# The Introduction of a Non-Linear Element

- For the first time in our discussion, have introduced a "non-linear" transverse magnetic field for explicit use in the accelerator system — sextuples for chromatic and/or chromaticity correction
- This opens the door to new and interesting phenomena:
  - phase space distortions
  - tune variation with amplitude
  - dynamic aperture

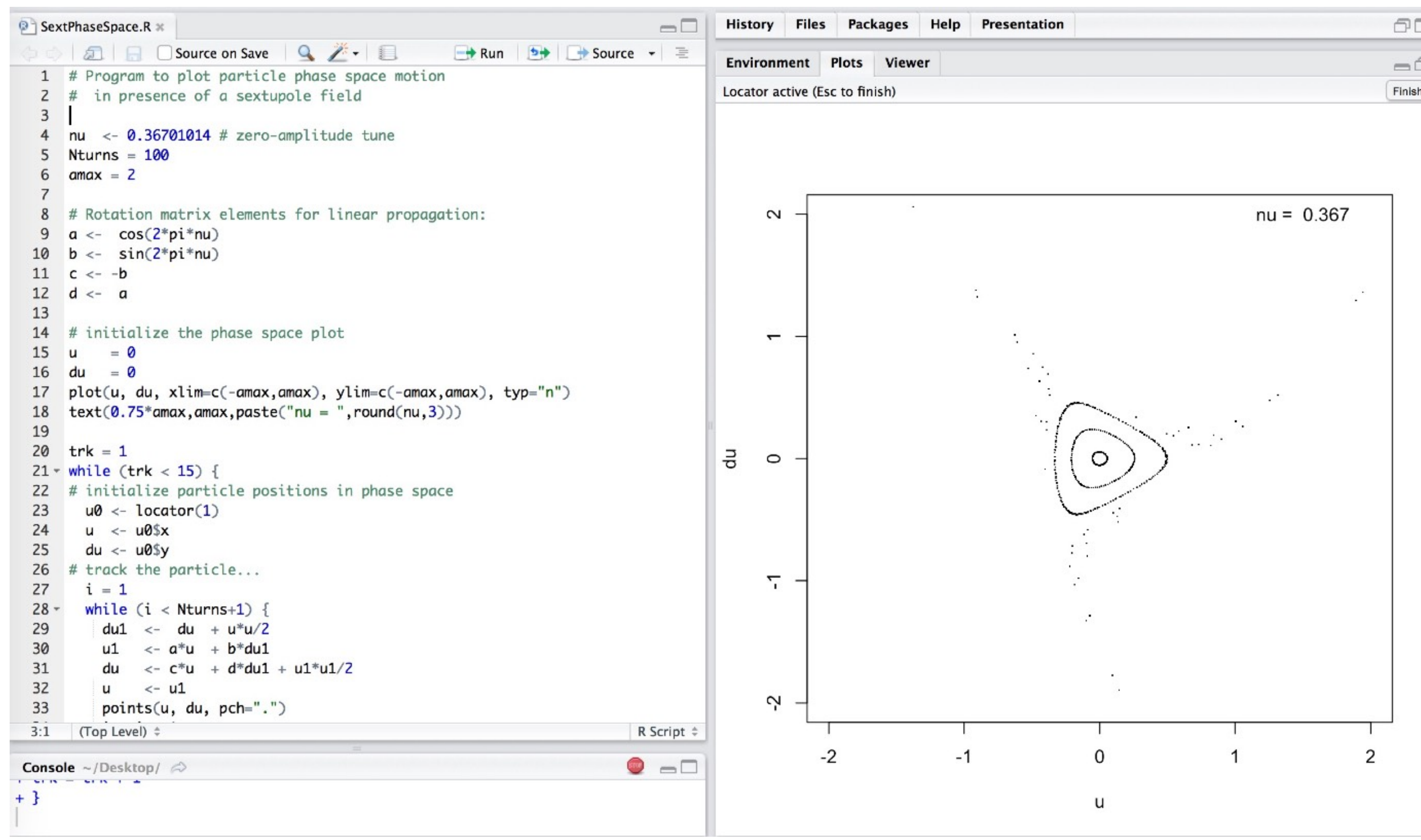
# Effect on Phase Space due to Single Sextuple

- Track the trajectory of a particle around an ideal ring, but include the kick from a single sextupole every revolution: $$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix} \begin{pmatrix} x \\ x' - Sx^2 \end{pmatrix}_n$$
  - transform to new coordinates: $p \equiv \alpha x + \beta x'$
    $$\begin{pmatrix} x \\ p \end{pmatrix}_{n+1} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} x \\ p - \beta S x^2 \end{pmatrix}_n$$
  - transform again:
    $u \equiv \beta S x, \qquad v \equiv \beta S p$
    $$\boxed{\begin{pmatrix} u \\ v \end{pmatrix}_{n+1} = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix} \begin{pmatrix} u \\ v - u^2 \end{pmatrix}_n}$$
- The topology of the phase space here only depends upon the choice of tune, $\nu$. Let's see what happens…

# Sextupole Tracking Code Demonstration

```
while (i < Nturns+1) {
  du1   <-  du  + u*u/2
   u1   <-  a*u  + b*du1
   du   <-  c*u  + d*du1 + u1*u1/2
    u   <-  u1
  points(u, du, pch=".")
  i = i + 1
}
```

Let's run a code…



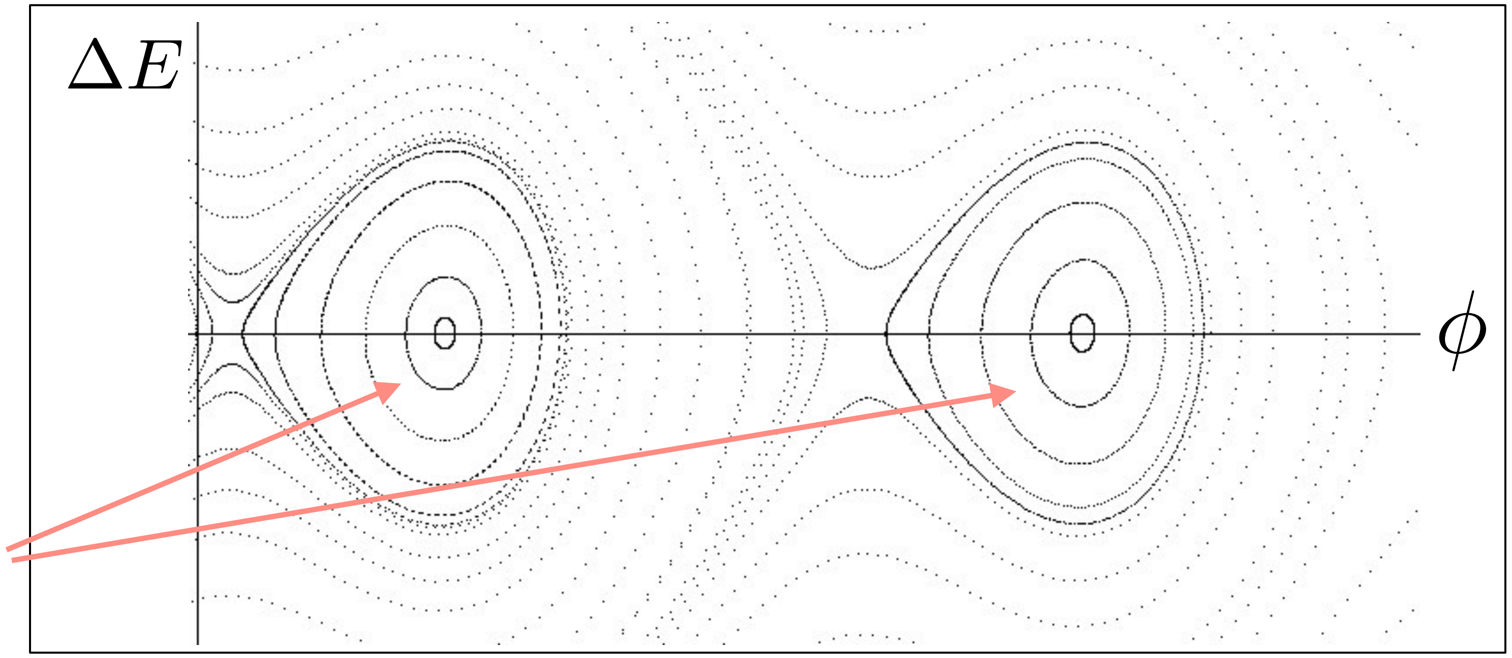
# Nonlinear Motion and Resonances

- Sources of nonlinear field perturbations
- Characteristics of nonlinear motion in phase space
- Longitudinal Motion
  - the Standard Map
- Transverse Motion
  - ex:  sextupole field
  - the driven harmonic oscillator
- Resonant Extraction
- Nonlinear coupled motion
  - sum and difference resonances
  - Carpet Plot

# Longitudinal Motion…

- Adiabatic (on scale of energy oscillation period) increase of the magnetic field moves the stable fixed points; particles continue to oscillate, follow along

*Have already seen an example of nonlinear motion*




*Stable Phase Space regions*

# Stability of Longitudinal Motion

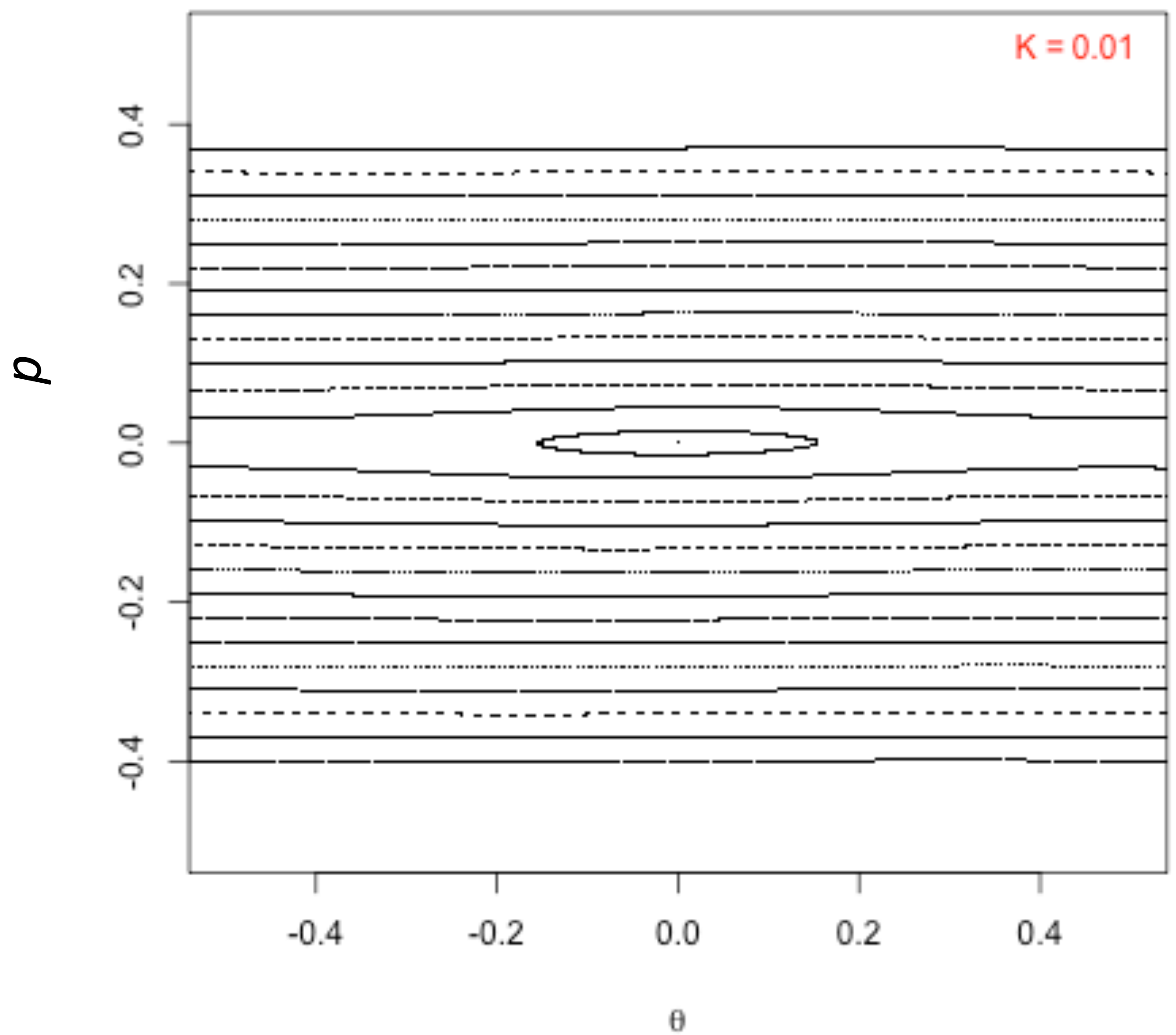
- Since longitudinal motion is "slow", can usually treat time as differential variable
- However, acceleration happens at a "point" (or limited number of points) in the synchrotron; perhaps more accurate to treat as a "map":

$$\begin{aligned} \Delta E_{n+1} &= \Delta E_n + eV(\sin\omega_{\mathrm{rf}}\Delta t_n - \sin\phi_s) \\ \Delta t_{n+1} &= \Delta t_n + k\,\Delta E_{n+1} \end{aligned}$$

- Essentially the "Standard Map" (when $\phi_s = 0$)
  - (or Chirikov-Taylor map, or Chirikov standard map)

$$\begin{aligned} p_{n+1} &= p_n - K\sin\theta_n \\ \theta_{n+1} &= \theta_n + p_{n+1} \end{aligned}$$

# Phase Space of the Standard Map




Each view uses the same initial conditions for 27 particles

Typical synchrotrons:

*K* ~ 0.0001 - 0.1

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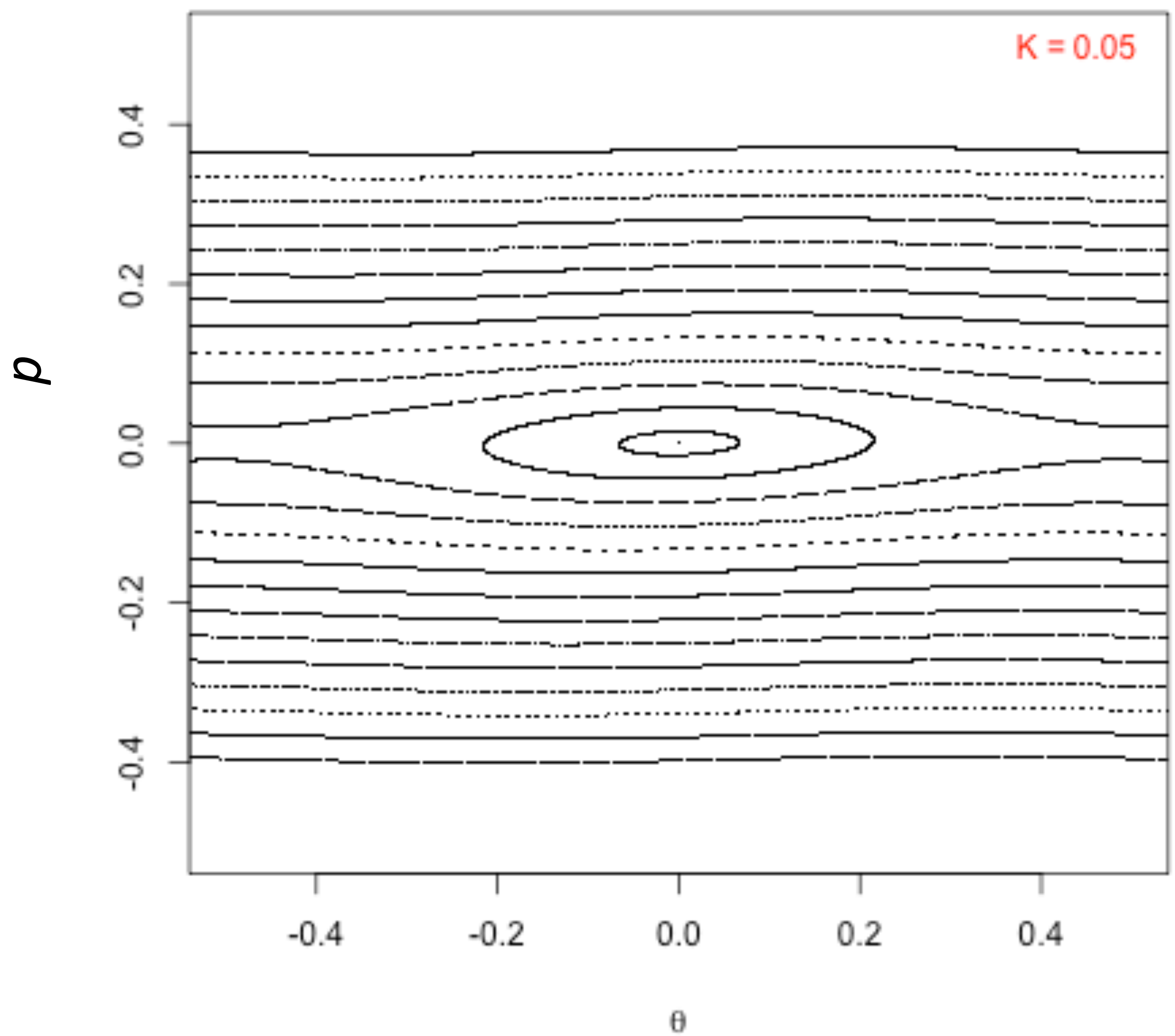


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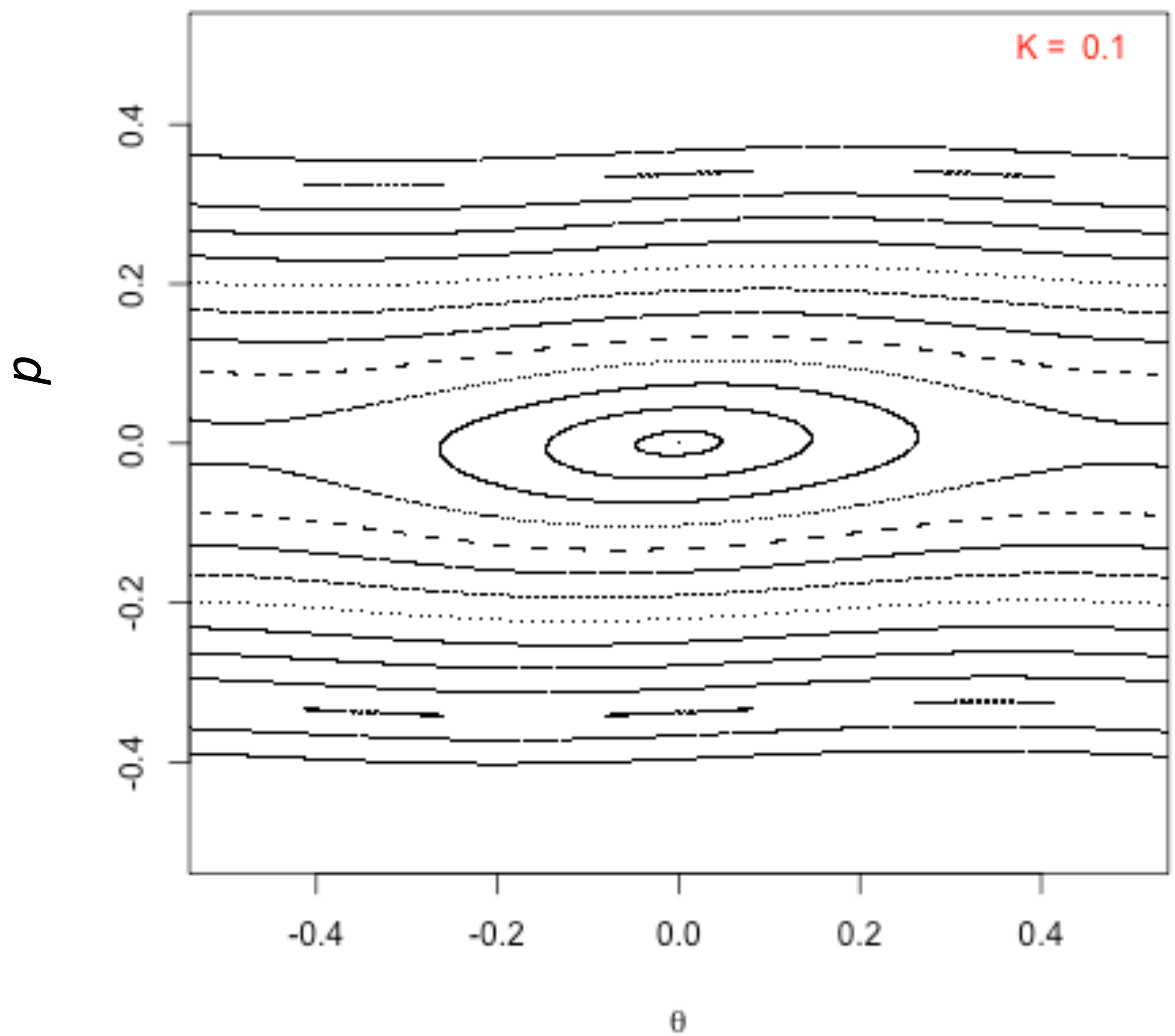
# Phase Space of the Standard Map

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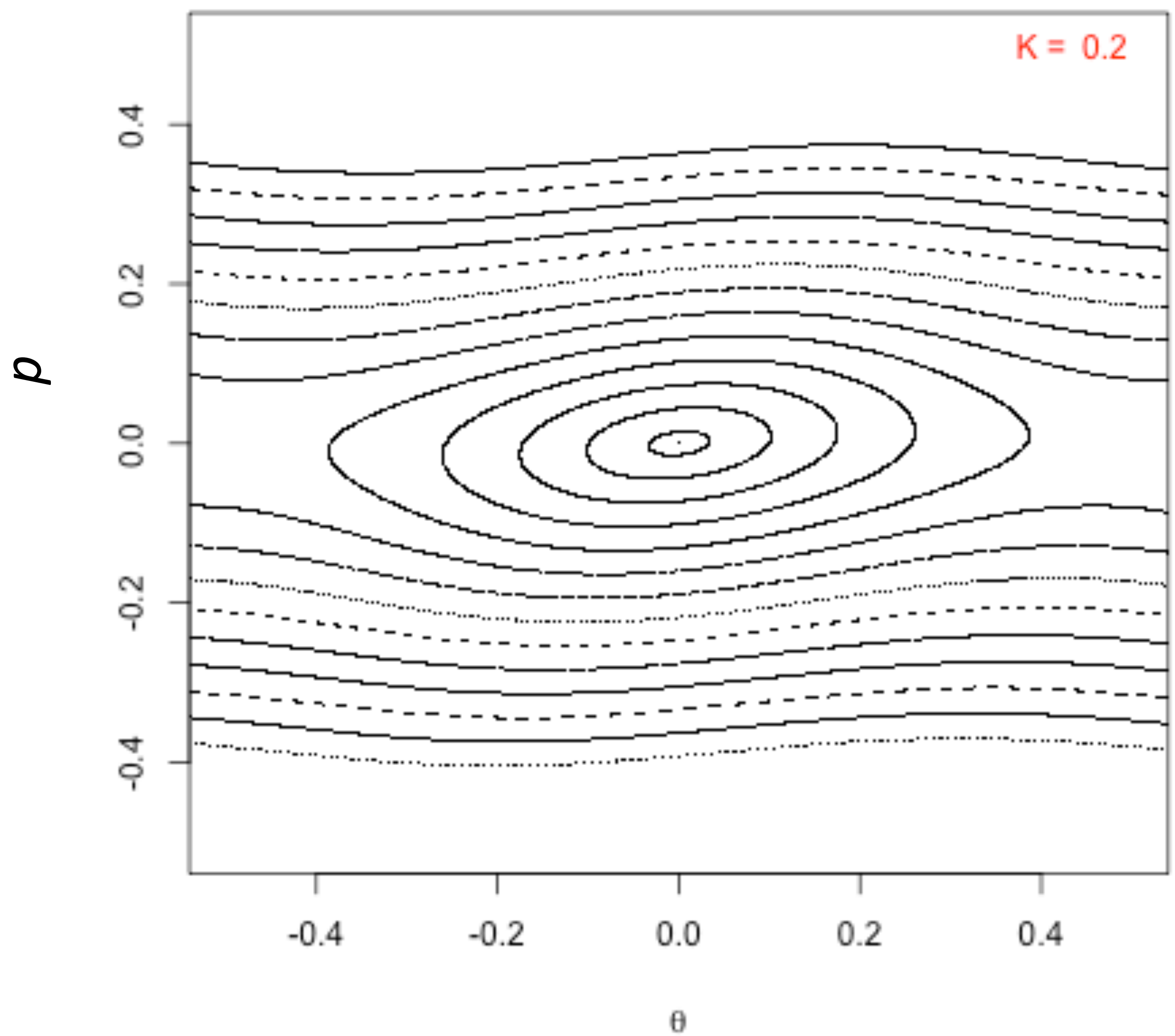


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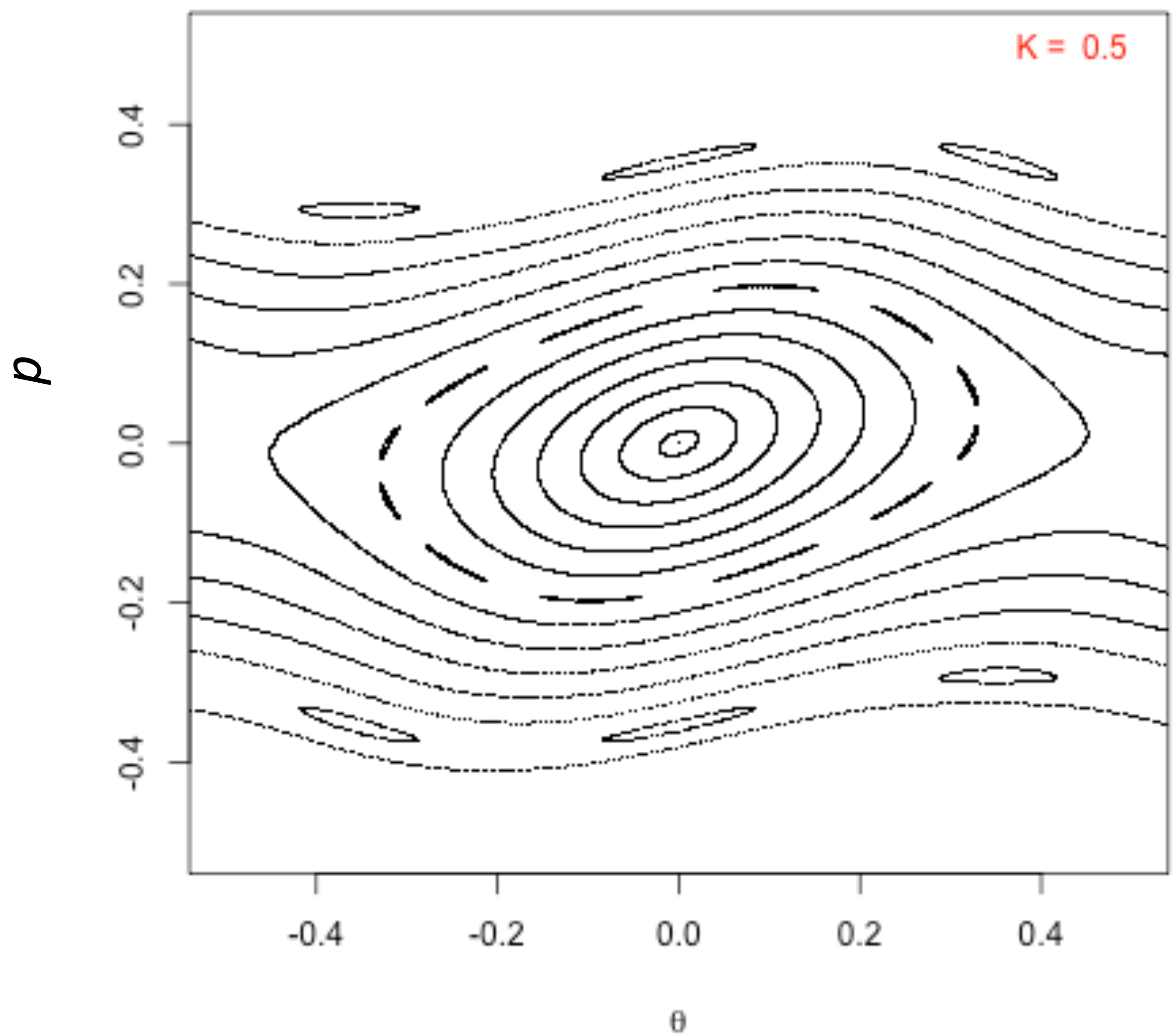


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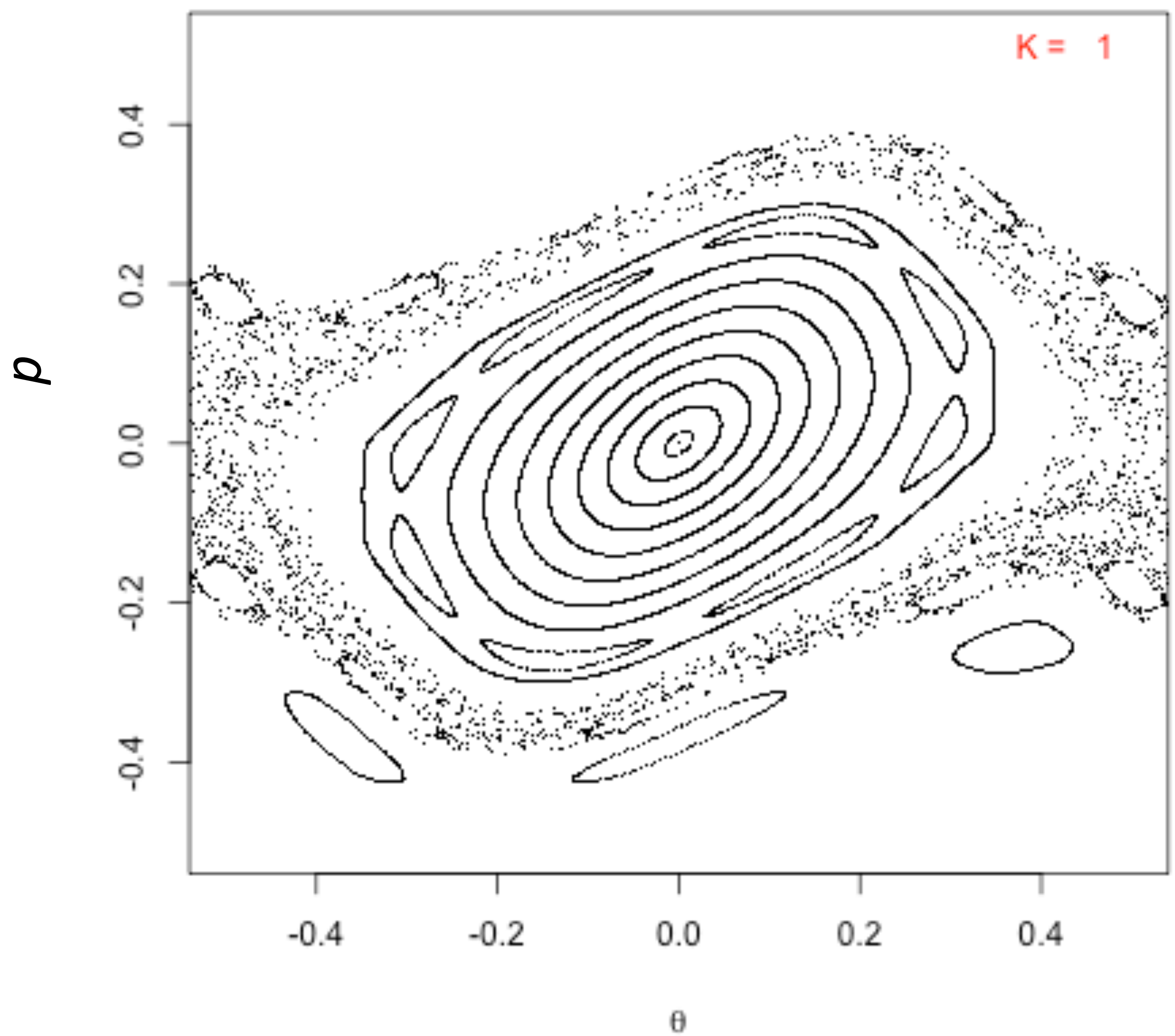


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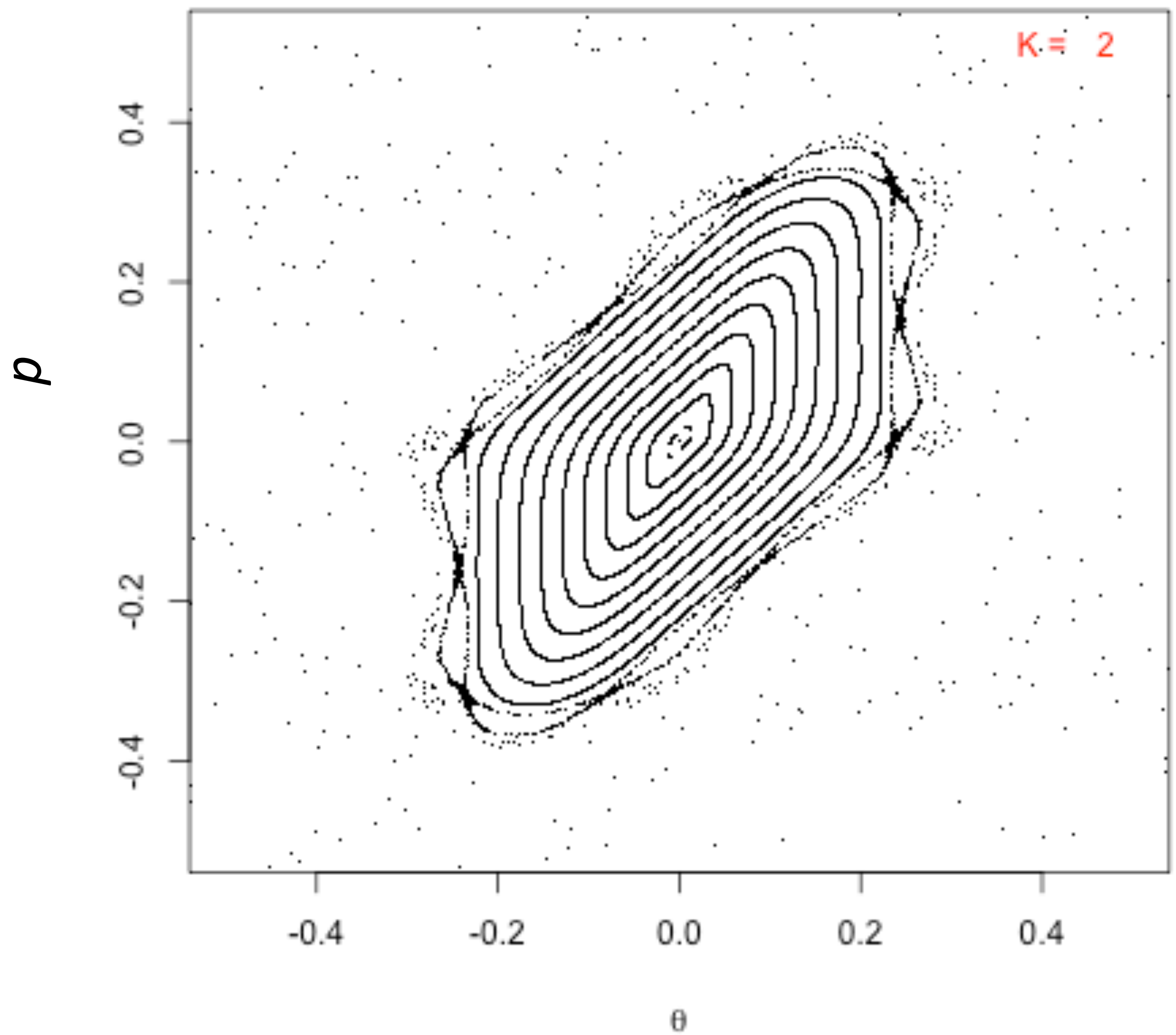


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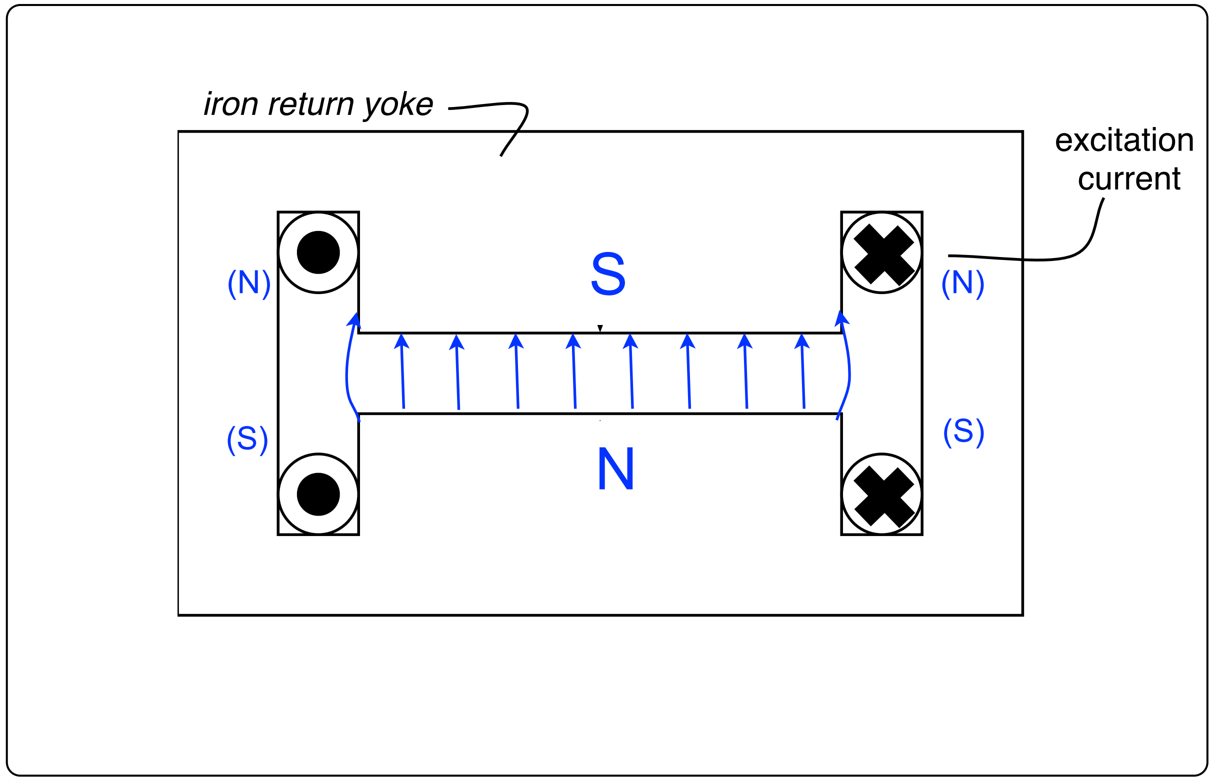
# Sources of Transverse Nonlinearities

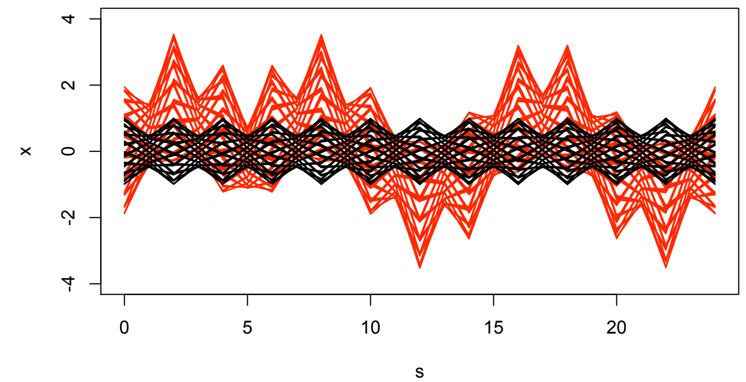
- *Real* accelerator magnets
  - Finite width of the field region in a dipole magnet produces a 6-pole (sextupole) term -- $B_{y(y=0)} \sim x^2$
  - Real magnets also have:
    - Systematic construction errors
    - Random construction errors
    - Eddy currents in vacuum
    - chambers as fields ramp up




- So, real life will introduce sources of linear *AND* nonlinear field perturbations which can affect the region of stable phase space ....

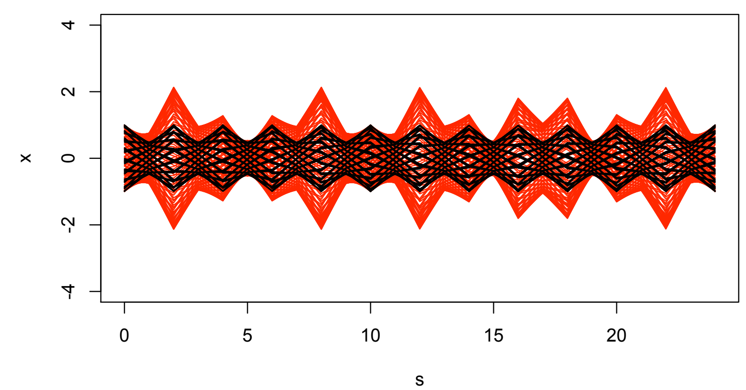
# Linear Resonances in Circular Accelerators

- Imperfections of the ideal "linear elements" lead to implications of the motion




black = ideal
red = distortion

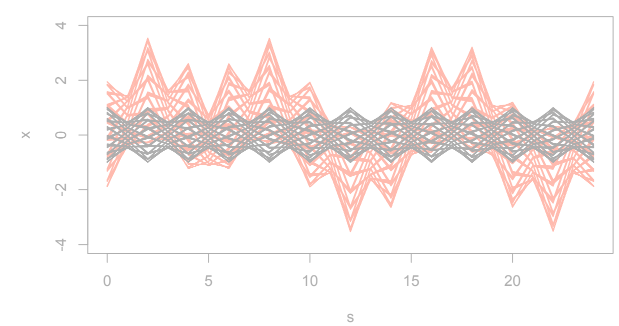
  - guide-field errors
    - the 'closed' trajectory about the synchrotron will become distorted -- average beam trajectory must be adjusted using small, corrector magnets




  - focusing field errors
    - distortions of the beam envelope
    - if too many, can have $|\mathrm{tr}M| > 2$ ==> entire accelerator is unstable

# Resonances and Tune Space

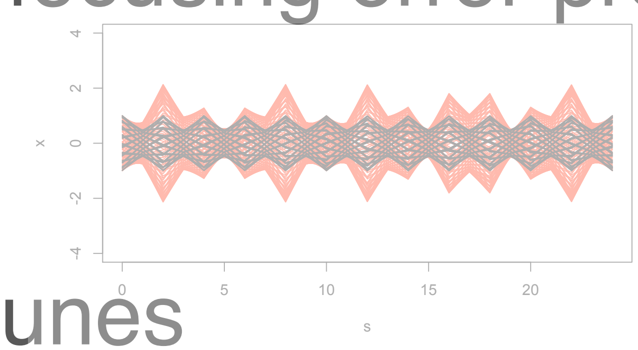
- Error fields are encountered repeatedly each revolution -- thus, can be resonant with the transverse oscillation frequency
- Let the "***tune***" $\nu$ = no. of oscillations per revolution
  - repeated encounter with a steering (dipole) error produces an orbit distortion:




$$\boxed{\Delta x \sim \frac{1}{\sin \pi\nu}}$$

    - thus, avoid integer tunes

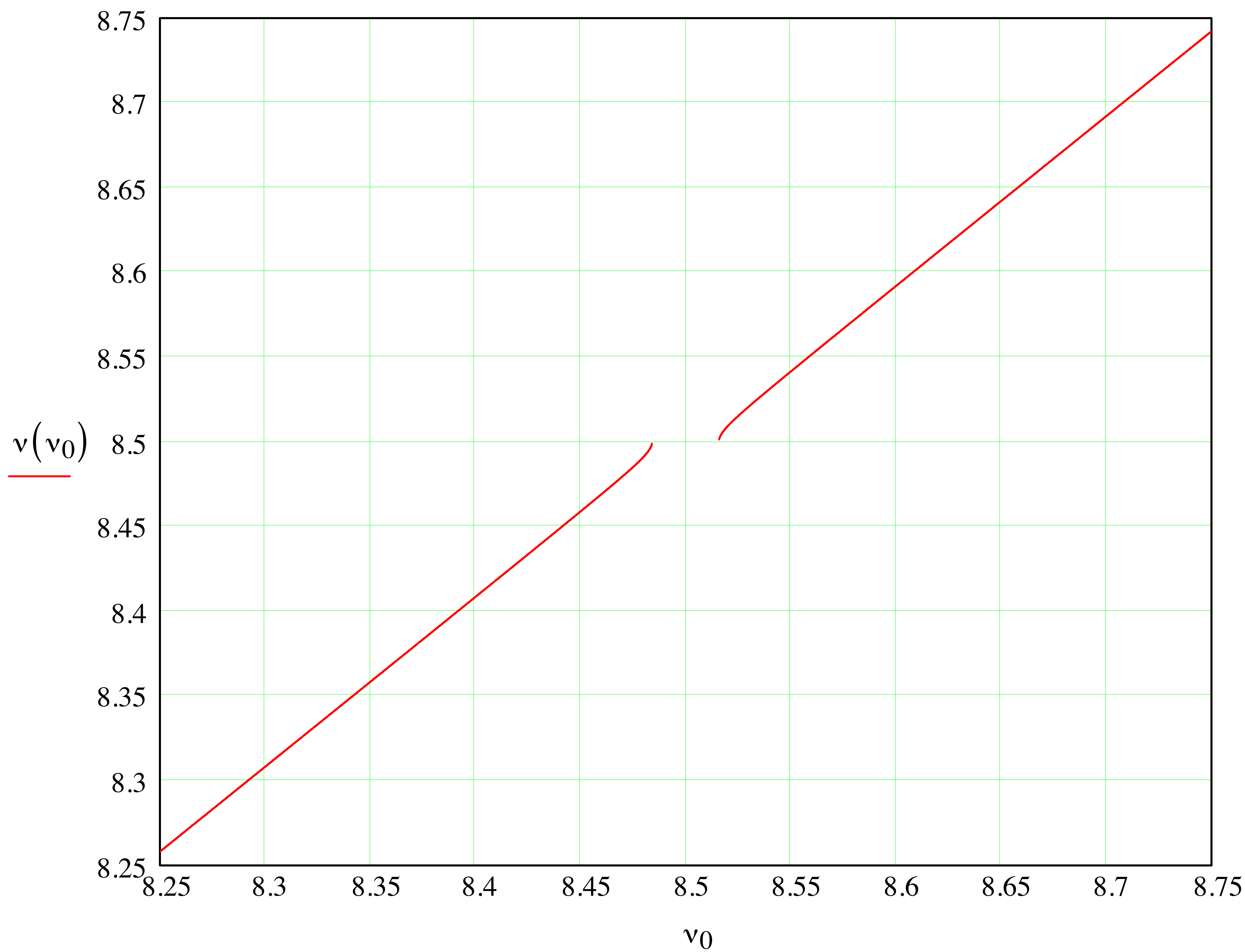
  - repeated encounter with a focusing error produces distortion of amplitude function, $\beta$:




$$\boxed{\Delta\beta/\beta \sim \frac{1}{\sin 2\pi\nu}}$$

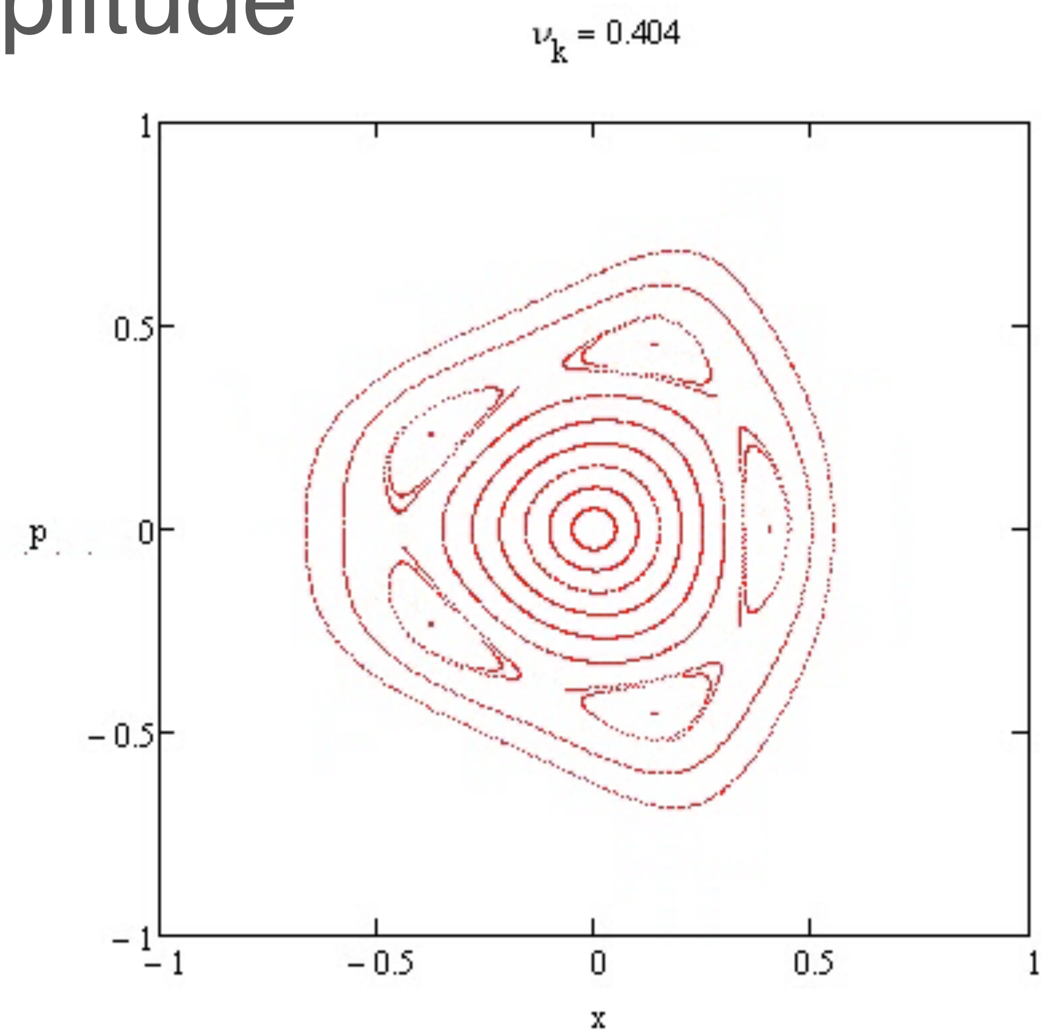
    - thus, avoid half-integer tunes

# Half-Integer Stop Band



# Nonlinear Resonances

- Phase space w/ sextupole field present (~$x^2$)
  - topology is tune dependent:
  - frequency depends upon amplitude
  - "dynamic aperture"




With sextupole field present, must avoid tunes:

- *integer*, *integer*/2, *integer*/3, ...

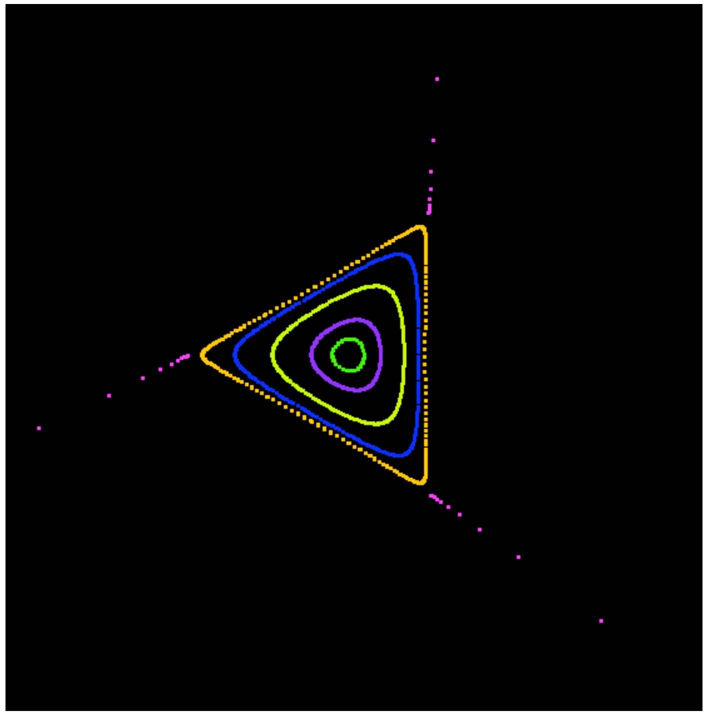
"normalized" phase space; ideal trajectories are circular

# An Application

- Put the transverse nonlinear fields to work for us
- Can pulse an electromagnet to send the particles out of the accelerator all at once; but Particle Physics experiments often desire smooth flow of particles from the accelerator toward their detectors
- Resonant Extraction
  - developed in 1960's, particles can be put "on resonance" in a controlled manner and slowly extracted
  - third-integer: carefully approach $\nu = k/3$
    - driven by sextupole fields
  - half-integer: carefully approach $\nu = k/2$
    - driven by quadrupole and octupole (8-pole) fields

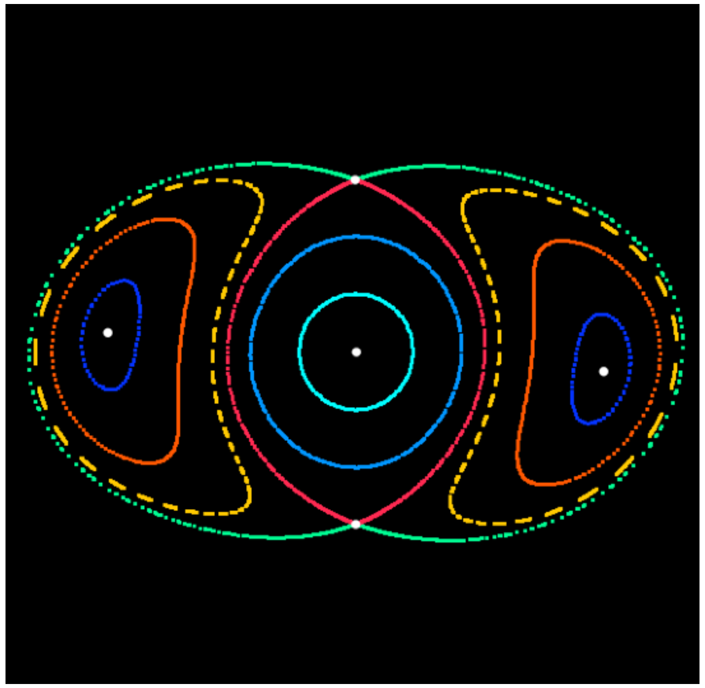
# Phase Space used for Extraction

- Linear restoring forces with Sextupole perturbation, running near a tune of *k*/3



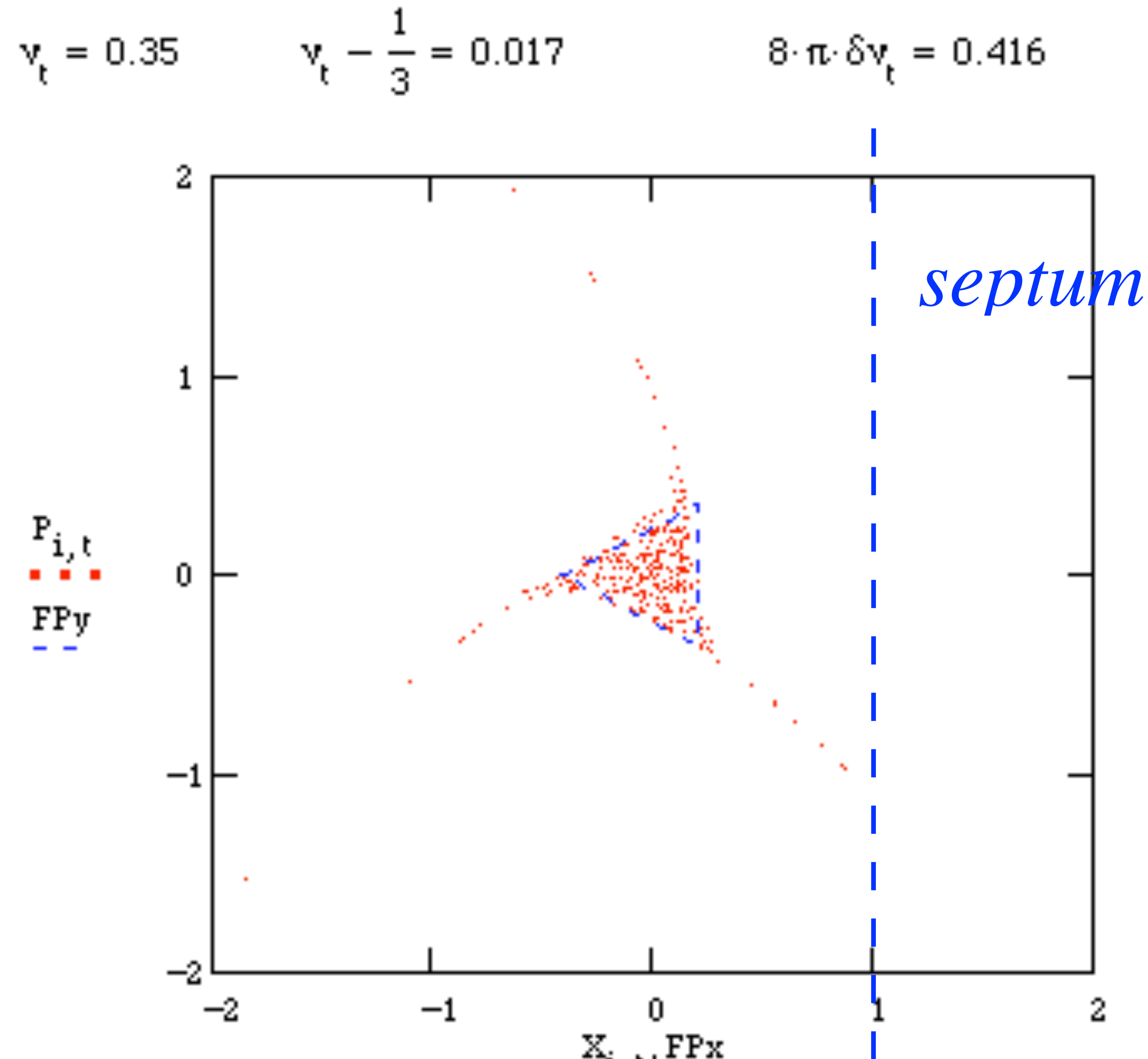
*k* = "integer"

- Linear restoring forces with Octupole (8-pole) and quadrupole perturbations, running near a tune of *k*/2



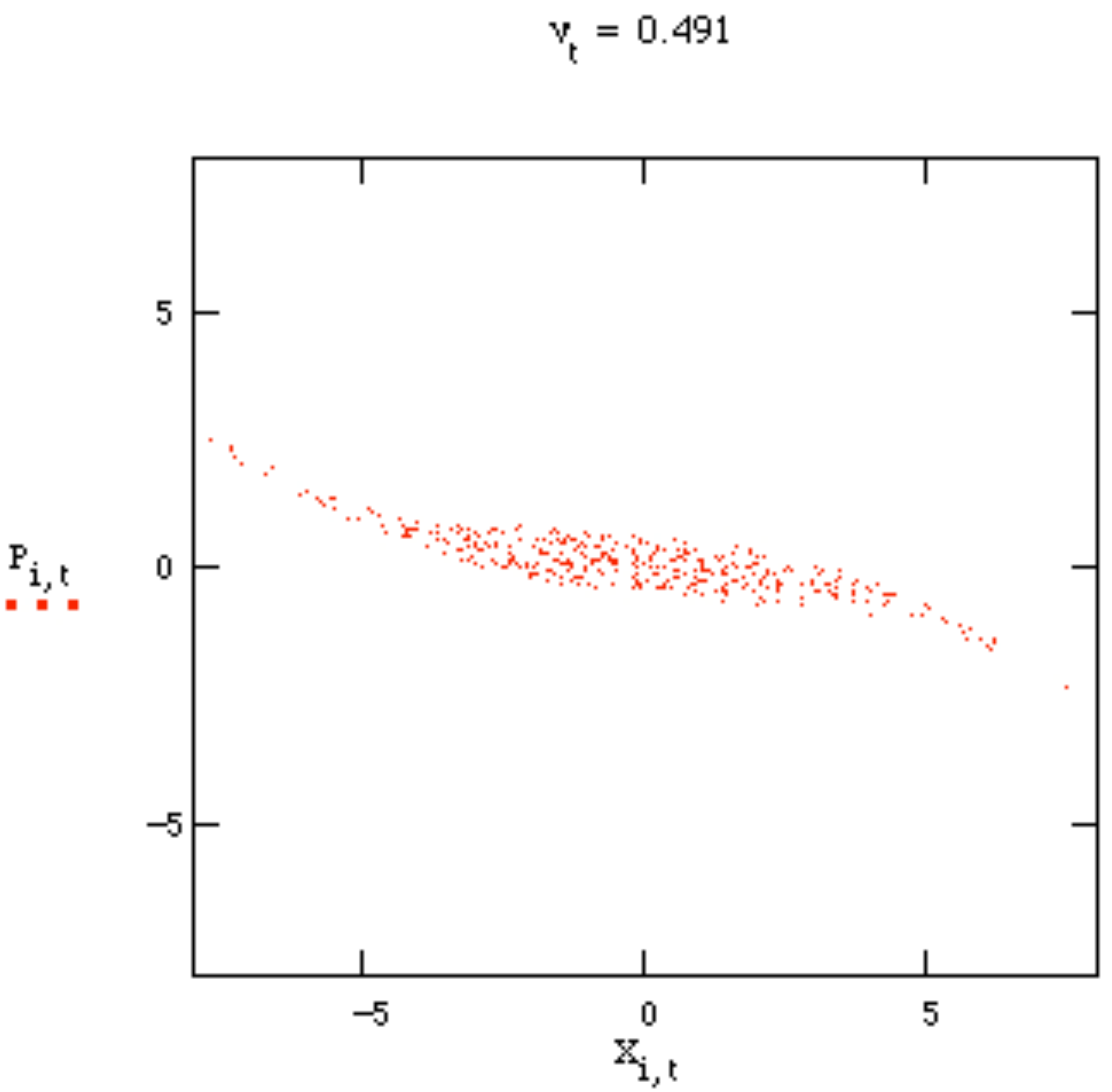
# Third-integer Extraction

- Example: particles oscillate in phase space in presence of a single sextupole
- Slowly adjust the tune toward a value of *k*/3
  - (here, *k*=1)
- Tune is exactly 1/3 *at* the separatrix
- The lines that appear are derived from a first-order perturbation calculation
- Particles stream away from the "unstable fixed points", stepping across a "septum" which leads out of the accelerator



# Half-integer Extraction

- Similar to last movie, but "ideal" accelerator has extra quadrupole and octupole (8-pole) fields
- Slowly adjust the tune toward a value of *k*/2
  - (here, *k*=1)
- Here, lowest-order separatrices defined by two intersecting circles
- Eventually, when very close to half-integer tune, entire phase space becomes unstable (|trM|>2)

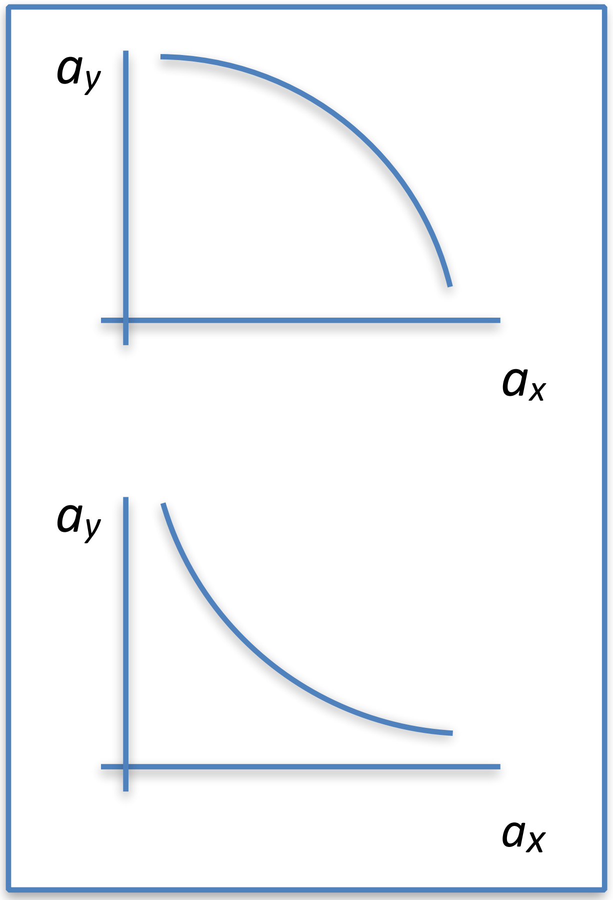


# Coupling Resonances

- We've seen that coupling produces conditions where the motion in one plane (x) can depend upon the motion in the other plane (y) and *vice versa*. When the frequencies of the coupled motion create integer relationships, then coupling resonances can occur:

$$\boxed{m\,\nu_x \pm n\,\nu_y = k}$$

  - In general, a "difference" resonance will simply exchange the energy between the two planes, back and forth, but the motion remains bounded
  - A "sum" resonance will exchange energy, but the overall motion can become unbounded



# Coupling Resonances

- Always “error fields” in the real accelerator
- “Skew” fields can couple the motion between the two transverse degrees of freedom
  - thus, can also generate coupling resonances
    - (sum/difference resonances)
  - in general, should avoid: $\boxed{m\,\nu_x \pm n\,\nu_y = k}$

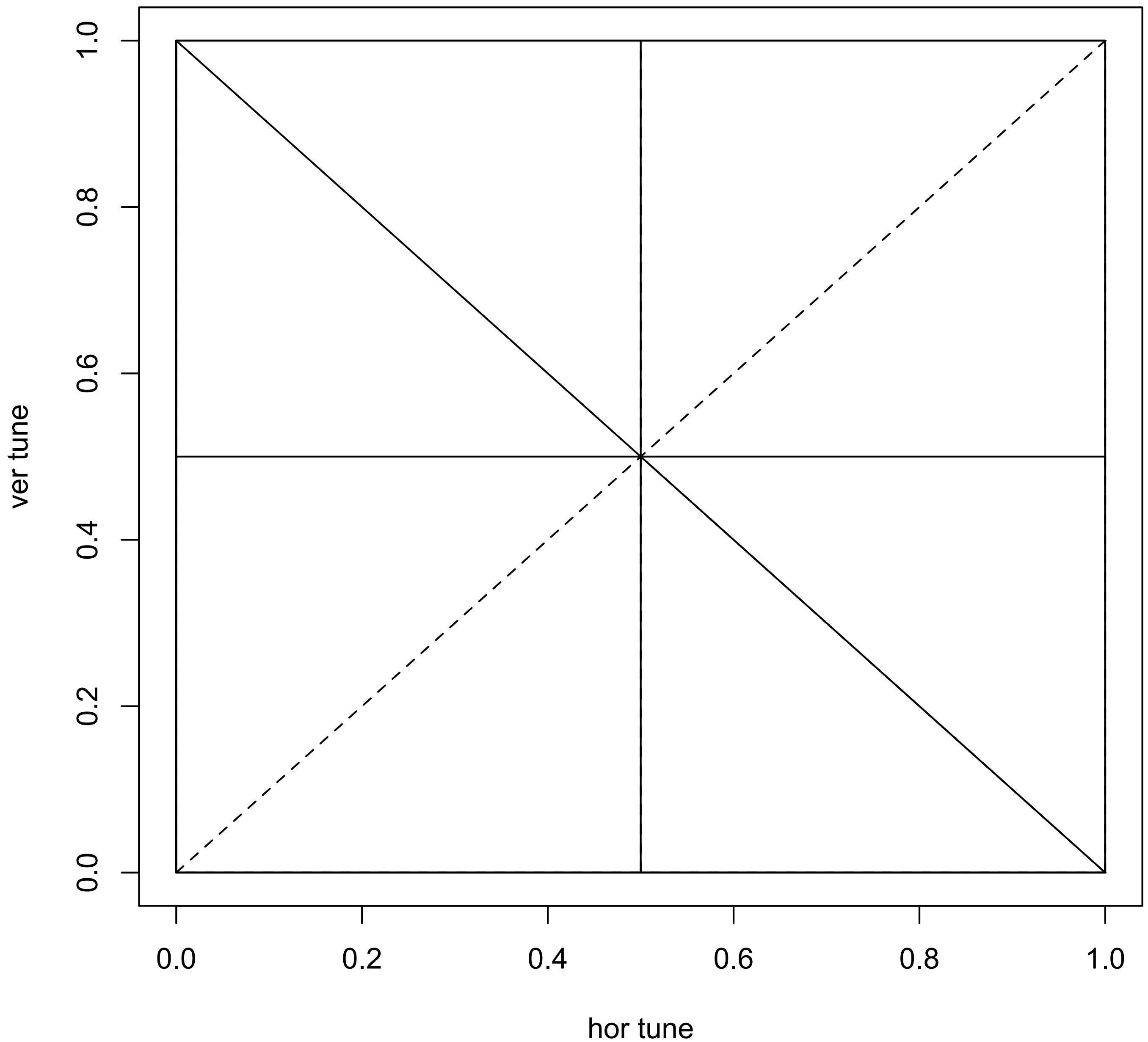
avoid ALL rational tunes???

# Tune Diagram

lines of $\boxed{m\,\nu_x \pm n\,\nu_y = k}$

Through order

k =2

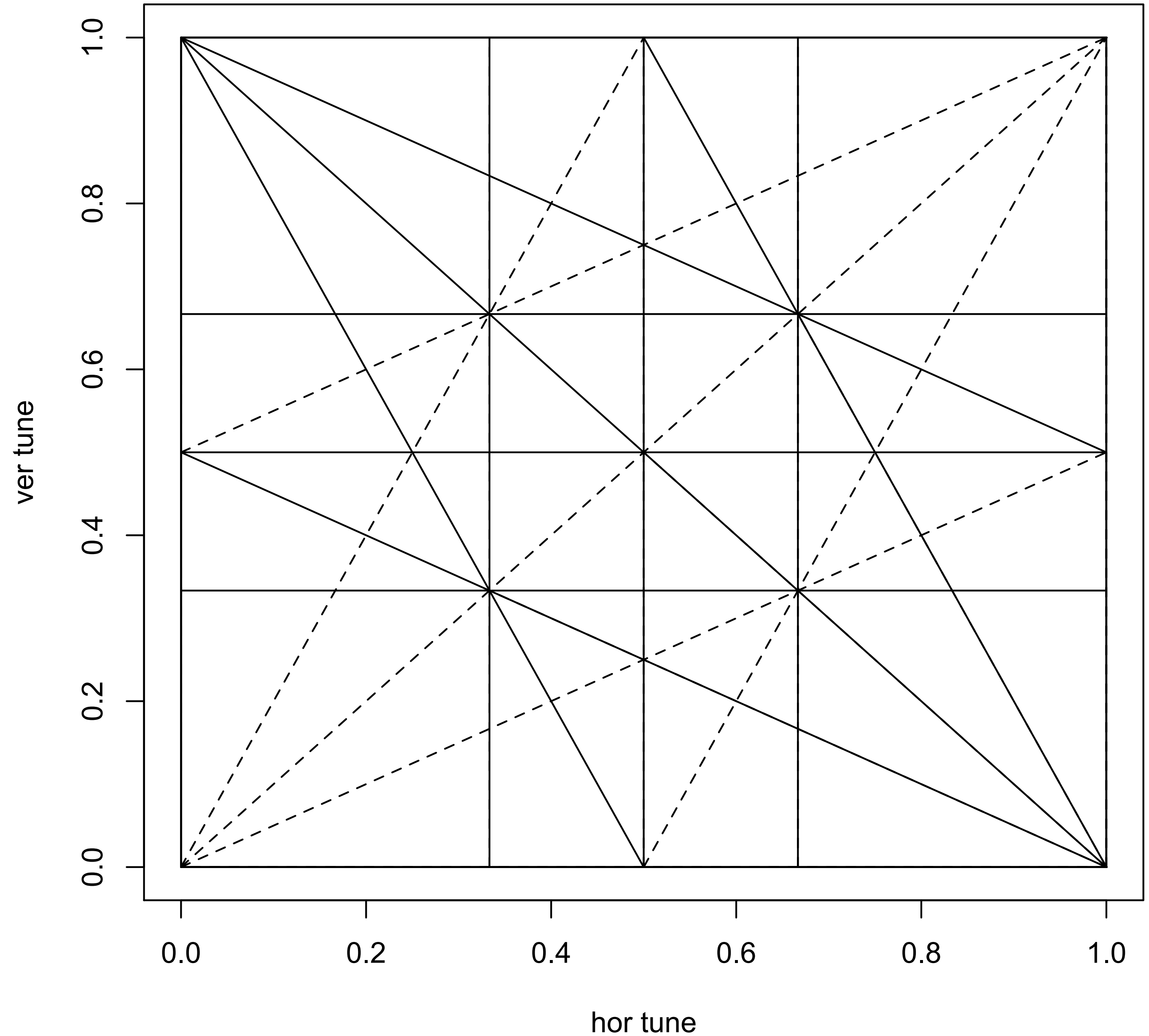


# Tune Diagram

lines of $\boxed{m\,\nu_x \pm n\,\nu_y = k}$

Through order

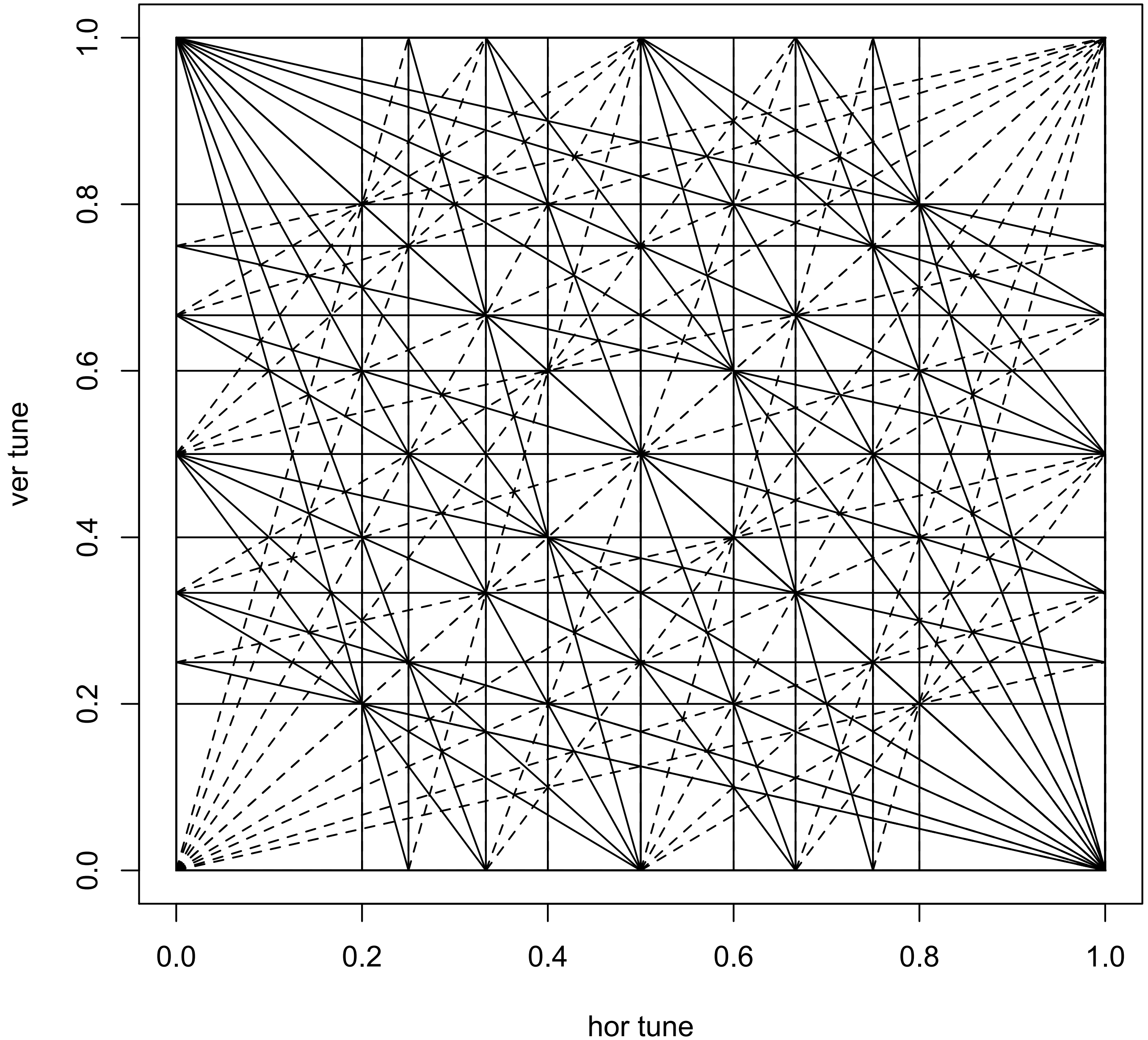
k =3



# Tune Diagram

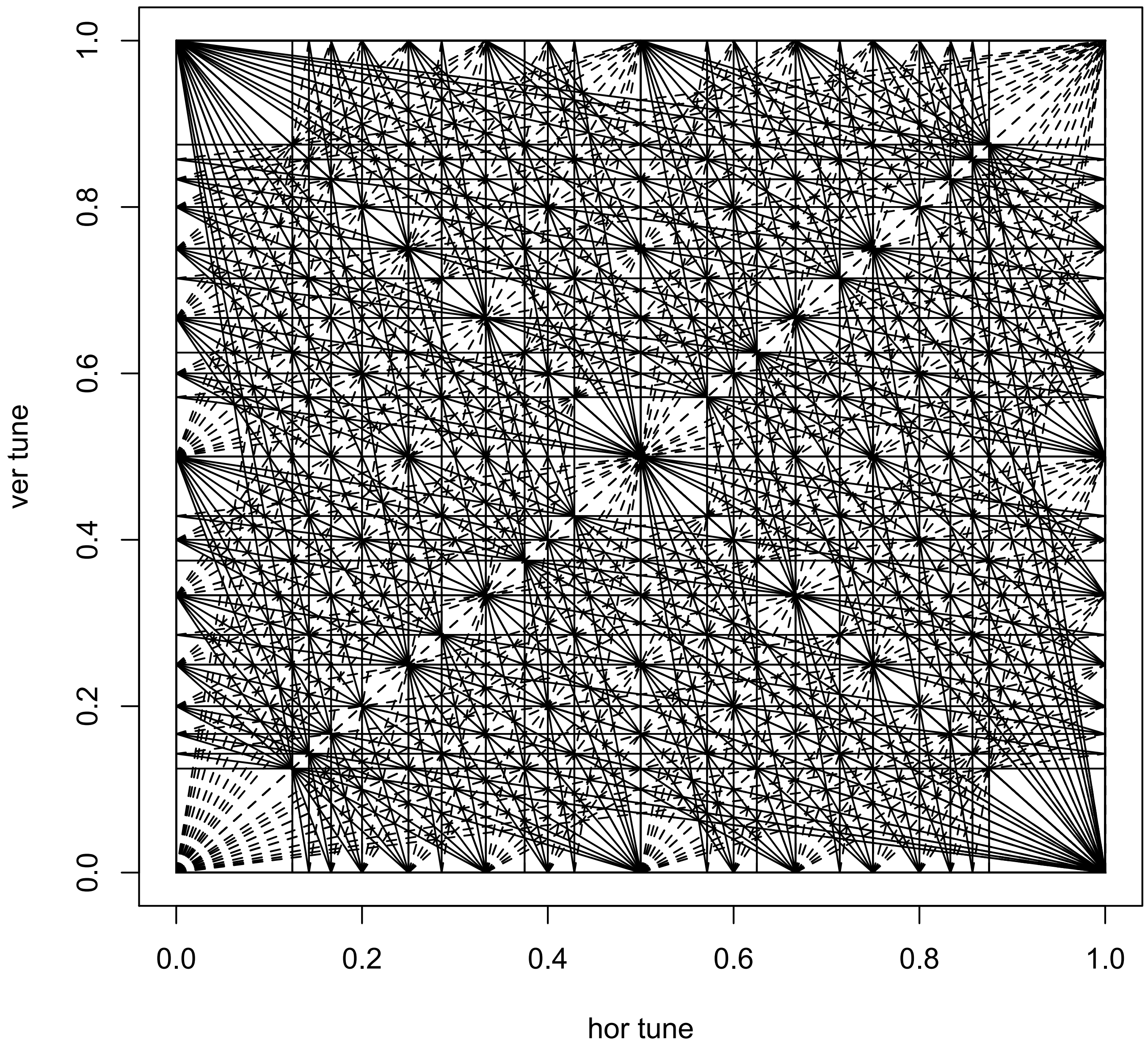
lines of $\boxed{m\,\nu_x \pm n\,\nu_y = k}$

Through order
k =5



# Tune Diagram

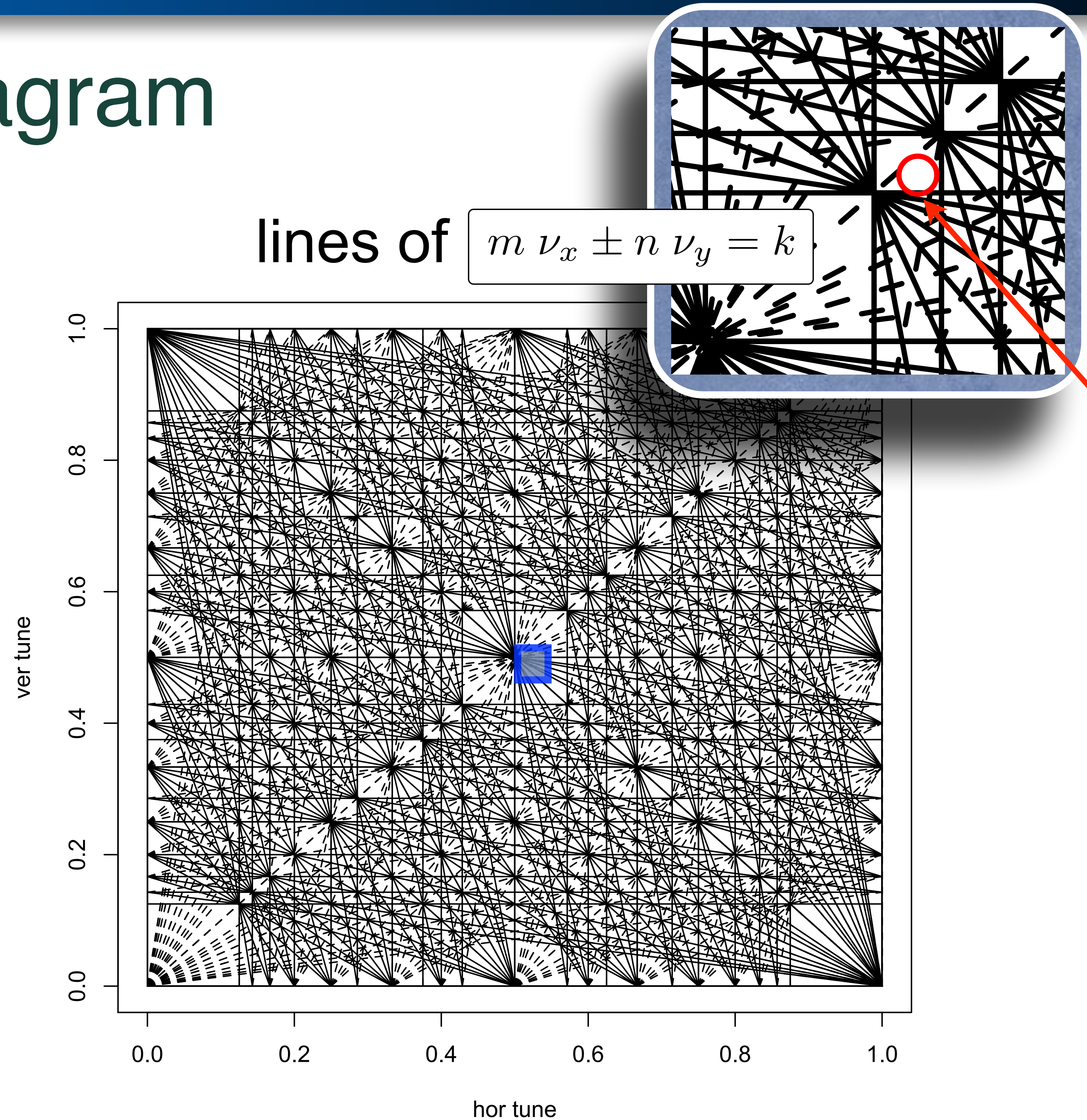
lines of $\boxed{m\,\nu_x \pm n\,\nu_y = k}$




Through order
k =8

# Tune Diagram

lines of $m\,\nu_x \pm n\,\nu_y = k$




width ~ 0.025

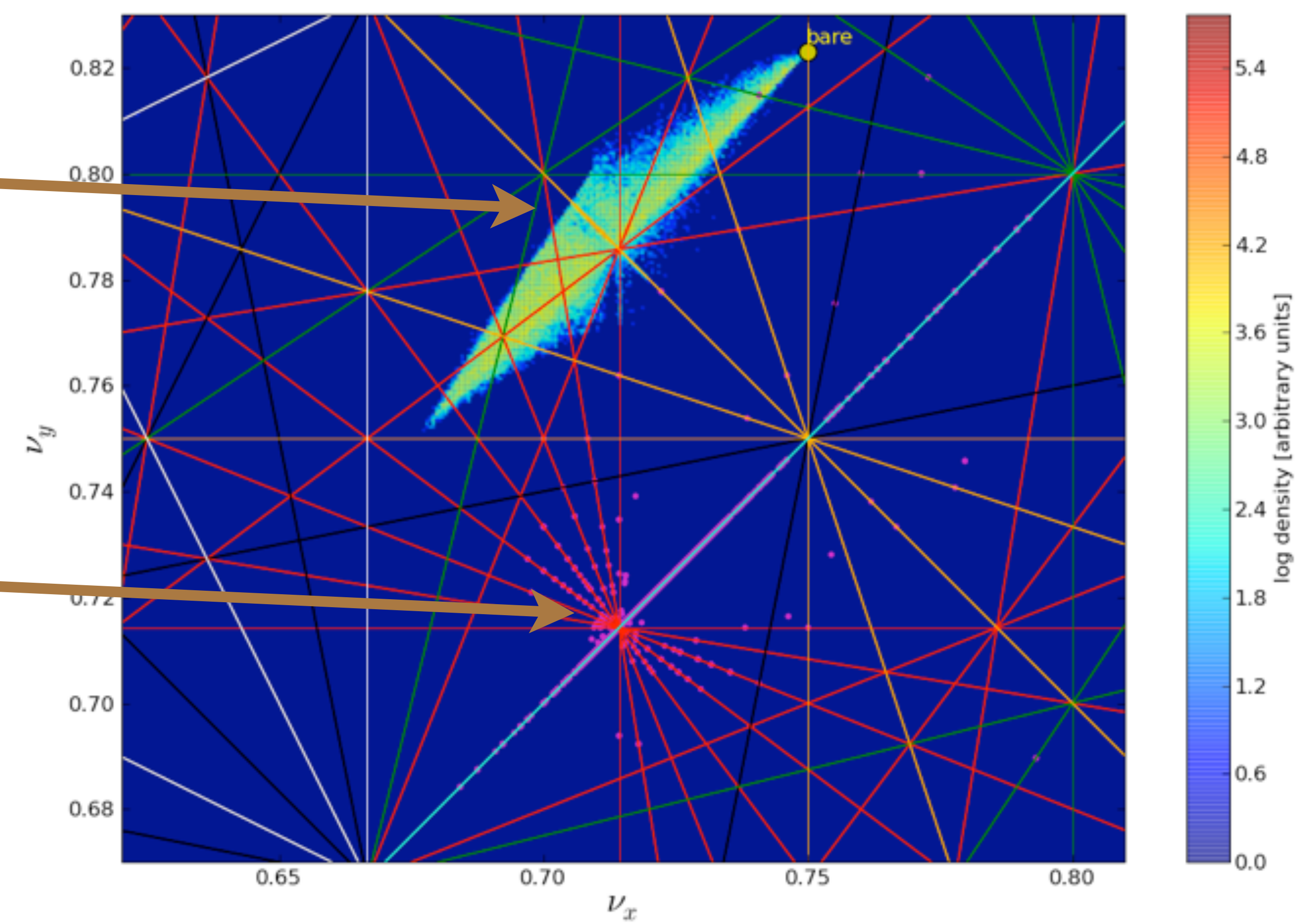
Note:
In Tevatron,
~20 oscillations
per revolution

# Phase Space Diffusion

- Overlapping resonance conditions can lead to particle diffusion
- Here, **simulation** of large initial tune distribution, due to high intensity beam

*initial distribution of tunes*

*final tunes when particles are "lost"*

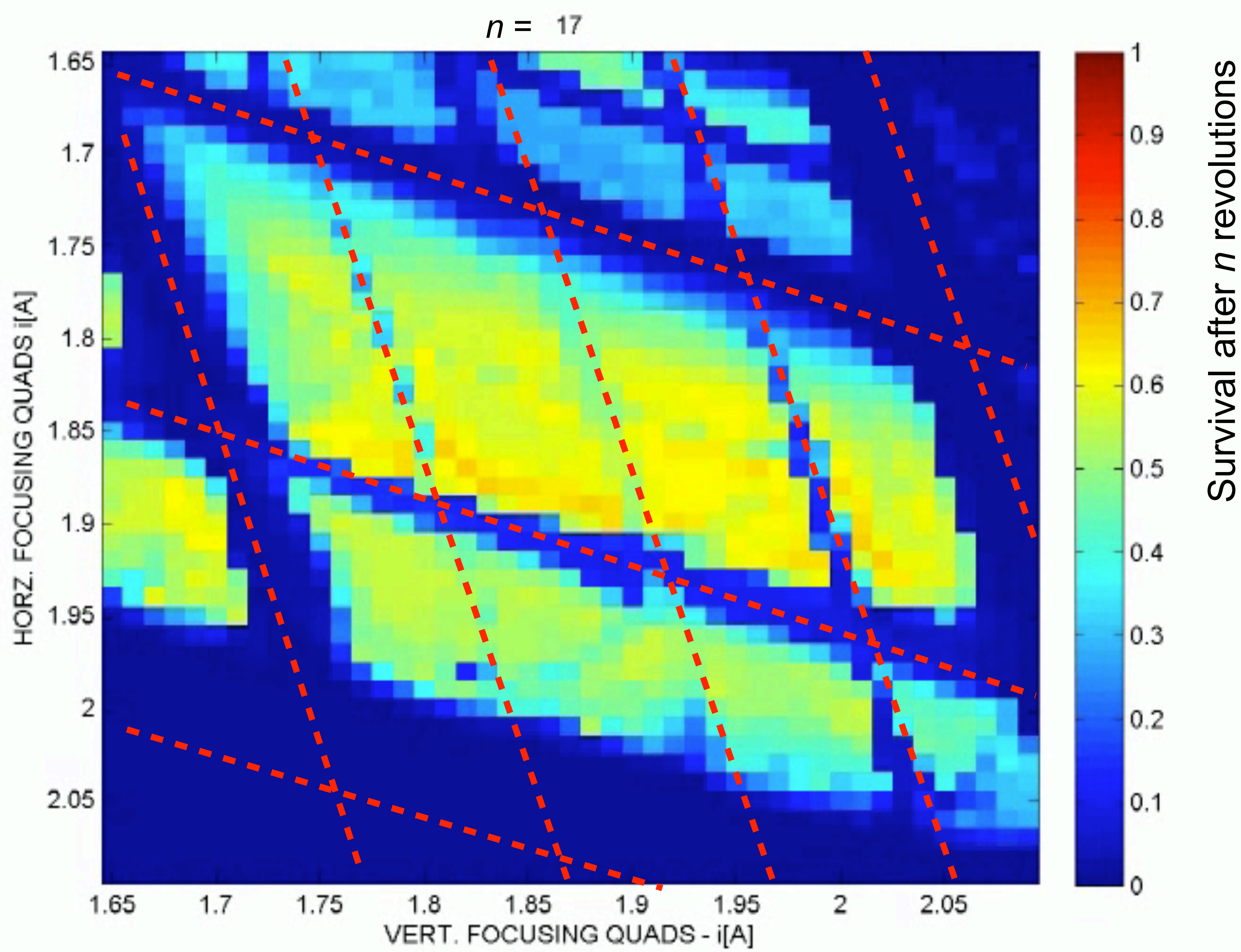



Amundson, Michelotti (FNAL)

# “Tune Scan”

Courtesy of
T. Koeth, *et al.*,
Univ. Maryland

- **Data** showing particle survival in a storage ring for various settings of the focusing magnets used for tune adjustment
- Note the appearance of the resonance lines...



# Large Accelerator Design

- Stability of Particle Motion in Large Colliders
  - Particles in the Tevatron were stored for well over 24 hours
  - at $C = 6$ km and $v = c$, this is ...
    - 24 • 3600 • (3x108 m/s)/(6 km) = $4 \times 10^9$ revolutions
    - *~ # times the earth has gone around the sun!*
- Typically, exact sources of nonlinearities -- at the level that determines the long-term behavior -- are not well known; fields typically known at the level of a few parts per $10^4$
- To predict long-term behavior of particle motion through these large machines, must perform simulations assuming various levels of field imperfections

# Dynamic Aperture and Design Criteria

- Computations of dynamic aperture began in earnest during the Tevatron design studies
- SSC Design Study,  LHC design study:

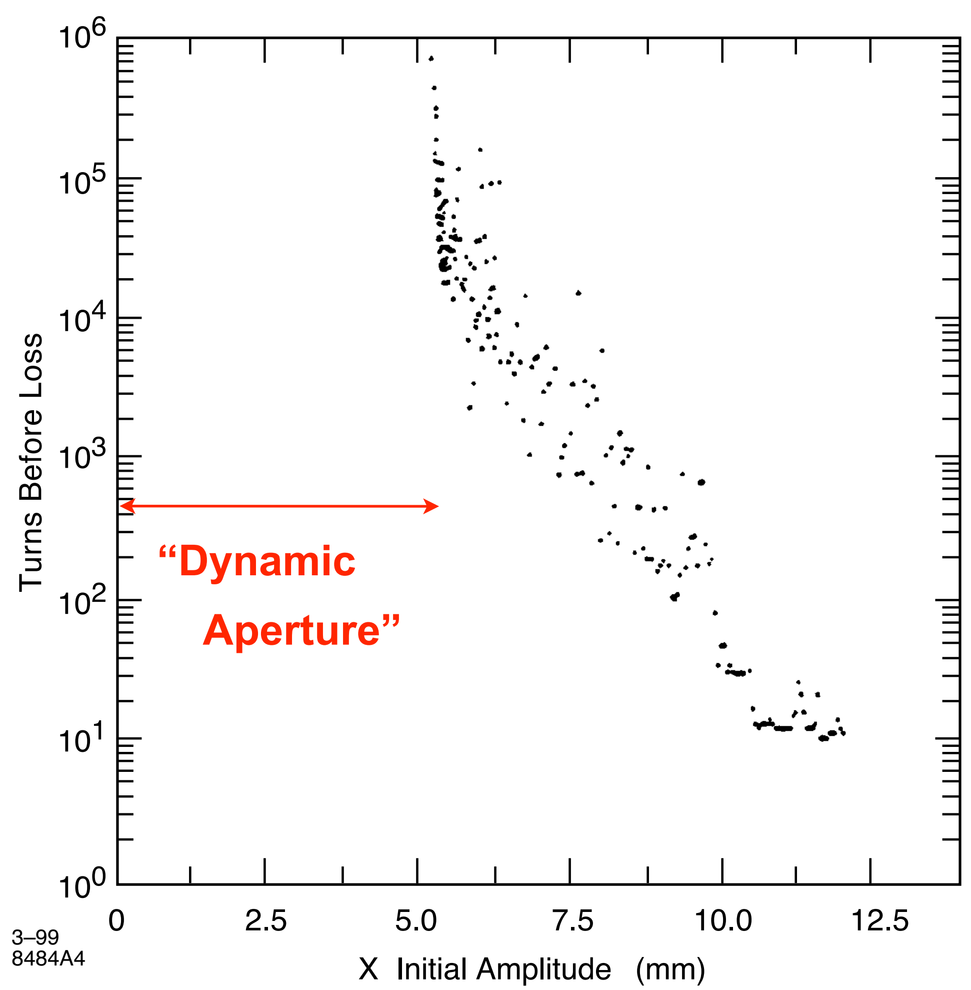



Figure 3. Survival plot for an SSC model.

- Generate a model of the accelerator with error fields
  - random, systematic
- "Track" particles with various initial oscillation amplitudes; record survival times

Y. Yan, *et al.*, SSCL Report 303 (1990)