The Introduction of a Non-Linear Element

• For the first time in our discussion, have introduced a “non-linear” transverse magnetic field for explicit use in the accelerator system — sextuples for chromatic and/or chromaticity correction

• This opens the door to new and interesting phenomena:
  ‣ phase space distortions
  ‣ tune variation with amplitude
  ‣ dynamic aperture
Effect on Phase Space due to Single Sextupole

• Track the trajectory of a particle around an ideal ring, but include the kick from a single sextupole every revolution:

\[
\begin{pmatrix}
x \\
x'
\end{pmatrix}_{n+1} = \begin{pmatrix}
\cos \mu + \alpha \sin \mu & \beta \sin \mu \\
-\gamma \sin \mu & \cos \mu - \alpha \sin \mu
\end{pmatrix} \begin{pmatrix}
x' - Sx^2 \n\end{pmatrix}_{n}
\]

‣ transform to new coordinates: \( p \equiv \alpha x + \beta x' \)

\[
\begin{pmatrix}
x \\
p
\end{pmatrix}_{n+1} = \begin{pmatrix}
\cos \mu & \sin \mu \\
-\sin \mu & \cos \mu
\end{pmatrix} \begin{pmatrix}
x - Sx^2 \n\end{pmatrix}_{n}
\]

‣ transform again:

\[
\begin{pmatrix}
u \\
v
\end{pmatrix}_{n+1} = \begin{pmatrix}
\cos 2\pi \nu & \sin 2\pi \nu \\
-\sin 2\pi \nu & \cos 2\pi \nu
\end{pmatrix} \begin{pmatrix}
u - u^2 \n\end{pmatrix}_{n}
\]

• The topology of the phase space here only depends upon the choice of tune, \( \nu \). Let’s see what happens…
Sextupole Tracking Code Demonstration

```R
while (i < Nturns+1) {
  du1 <- du + u*u/2
  u1  <- a*u + b*du1
  du  <- c*u + d*du1 + u1*u1/2
  u    <- u1
  points(u, du, pch=".")
  i = i + 1
}
```

Let’s run a code…
Nonlinear Motion and Resonances

- Sources of nonlinear field perturbations
- Characteristics of nonlinear motion in phase space
- Longitudinal Motion
  - the Standard Map
- Transverse Motion
  - ex: sextupole field
  - the driven harmonic oscillator
- Resonant Extraction
- Nonlinear coupled motion
  - sum and difference resonances
  - Carpet Plot
Longitudinal Motion…

- Adiabatic (on scale of energy oscillation period) increase of the magnetic field moves the stable fixed points; particles continue to oscillate, follow along

Have already seen an example of nonlinear motion
Stability of Longitudinal Motion

- Since longitudinal motion is “slow”, can usually treat time as differential variable
- However, acceleration happens at a “point” (or limited number of points) in the synchrotron; perhaps more accurate to treat as a “map”:

\[
\begin{align*}
\Delta E_{n+1} &= \Delta E_n + eV (\sin \omega_{rf} \Delta t_n - \sin \phi_s) \\
\Delta t_{n+1} &= \Delta t_n + k \Delta E_{n+1}
\end{align*}
\]

- Essentially the “Standard Map” (when \( \phi_s = 0 \))
  - (or Chirikov-Taylor map, or Chirikov standard map)

\[
\begin{align*}
p_{n+1} &= p_n - K \sin \theta_n \\
\theta_{n+1} &= \theta_n + p_{n+1}
\end{align*}
\]
Phase Space of the Standard Map

Each view uses the same initial conditions for 27 particles.

Typical synchrotrons:
\[ K \sim 0.0001 - 0.1 \]

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Typical synchrotrons:

$$K \sim 0.0001 - 0.1$$

\[
\begin{align*}
\frac{\theta_{n+1}}{\theta_n} &= 1 + \frac{p_n}{\sigma_n} \\
\frac{p_{n+1}}{p_n} &= 1 + K \sin \theta_n
\end{align*}
\]
Phase Space of the Standard Map

Each view uses the same initial conditions for 27 particles

Typical synchrotrons: $K \sim 0.0001 - 0.1$

\[
\begin{align*}
    p_{n+1} &= p_n - K \sin \theta_n \\
    \theta_{n+1} &= \theta_n + p_{n+1}
\end{align*}
\]
Phase Space of the Standard Map

Each view uses the same initial conditions for 27 particles

Typical synchrotrons:
\[ K \approx 0.0001 - 0.1 \]

\[ p_{n+1} = p_n - K \sin \theta_n \]
\[ \theta_{n+1} = \theta_n + p_{n+1} \]
Sources of Transverse Nonlinearities

- Real accelerator magnets
  - Finite width of the field region in a dipole magnet produces a 6-pole (sextupole) term \( \sim B_{y(y=0)} \sim x^2 \)
  - Real magnets also have:
    - Systematic construction errors
    - Random construction errors
    - Eddy currents in vacuum
    - Chambers as fields ramp up

- So, real life will introduce sources of linear AND nonlinear field perturbations which can affect the region of stable phase space ....
Linear Resonances in Circular Accelerators

- Imperfections of the ideal "linear elements" lead to implications of the motion
  - guide-field errors
    - the 'closed' trajectory about the synchrotron will become distorted -- average beam trajectory must be adjusted using small, corrector magnets
  - focusing field errors
    - distortions of the beam envelope
    - if too many, can have $|\text{ltr}M| > 2 \implies$ entire accelerator is unstable

![Graph showing ideal and distorted trajectories](image)
Resonances and Tune Space

• Error fields are encountered repeatedly each revolution -- thus, can be resonant with the transverse oscillation frequency

• Let the "tune" $\nu = \text{no. of oscillations per revolution}$
  - repeated encounter with a steering (dipole) error produces an orbit distortion:
    - thus, avoid integer tunes

  \[ \Delta x \sim \frac{1}{\sin \pi \nu} \]

• repeated encounter with a focusing error produces distortion of amplitude function, $\beta$:
  - thus, avoid half-integer tunes

  \[ \Delta \beta/\beta \sim \frac{1}{\sin 2\pi \nu} \]
Half-Integer Stop Band

\[
T_{\text{M}}(\nu_0) = \frac{1}{2} \pi \beta_0 q \sin 2\pi \nu_0 - \left( \frac{1}{8} \pi \beta_0 q \sin 2\pi (\nu_0 + \nu_0 - \nu_0) \right)
\]

\[
\nu(\nu_0) = \frac{1}{2} \pi \beta_0 q \sin 2\pi (1 - \nu_0) - \frac{1}{8} \pi \beta_0 q \sin 2\pi (\nu_0 + 1 - \nu_0)
\]

\[
\nu(\nu_0) = 8.25, 8.3, 8.35, 8.4, 8.45, 8.5, 8.55, 8.6, 8.65, 8.7, 8.75
\]

\[
\nu_0 = \frac{1}{2} \pi \beta_0 q \sin 2\pi (\nu_0 + 1 - \nu_0)
\]

\[
\nu_0 = \frac{1}{2} \pi \beta_0 q \sin 2\pi (\nu_0 + 1 - \nu_0)
\]
Nonlinear Resonances

- Phase space w/ sextupole field present ($\sim x^2$)
  - topology is tune dependent:
  - frequency depends upon amplitude
  - “dynamic aperture”

- With sextupole field present, must avoid tunes:
  - integer, integer/2, integer/3, ...

“normalized” phase space; ideal trajectories are circular
An Application

• Put the transverse nonlinear fields to work for us
• Can pulse an electromagnet to send the particles out of the accelerator all at once; but Particle Physics experiments often desire smooth flow of particles from the accelerator toward their detectors

• Resonant Extraction
  ‣ developed in 1960’s, particles can be put “on resonance” in a controlled manner and slowly extracted
  ‣ third-integer: carefully approach \( v = k/3 \)
    - driven by sextupole fields
  ‣ half-integer: carefully approach \( v = k/2 \)
    - driven by quadrupole and octupole (8-pole) fields
Phase Space used for Extraction

- Linear restoring forces with Sextupole perturbation, running near a tune of $k/3$
  
  $k = \text{"integer"}$

- Linear restoring forces with Octupole (8-pole) and quadrupole perturbations, running near a tune of $k/2$
Third-integer Extraction

- Example: particles oscillate in phase space in presence of a single sextupole
  - (here, \( k=1 \))
- Slowly adjust the tune toward a value of \( k/3 \)
- Tune is exactly \( 1/3 \) at the separatrix
- The lines that appear are derived from a first-order perturbation calculation
- Particles stream away from the “unstable fixed points”, stepping across a “septum” which leads out of the accelerator

\[
\begin{align*}
\nu_t &= 0.35 \\
\nu_t - \frac{1}{3} &= 0.017 \\
8\pi \delta \nu_t &= 0.416
\end{align*}
\]
Half-integer Extraction

- Similar to last movie, but “ideal” accelerator has extra quadrupole and octupole (8-pole) fields
  - Slowly adjust the tune toward a value of $k/2$
    - (here, $k=1$)
- Here, lowest-order separatrices defined by two intersecting circles
- Eventually, when very close to half-integer tune, entire phase space becomes unstable ($|\text{tr}M|>2$)
Coupling Resonances

- We’ve seen that coupling produces conditions where the motion in one plane \((x)\) can depend upon the motion in the other plane \((y)\) and *vice versa*. When the frequencies of the coupled motion create integer relationships, then coupling resonances can occur:

\[ m \nu_x \pm n \nu_y = k \]

- In general, a “difference” resonance will simply exchange the energy between the two planes, back and forth, but the motion remains bounded.

- A “sum” resonance will exchange energy, but the overall motion can become unbounded.
Coupling Resonances

- Always “error fields” in the real accelerator
- “Skew” fields can couple the motion between the two transverse degrees of freedom
  - thus, can also generate coupling resonances
    - (sum/difference resonances)

- in general, should avoid: \( m \nu_x \pm n \nu_y = k \)

avoid ALL rational tunes???
Tune Diagram

Through order $k = 2$

Lines of $m \nu_x \pm n \nu_y = k$
Tune Diagram

Through order
k = 3

lines of $m \nu_x \pm n \nu_y = k$
Tune Diagram

Through order
$k = 5$

lines of $m \nu_x \pm n \nu_y = k$
Tune Diagram

Through order

$k = 8$

lines of $m \nu_x \pm n \nu_y = k$
Tune Diagram

lines of \( m \nu_x \pm n \nu_y = k \),

Note:
In Tevatron, ~20 oscillations per revolution

width \( \sim 0.025 \)
Phase Space Diffusion

- Overlapping resonance conditions can lead to particle diffusion
- Here, simulation of large initial tune distribution, due to high intensity beam

```
initial distribution of tunes
```

```
final tunes when particles are “lost”
```
“Tune Scan”

- **Data** showing particle survival in a storage ring for various settings of the focusing magnets used for tune adjustment.
- Note the appearance of the resonance lines...

![Image of a graph or diagram demonstrating tune scan results, showing survival after n revolutions.](image-url)

*Courtesy of T. Koeth, et al., Univ. Maryland*
Large Accelerator Design

- Stability of Particle Motion in Large Colliders
  - Particles in the Tevatron were stored for well over 24 hours
  - at $C = 6 \text{ km}$ and $v = c$, this is ...
    - $24 \cdot 3600 \cdot (3 \times 10^8 \text{ m/s})/(6 \text{ km}) = 4 \times 10^9$ revolutions
    - $\sim \# \text{ times the earth has gone around the sun!}$

- Typically, exact sources of nonlinearities -- at the level that determines the long-term behavior -- are not well known; fields typically known at the level of a few parts per $10^4$

- To predict long-term behavior of particle motion through these large machines, must perform simulations assuming various levels of field imperfections
Dynamic Aperture and Design Criteria

- Computations of dynamic aperture began in earnest during the Tevatron design studies
- SSC Design Study, LHC design study:
  - Generate a model of the accelerator with error fields
    - random, systematic
  - “Track” particles with various initial oscillation amplitudes; record survival times

![Graph showing the dynamic aperture and survival plot for an SSC model.](image-url)