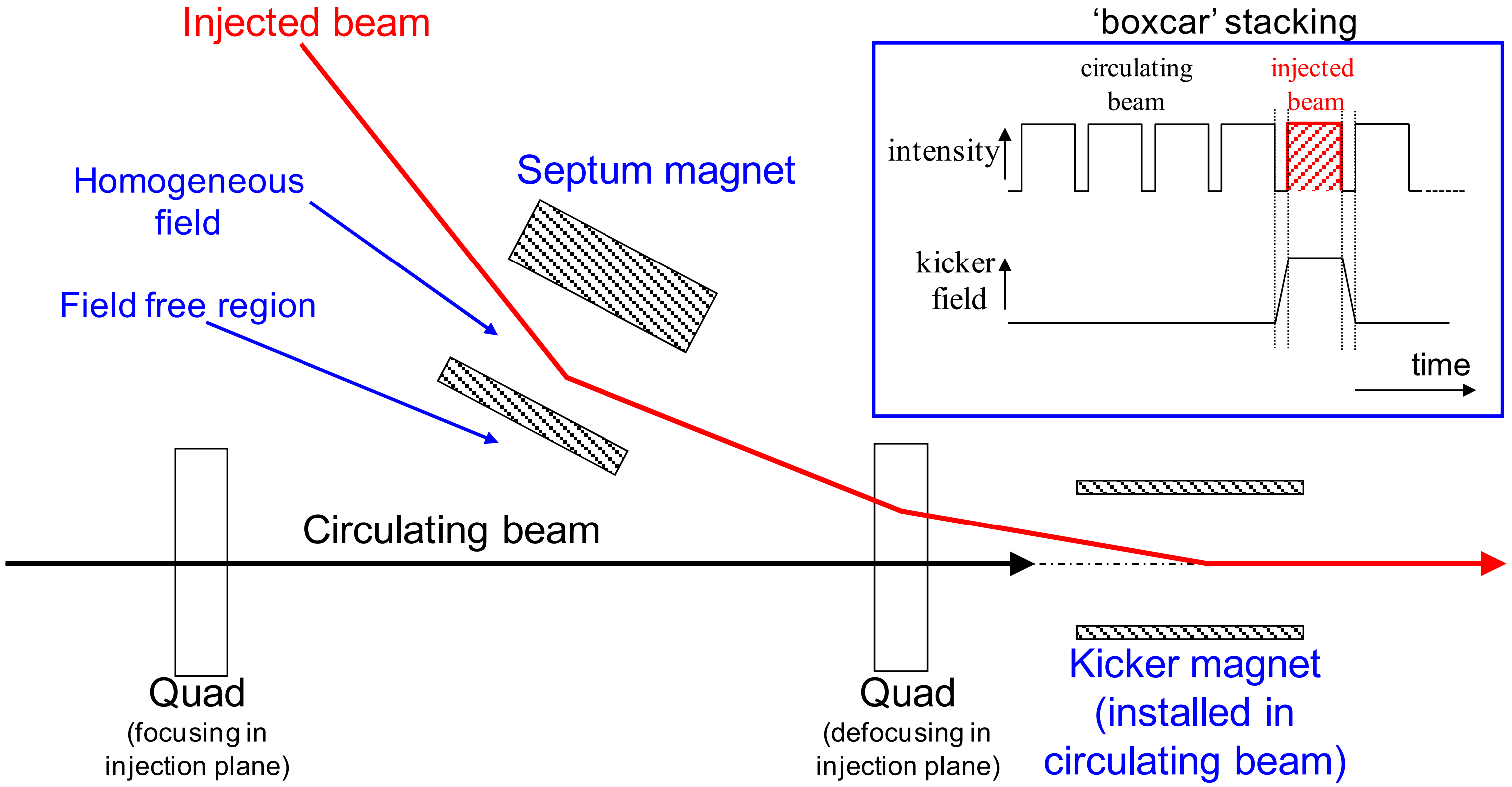


figure courtesy of M.J. Barnes, et al., CERN

- Can also be used for diagnostic purposes, by intentionally inducing a betatron oscillation in the beam and observing downstream reaction

Non-adiabatic Disturbances

Example: Discharge of a beam kicker in a synchrotron

- Initially, the distribution is simply "displaced" by the action of the kick:

- Nonlinearities will yield:
  - tune vs. amplitude
  - decoherence
  - *filamentation*
  - emittance growth

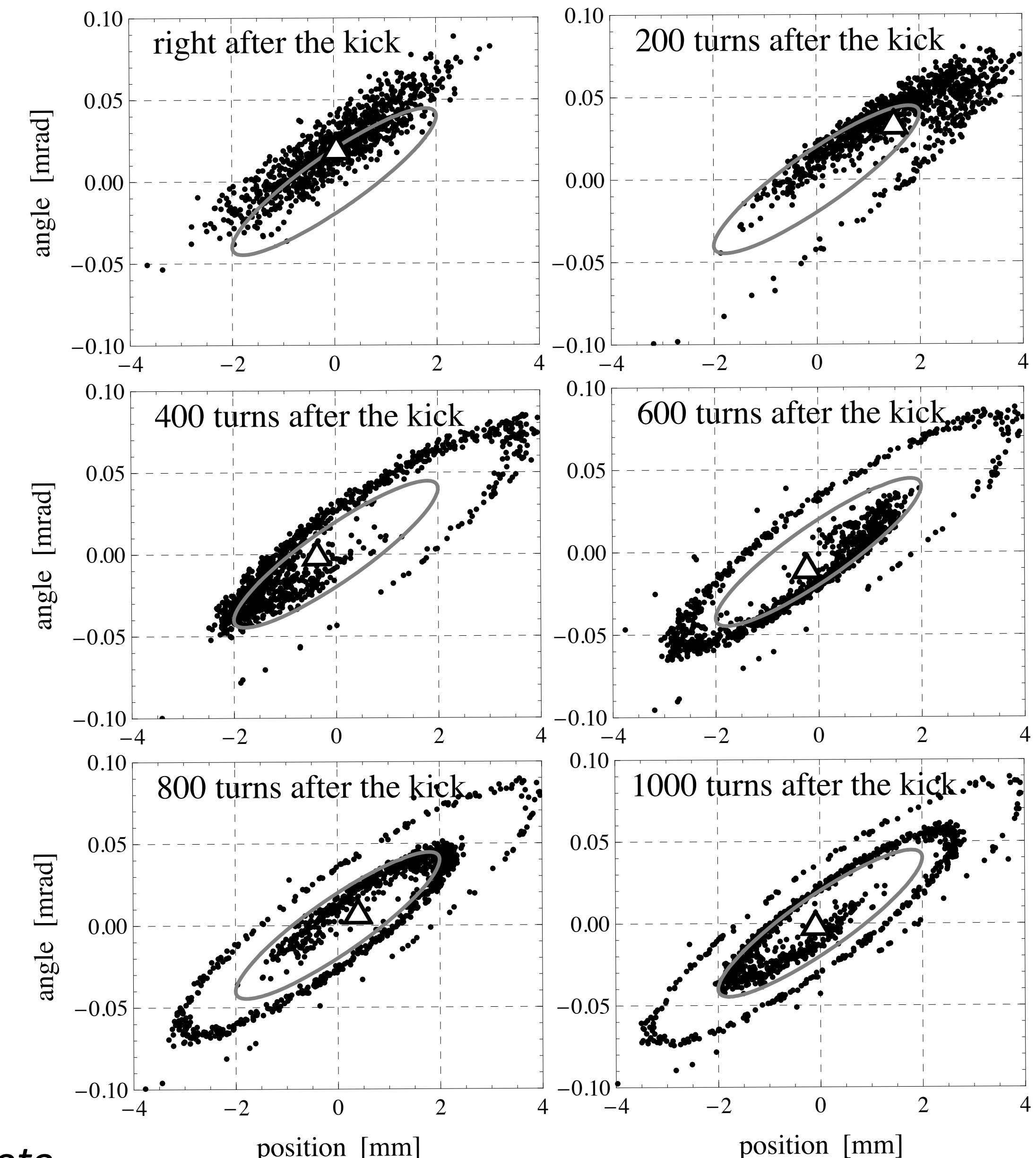
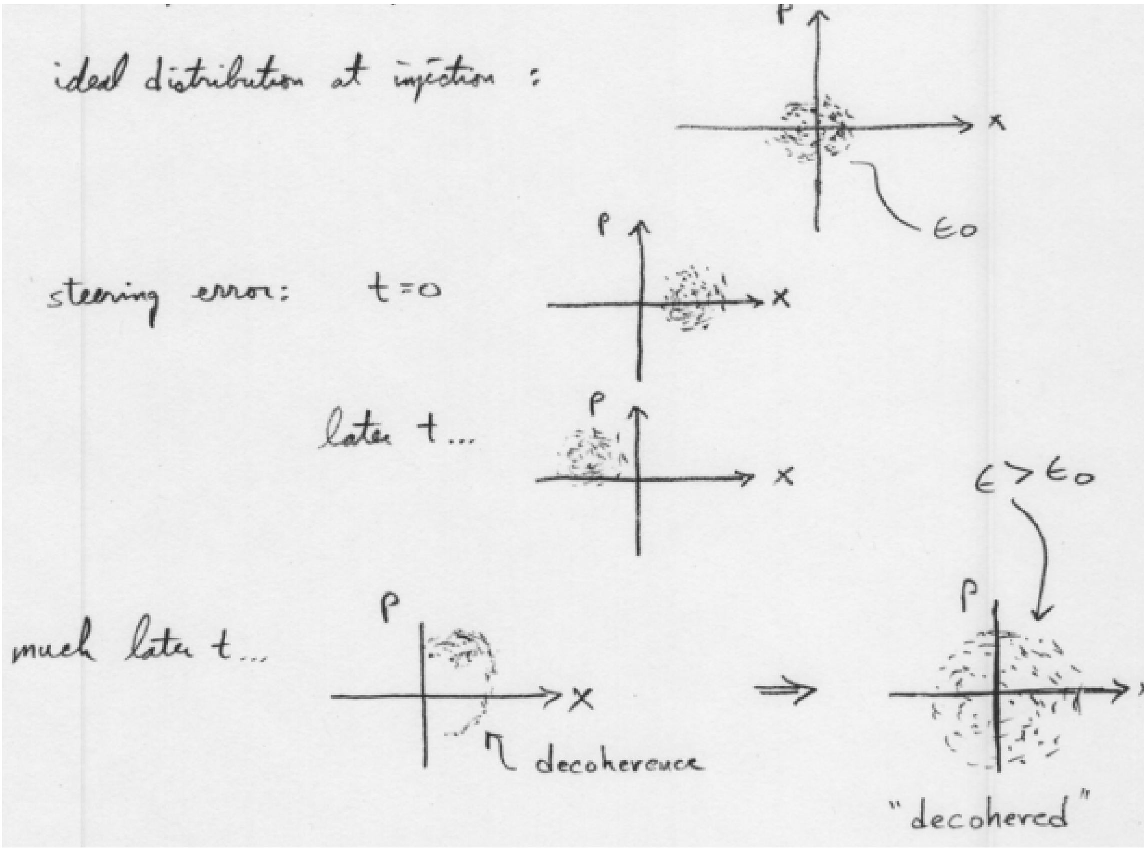


figure by R. Miyamoto

Accelerator Model

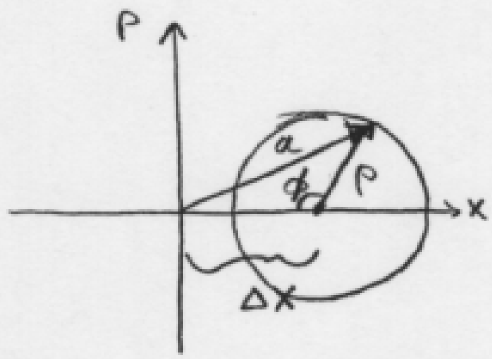
- So we will model these effects by assuming the distribution will oscillate about the closed orbit, and that the oscillation frequencies of the particles will depend upon the amplitude of their oscillations
  - typically: $\nu \approx \nu_0 + ka^2$

    nonlinear tune shift
  - coherent at first,
  - then “decoheres”
    - leads to filamentation
  - eventually: larger emittance



Example: Injection Steering Mismatch

how does rms evolve ?



if $\Delta x = 0$, would have

$$\langle x^2 \rangle = \tfrac{1}{2}\rho^2 \equiv \sigma_0^2$$

but here,

$$a^2 = \rho^2 + \Delta x^2 - 2\Delta x\, \rho \cos\phi$$

average over all particles – $\langle a^2 \rangle = \langle \rho^2 \rangle + \Delta x^2 - 2\Delta x \underbrace{\langle \rho \cos\phi \rangle}_{\sim \phi}$

$\Rightarrow \langle a^2 \rangle = \langle \rho^2 \rangle + \Delta x^2$ after decoherence

$\Rightarrow \langle x^2 \rangle = \sigma_0^2 + \tfrac{1}{2}\Delta x^2$ "

if $\epsilon \equiv \frac{6\pi\sigma^2}{\beta}\cdot(\gamma\beta)$ $\longrightarrow$ $\boxed{\Delta\epsilon_N = \frac{3\pi\Delta x^2}{\beta}(\gamma\beta)}$ or, $\boxed{\epsilon/\epsilon_0 = 1 + \frac{1}{2}\left(\frac{\Delta x}{\sigma_0}\right)^2}$

β @ location of Δx

95% emittance; no "6" if rms emittance

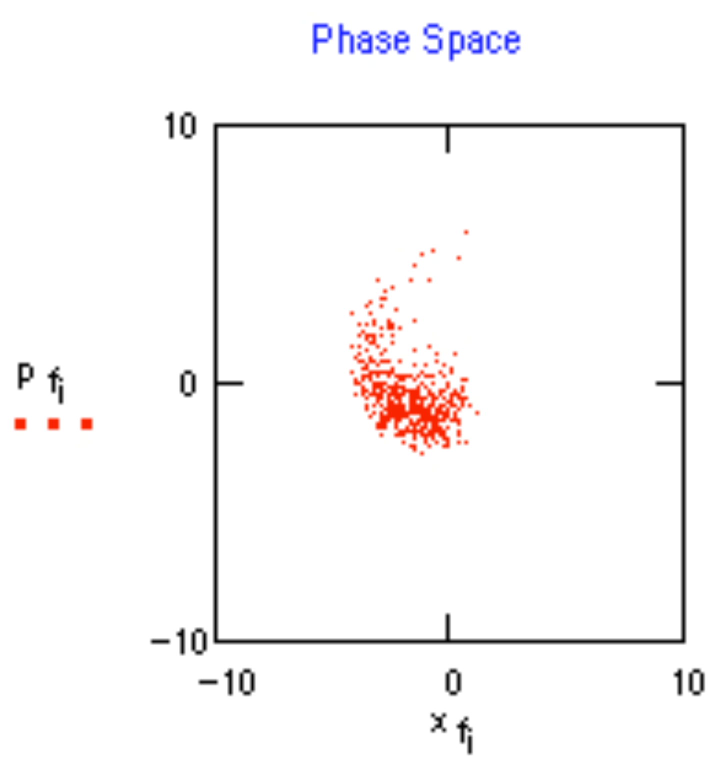
Note:

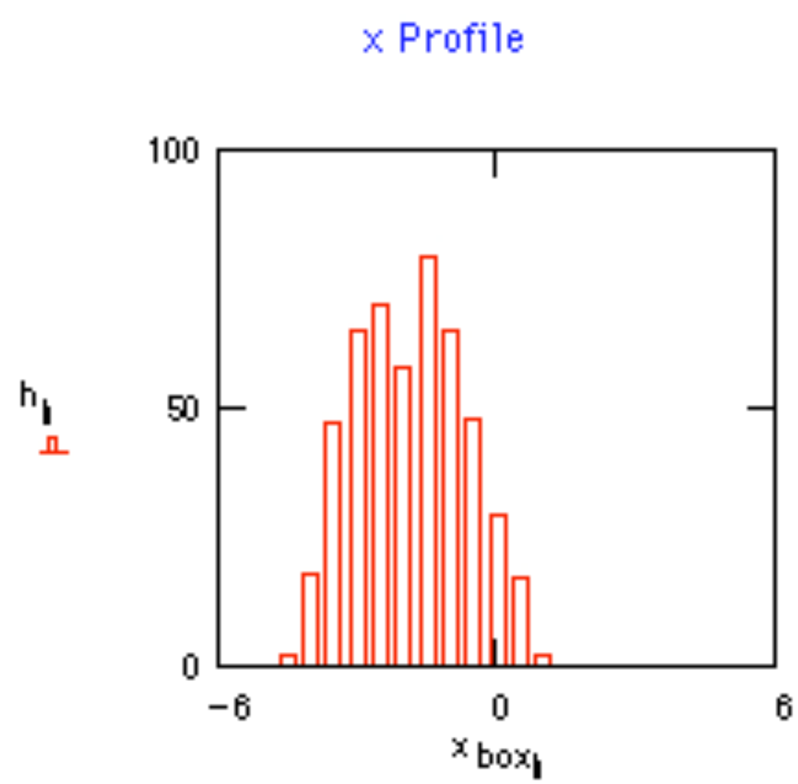
injection errors – $\Delta\epsilon_N \propto (\gamma\beta)$, ∴ more important @ higher energies

let $\Delta x = 1$ mm @ injection to

	$\hat{\beta}$	$\gamma\beta$	$\Delta\epsilon_N$
BOO	30m	1.017	0.10 π mm-mr
MI	50 m	9.48	0.57 π mm-mr
TEV	100m	160	4.8 π mm-mr ☆

Injection Mismatch





$\mathrm{mean}(x_f) = -1.609$ $\qquad$ $\mathrm{stdev}(x_f) = 1.179$

Emittance Increase: $\mathrm{stdev}(x_f)^2 = 1.39$

Predicted "typical" values:

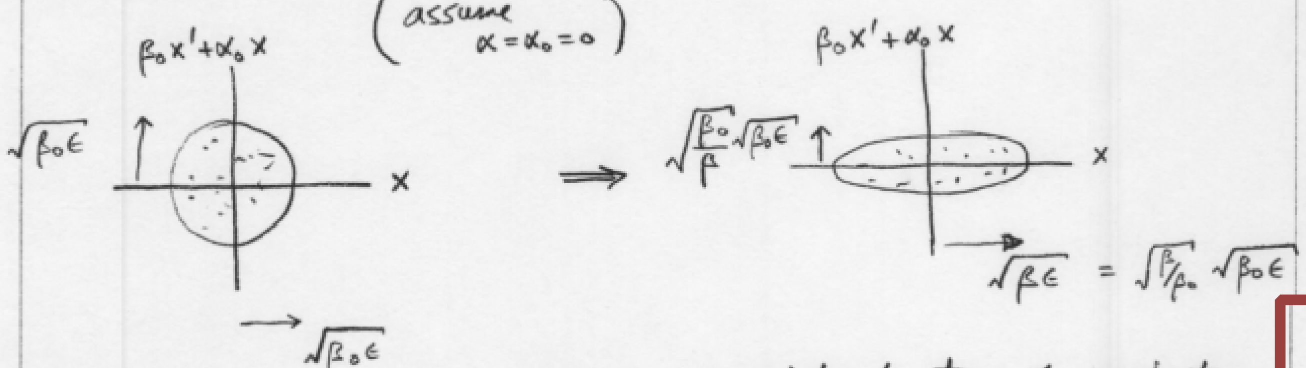
(Steering Mismatch) $\quad 1 + \frac{1}{2}\cdot\Delta x^2 = 3$

FRAME = 8 $\quad$ (Amplitude function Mismatch) $\quad \frac{r_\beta^2 + 1}{2\cdot r_\beta} = 1$

Injection “Beta” Mismatch

- We imagine a ring with an ideal amplitude function, β, at an injection point. But, suppose the beam line transporting beam from an upstream injector delivers the wrong β function:

$\therefore$ in "circular" phase space ... (assume $\alpha = \alpha_0 = 0$)



β_0, α_0 : periodic functions of synchrotron

β, α : delivered by the beamline

then, after the distribution tumbles and filaments in phase space, the emittance will grow...

$$a^2 = x^2 + (\beta_0 x' + \alpha_0 x)^2 \equiv x^2 + p^2$$

let $b \equiv \beta/\beta_0$ $\Rightarrow$

$$\langle a^2 \rangle = \langle x^2 \rangle + \langle p^2 \rangle = (\sqrt{b}\,\sigma_0)^2 + \left(\frac{1}{\sqrt{b}}\,\sigma_0\right)^2$$

$$\Rightarrow \quad 2\sigma^2 = \frac{b^2+1}{b}\,\sigma_0^2 \quad \Rightarrow \quad \boxed{\epsilon/\epsilon_0 = \frac{b^2+1}{2b}}$$

Injection "Beta" Mismatch

- Can write a more general result in terms of the "mismatch" invariant:
  - det(ΔJ) = $|\Delta\beta\Delta\gamma - \Delta\alpha^2|$ = *invariant*

- If inject with *"beam"* parameters α, β, γ, whereas the ring has periodic parameters α_0, β_0, γ_0, then…

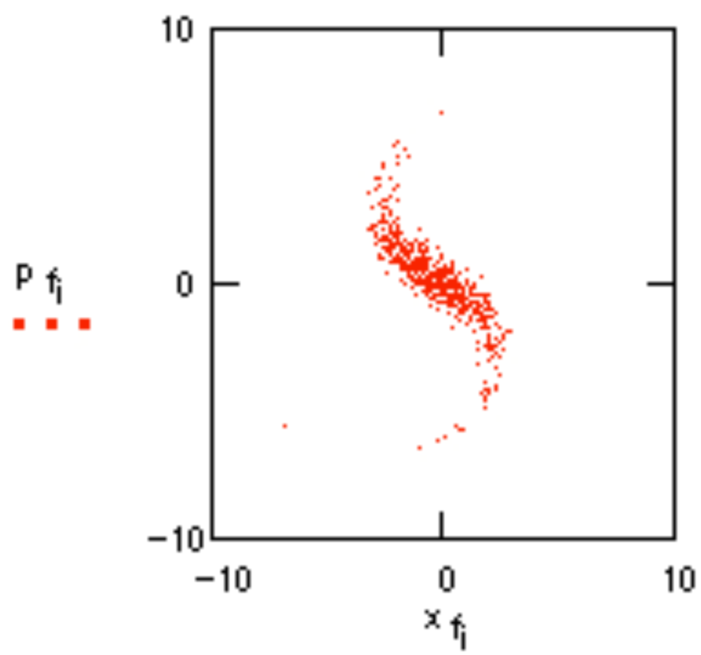
- … after filamentation, the final emittance will be given by

$$\boxed{\epsilon/\epsilon_0 = 1 + \frac{1}{2}|\det \Delta J|}$$

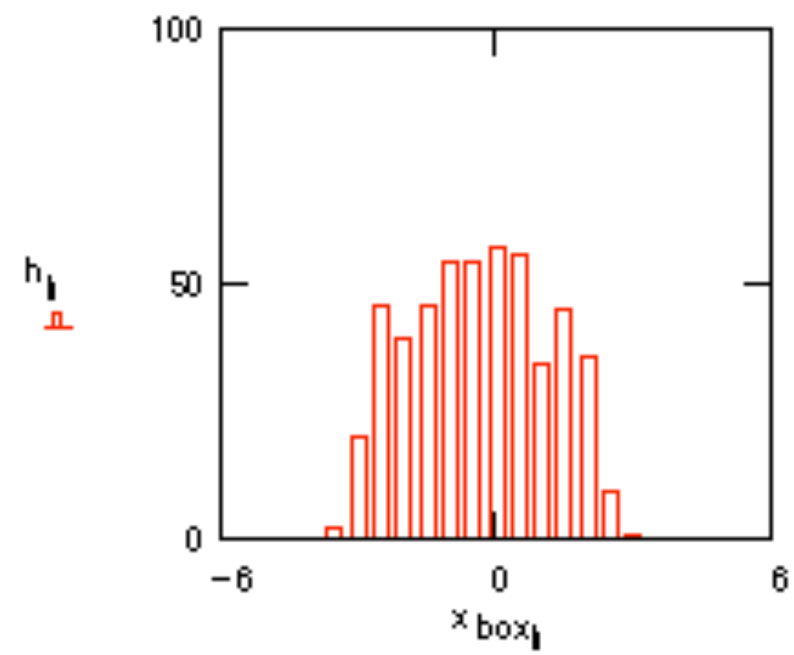
$$\Delta J = \begin{pmatrix} \Delta\alpha & \Delta\beta \\ -\Delta\gamma & -\Delta\alpha \end{pmatrix}$$

Injection Mismatch

Phase Space



x Profile



$\mathrm{mean}(x_f) = -0.12$ $\mathrm{stdev}(x_f) = 1.515$

Emittance Increase: $\mathrm{stdev}(x_f)^2 = 2.295$

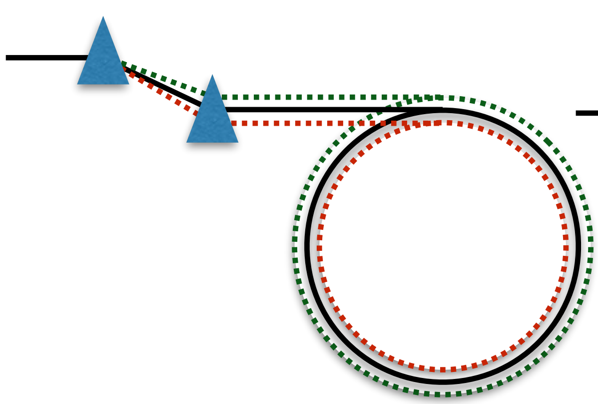
Predicted "typical" values: (Steering Mismatch) $1 + \frac{1}{2}\cdot\Delta x^2 = 1$

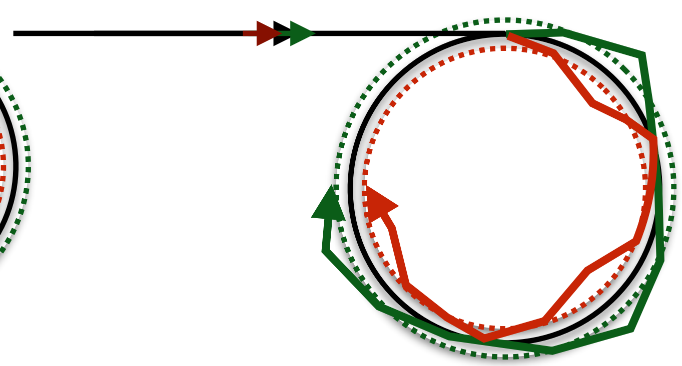
FRAME = 2 (Amplitude function Mismatch) $1 + \frac{1}{2}\cdot\frac{\delta\beta^2}{1+\delta\beta} = 2.6$

Mismatch of the Dispersion Function

- Can also imagine having the dispersion function entering the accelerator from a beam line having the wrong value
  - amounts to an injection steering error for an off-momentum particle — similar analysis as before

÷ 6 for rms emittance growth



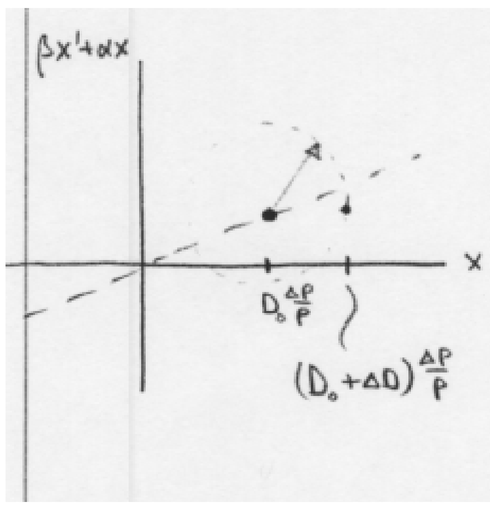


and so, $\boxed{\Delta\epsilon_N = \frac{3\pi\,\Delta D^2(s)}{\beta(s)}\left(\frac{\sigma_p}{p}\right)^2(\gamma\beta)}$

worse for larger σ_p/p (coalesced!)

ex: coalesced beam, w/ $\frac{\sigma_p}{p} = 5\times10^{-4}$, and $\Delta D \approx 1$ m

$$\Rightarrow \Delta\epsilon_N \approx \frac{3\pi(1\,\mathrm{m})^2}{100\,\mathrm{m}}(0.5\times10^{-3})^2(160) = 1.2\ \pi\ \text{mm-mrad}$$



each particle of momentum $\frac{\Delta p}{p}$ will see an injection steering error of amplitude $x_p = \Delta D\,\frac{\Delta p}{p}$

$$\Rightarrow a^2 = \rho^2 + \Delta D^2\left(\frac{\Delta p}{p}\right)^2$$

$$\Rightarrow \langle a^2\rangle = \langle\rho^2\rangle + \Delta D^2\left\langle\left(\frac{\Delta p}{p}\right)^2\right\rangle$$

important if the incoming beam has a high momentum spread

Emittance Growth from Diffusive Processes

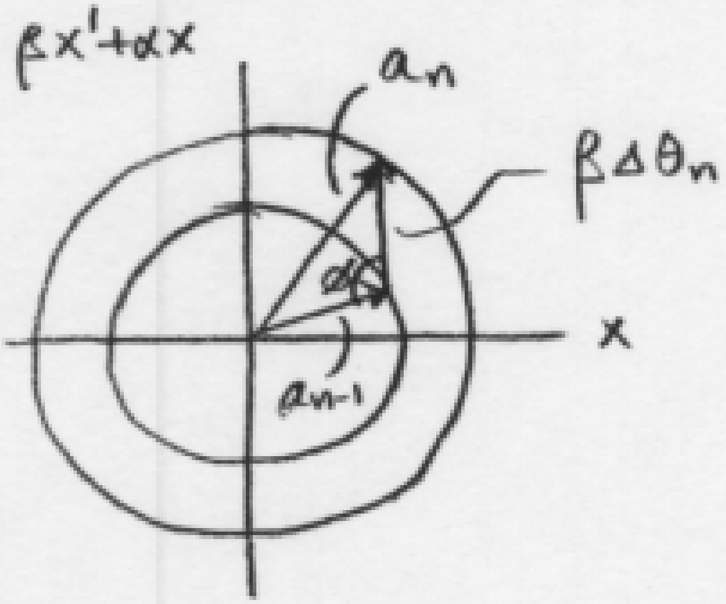
- So far have looked at single, non-adiabatic disturbances of our initial particle distribution
- Next, we look at the effect of repetitive random disturbances of individual particles, leading to diffusion
  - examples: scattering of particles off of the residual gas in the vacuum chamber; power supply noise; RF noise; continuous mechanical vibrations, …
- This amounts to continuous, random events taking place to alter the transverse amplitudes of the motion of individual particles

Repetitive Random Disturbances

- Estimations from a phase space perspective…

Next, consider diffusive mechanisms...

each particle's amplitude is changed "continuously" by a random process...



$$a_1^2 = a_0^2 + \beta^2\Delta\theta_1^2 - 2a_0\beta\theta_1\cos\phi_1$$

$$a_n^2 = a_{n-1}^2 + \beta^2\Delta\theta_n^2 - 2a_{n-1}\beta\theta_n\cos\phi_n$$

average over all particles ...

$$\langle a^2\rangle_n = \langle a^2\rangle_{n-1} + \beta^2\langle\Delta\theta^2\rangle_n$$

$$\cancel{- 2\beta\langle a\rangle\langle\theta\rangle_n\langle\cos\phi\rangle_n} \rightarrow 0$$

$$\therefore\ d\langle a^2\rangle = \beta^2\theta_{rms}^2\, dn$$

Repetitive Random Disturbances

- Look again at "vacuum" problem examined earlier
- Suppose circulating in a synchrotron w/ P=10^{-6} torr:
  - suppose $<\beta>$ = 20 m around the circumference, and that E = 13.6 GeV
  - $d<a^2> = 2\, d<x^2> = 2 <\beta>\; d(\epsilon/\pi) = <\beta>^2\, \theta_{rms}{}^2\; dn$
  - $d\epsilon/dt = \pi/2 <\beta>\; \theta_{rms}{}^2\; f_0 = \pi/2 <\beta>\; (0.0136/E)\; v/L_{rad}$
  - $= \pi/2$ (20 m) (10^{-3}) (3x10^{8}/2.3x10^{11})
  - $= \pi$ (13x10^{-6} m/s) = 13 π mm-mr/s !
  - so, might need much better vacuum here!

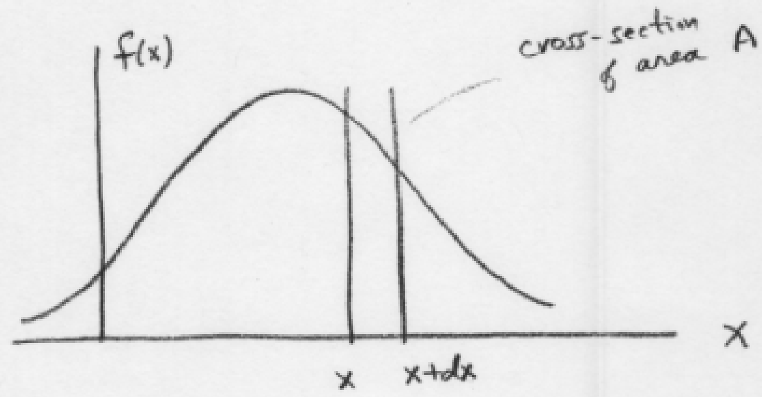
Repetitive Random Disturbances

- So, we see that in repetitive systems such random scattering events and other similar disturbances can cause emittance growth over time
- Wish to analyze such conditions
  - analytical approaches
  - simulations

The Diffusion Equation

Diffusion Equation

Look @ 1-D first:



$$\frac{\partial}{\partial t}(f \cdot A \cdot dx) = \underbrace{A \cdot J(x)}_{\#\ \text{into}} - \underbrace{A \cdot J(x+dx)}_{\#\ \text{out of}}$$

J = average # particles per unit time passing position x

$$\Rightarrow \quad \frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x}$$

if f uniform, then $J = 0$; otherwise $J \propto -\frac{\partial f}{\partial x}$

$$\therefore \quad \boxed{\frac{\partial f}{\partial t} = C \cdot \frac{\partial^2 f}{\partial x^2}} \qquad C = \text{constant}$$

more general 3-D case : $\boxed{\frac{\partial f}{\partial t} = C\ \nabla^2 f}$

can solve analytically

- particle velocities are randomly altered
- particles will move from one region into another
- the rate at which particles cross into or out of a region depends on the *slope* of the distribution function

The Diffusion Equation

- think of betatron motion in terms of coordinates:
  - *x*, *αx+βx'* (*circular)*
- use cylindrical coordinates for the Diffusion Equation
- re-cast in terms of an emittance ~ r^2/β
- with appropriate scaling, can write a dimensionless equation for the distribution function. Emittance is now scaled by the aperture acceptance
- apply boundary conditions

$$\frac{\partial f}{\partial t} = C \cdot \nabla^2 f$$

$$\frac{\partial f}{\partial t} = C \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)$$

$$\frac{\partial f}{\partial t} = R \frac{\partial}{\partial \epsilon_p}\left(\epsilon_p \frac{\partial f}{\partial \epsilon_p}\right)$$

$$\epsilon_p \equiv r^2/\beta$$

$$R \equiv \frac{\partial}{\partial t}\langle r^2/\beta\rangle$$

$$W \equiv a^2/\beta \quad \sim \quad \text{“acceptance”}$$

$$\tau \equiv (R/W)t$$

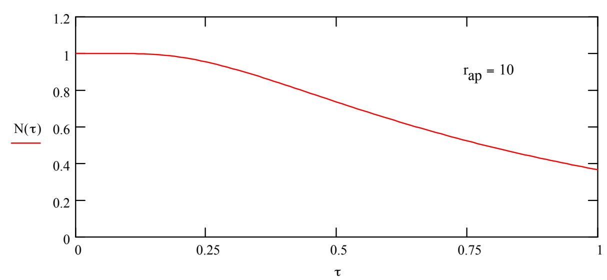
$$Z \equiv \epsilon_p/W$$

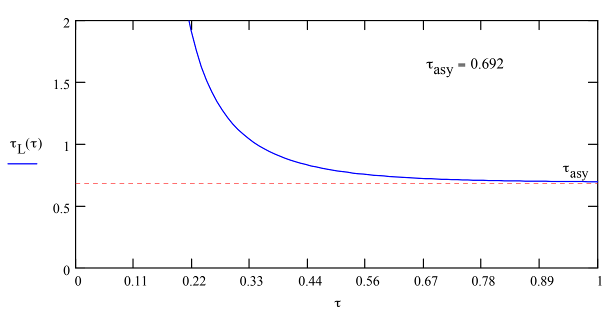
$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial Z}\left(\frac{\partial f}{\partial Z}\right)$$

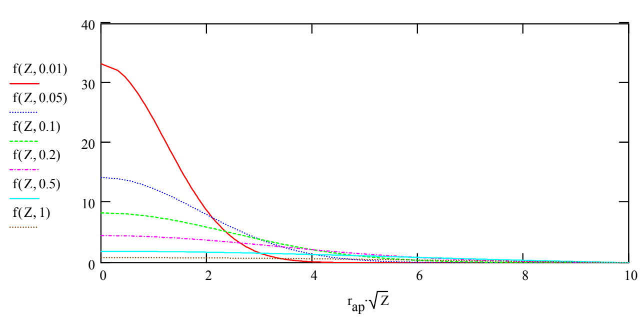
$$f(Z,0) = f_0(Z)$$
$$f(1,\tau) = 0$$

The Diffusion Equation

- Analytical Calculations:
  - solve
    and make plots:



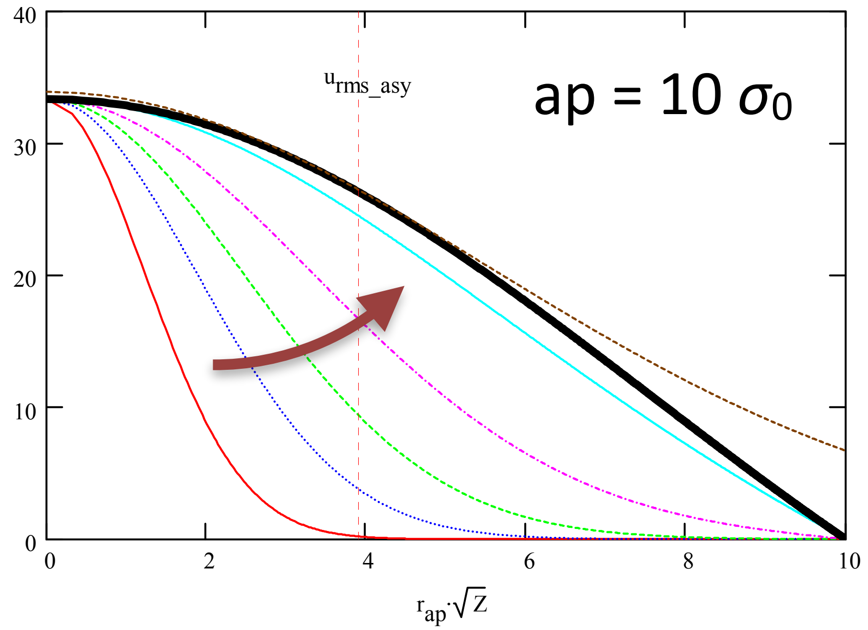


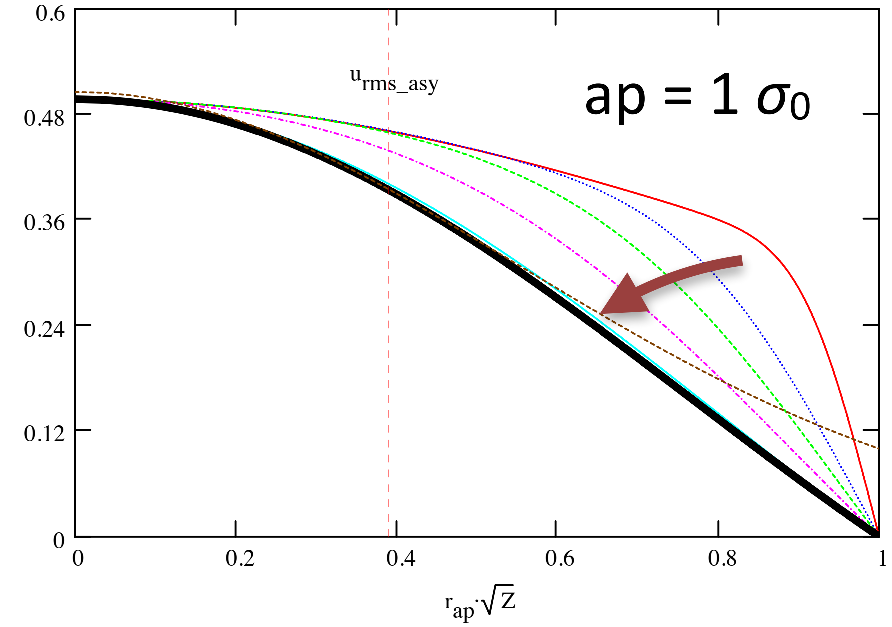


- Numerical Simulations
  - give particles random kicks over time, track in phase space, and plot distribution, etc.

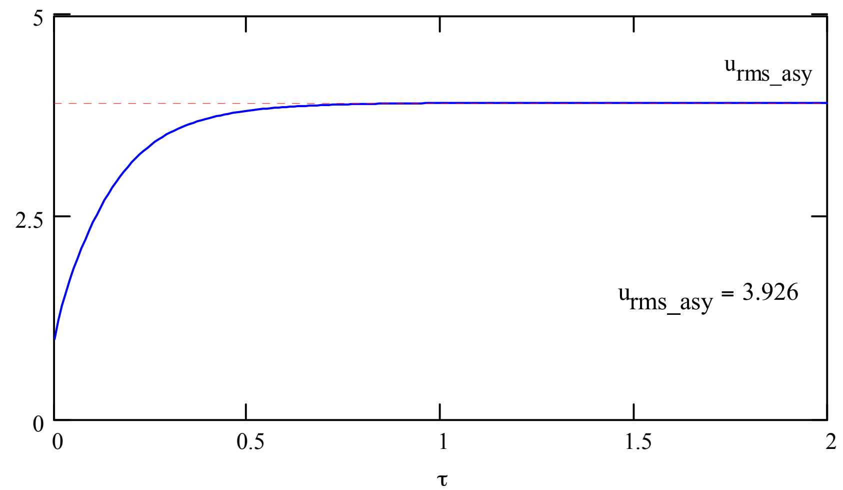
$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = M_{2\pi\nu} \begin{pmatrix} x_n \\ x'_n + \Delta\theta_n \end{pmatrix}$$

random number each time, determined by the process

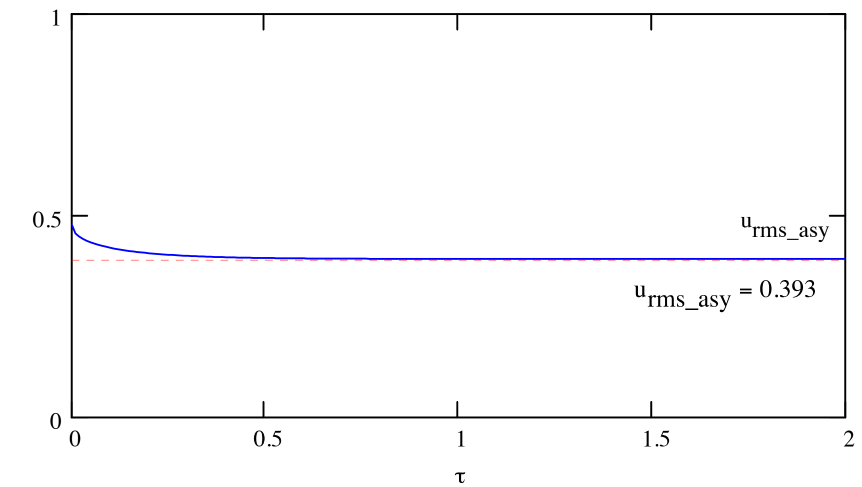




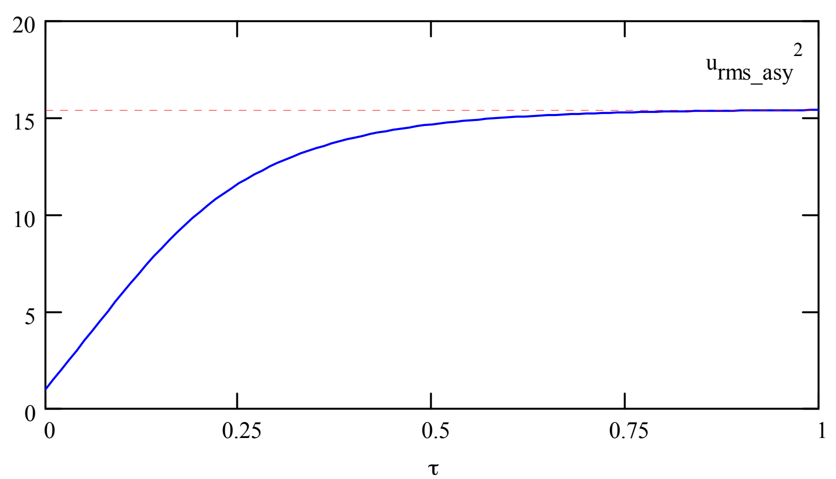
Beam Size



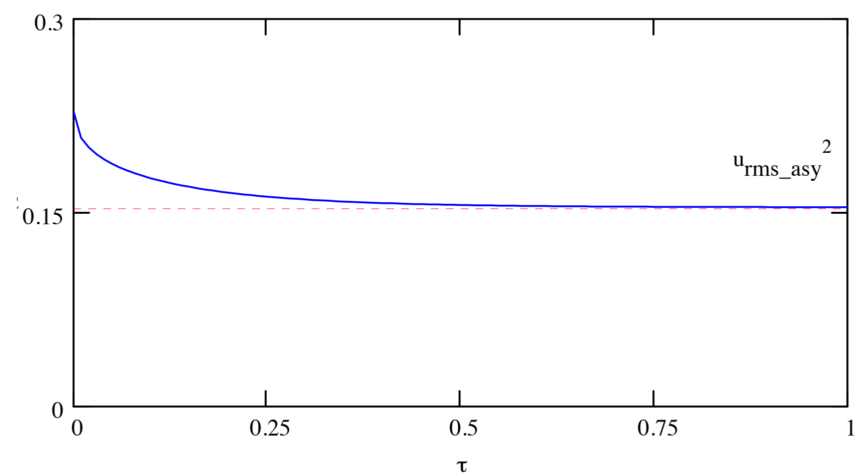
Beam Size



“Emittance”

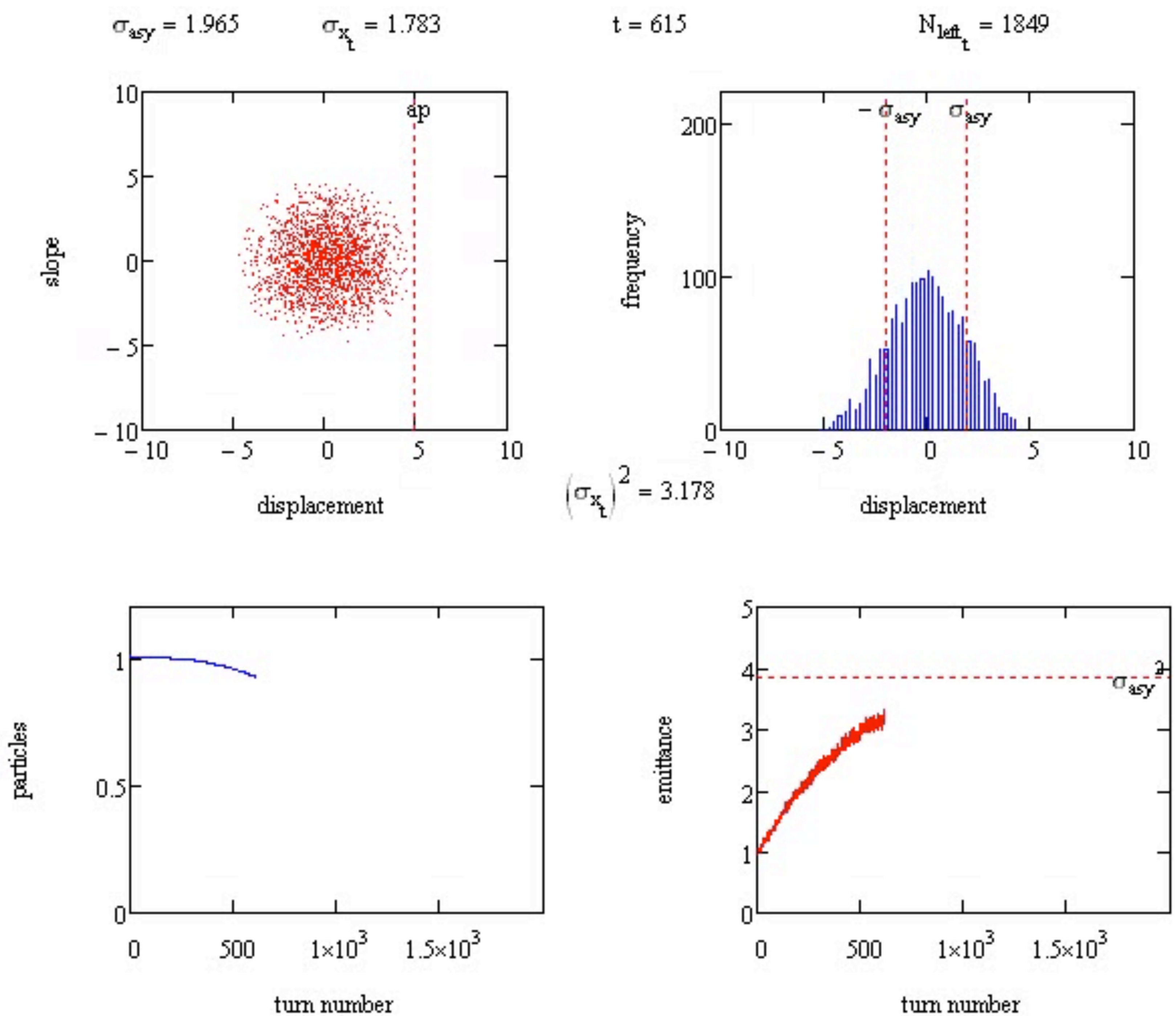


“Emittance”



Transverse Diffusion — Scattering

each particle gets a random "kick" in *x′* each turn, taken from a Gaussian distribution with rms value of θ_{rms}



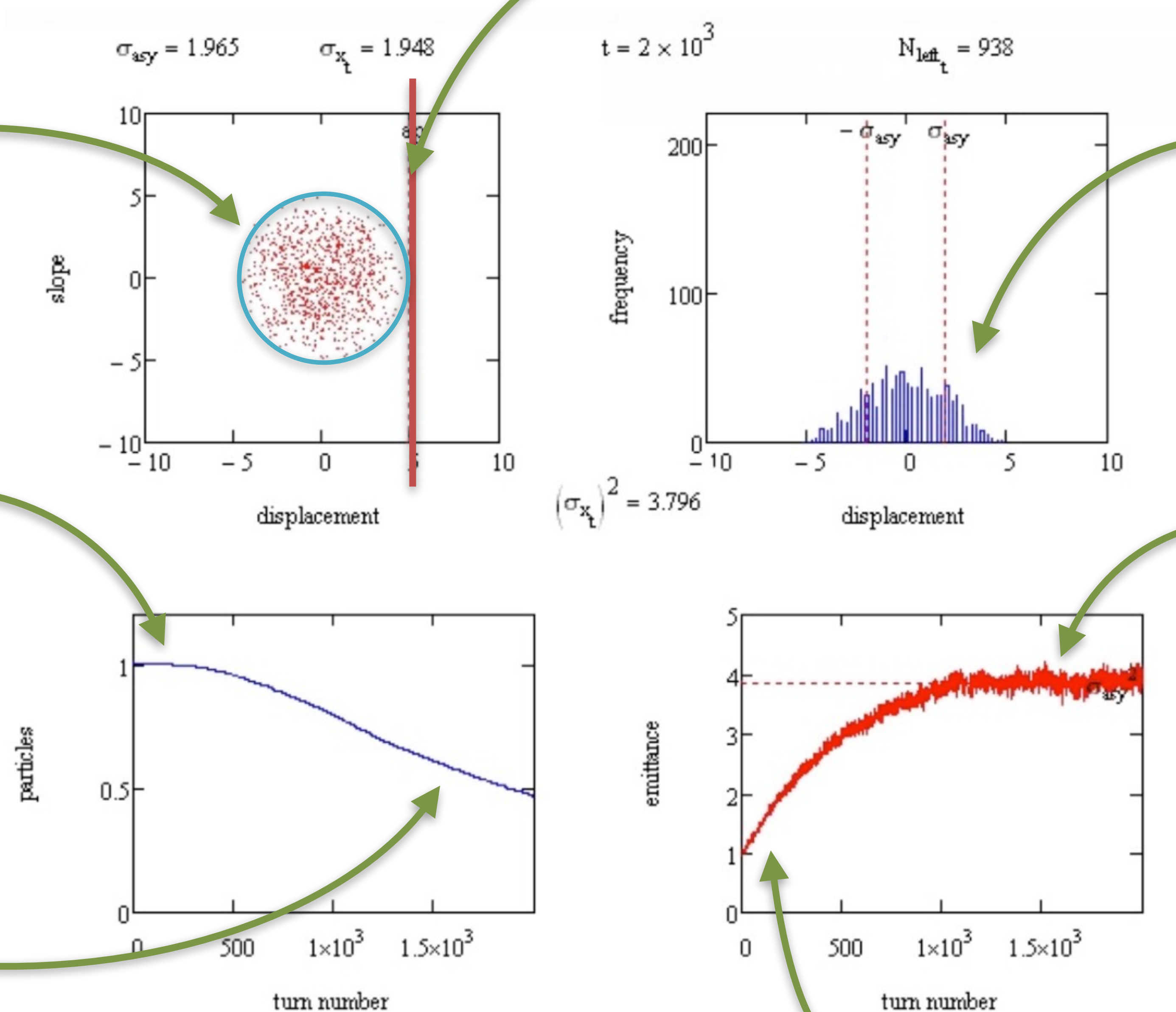
Transverse Diffusion

aperture at x = a

$W = \pi a^2/\beta$

final equilibrium distribution is well-defined

beam has not reached the aperture yet



no *apparent* emittance growth, but particle amplitudes are indeed growing, and particles are being lost

final $N(t) \sim e^{-t/\tau}$

$R = d\epsilon/dt$ = constant *(given by mechanism)*

$\tau = (4/2.405^2)(W/R)$

Some Comments

- The emittance may be growing, but the intensity will not decrease until the beam reaches an aperture
- The beam size may stop growing, but that does not mean that the individual particle amplitudes are no longer growing — just that the aperture was reached
- The long-term exponential decay of the beam intensity can tell you what the emittance growth rate is, if you know the transverse acceptance
- A beam with an initially more uniform distribution can actually have its “rms” value decrease until equilibrium is reached — it is NOT being “cooled”

Comments on “Beam Cooling”

- Stochastic Beam Cooling
- Electron Cooling of Hadron Beams
- Ionization Cooling

Example: Longitudinal Diffusion due to RF Noise

Here, a random phase error is given to each particle every turn