Week Two

- **Mon**:
  - Effects of Linear and Nonlinear Errors
- **Tue**:
  - Synchrotron Radiation and Light Sources
- **Wed**:
  - Emittance Preservation; Intensity Effects; Instrumentation/Diagnostics
- **Thu**:
  - Facilities; Special Topic(?); Outlook for the Field
- **Fri**:
  - ***Final Exam***

Add Some Realism to our Ideal Accelerator

- Steering (dipole) Errors
- Focusing (quadrupole) Errors
- Errors creating Linear Coupling
- Chromatic (momentum) Effects
- Nonlinear Motion and Resonances

- Not only will errors create perturbations in the beam size, etc., but they will also tend to identify operational considerations, such as frequency choices, corrector placement, alignment tolerances, power supply specifications, etc.

Steering (dipole) Errors

- dipole field error: $B_y = B_0 \quad \longrightarrow \quad B_y = B_0 + \Delta B$
  - manufacturing; powering; control setting, …

$$\Delta x' = -\frac{\Delta B \ell}{B\rho}$$

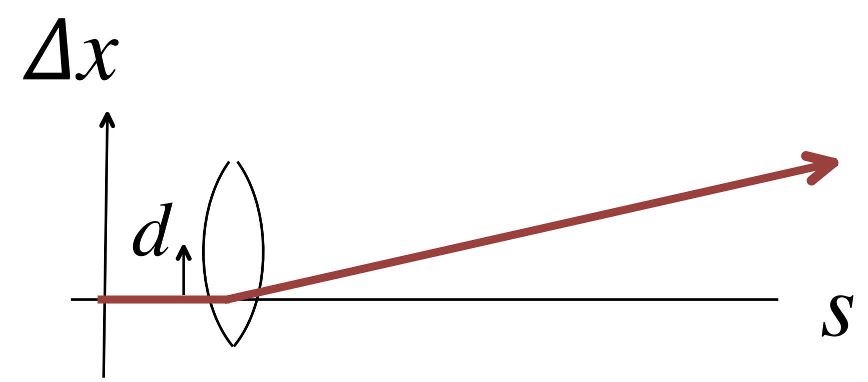
- dipole field "roll" (about the longitudinal axis)

$$B_y = B_0, \quad B_x = 0 \quad \longrightarrow \quad \begin{aligned} B_y &= B_0 \cos\phi \approx B_0 \\ B_x &= B_0 \sin\phi \approx \phi B_0 \end{aligned}$$

$$\Delta y' = \phi \frac{B_0 \ell}{B\rho} = \phi\, \theta_0$$

- Quadrupole misalignment:

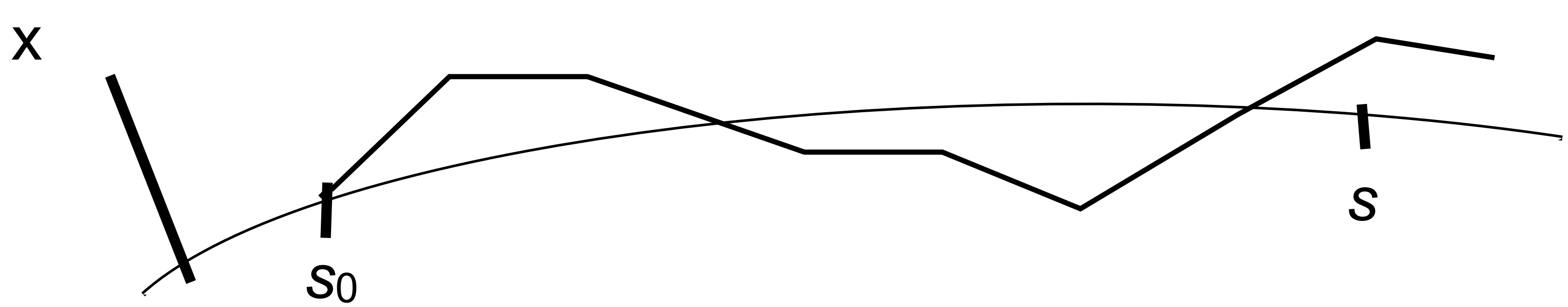
$$\Delta x' = \frac{d}{F}$$



Steering (dipole) Errors

- A field error creates a betatron oscillation…

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$



due to the small error field:

$$\Delta x' = x_0' = \Delta\theta$$

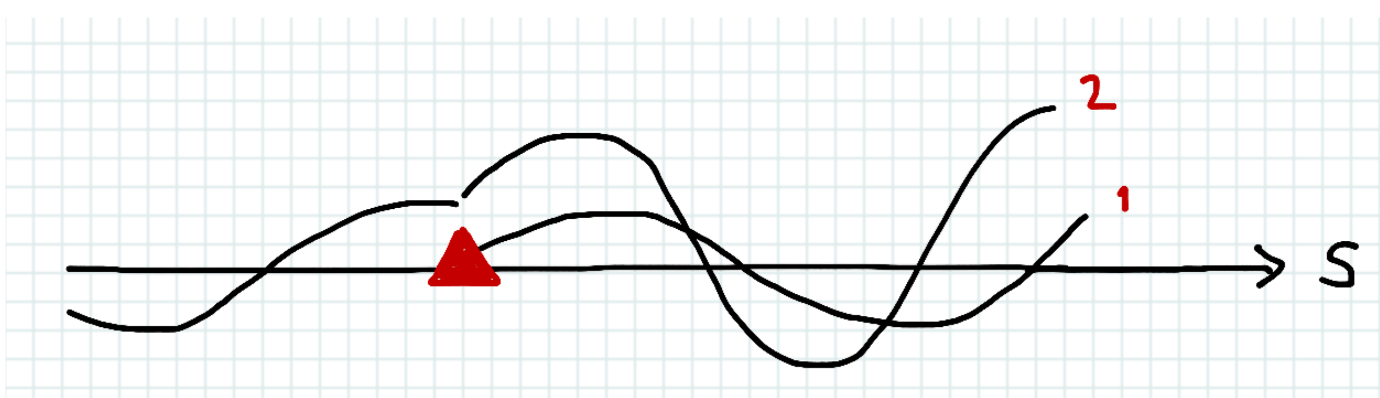
$$\Delta x = x_0 = 0$$

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi$$

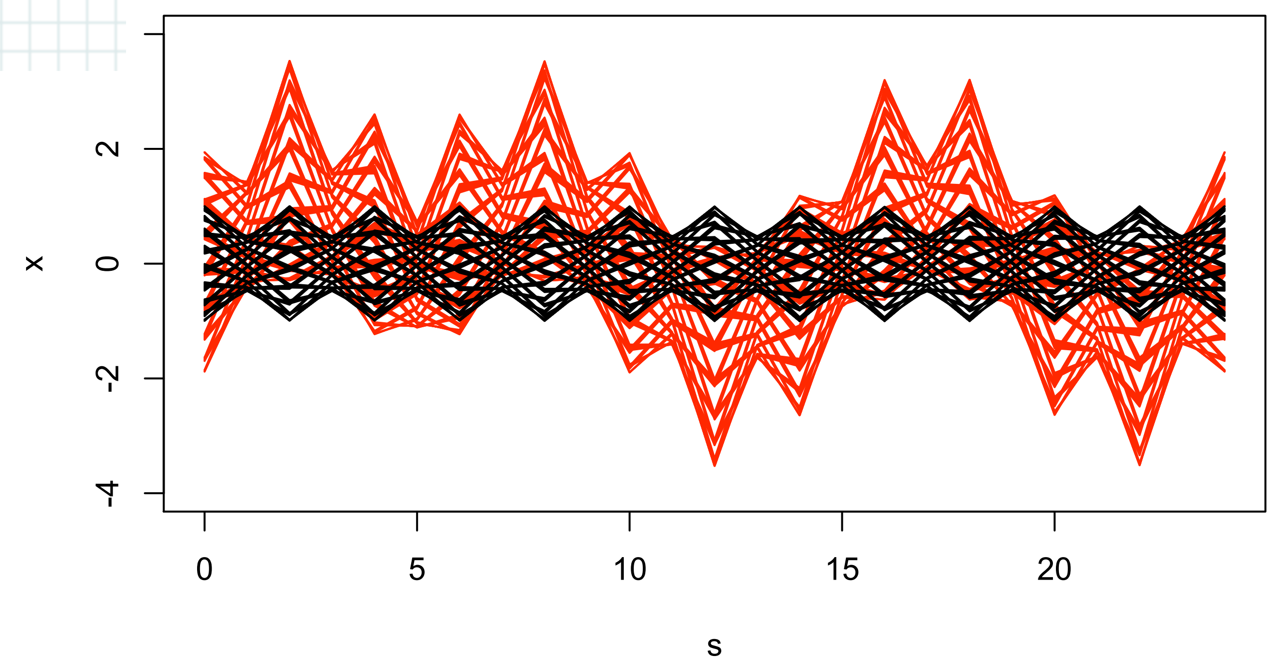
Steering (dipole) Errors

- Closed orbit distortions in a circular accelerator
  - These are not "one-time" kicks; they affect the particle motion every revolution

see ClosedOrbit.R



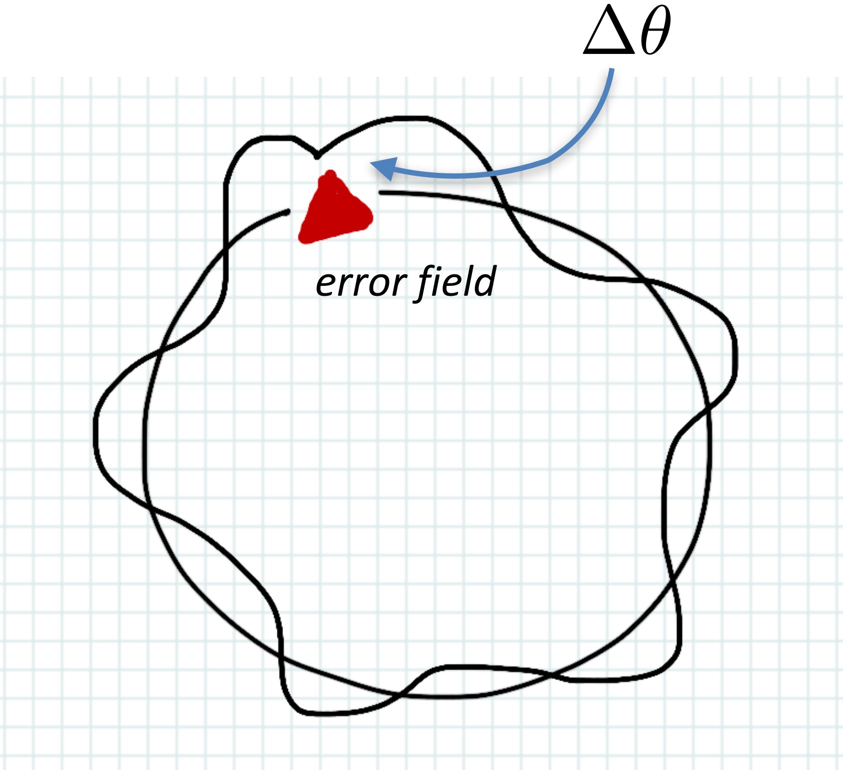
black = nominal
red = w/ error field



The trajectory of each particle will be altered by the angle $\Delta\theta$ every time it passes through the error field

The Closed Orbit

- Want to find the one trajectory which, upon passing through the error field, will come back upon itself
  - this is the "closed" trajectory, or ***closed orbit***



$$M_0 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M_0)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

- When find x_0, x'_0, can find x,x' downstream:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2} (\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Closed Orbit Distortion from Single Error

$$\begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} = (I - M_0)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$
$$= (I - e^{J_0\mu})^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$
$$= \left[e^{J_0\mu/2}(e^{-J_0\mu/2} - e^{J_0\mu/2})\right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$
$$= \left[e^{J_0\mu/2}(-2J_0 \sin \mu/2)\right]^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$
$$= -\frac{1}{2\sin \pi\nu} J_0^{-1} e^{-J_0\mu/2} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$
$$= \frac{1}{2\sin \pi\nu} J_0 (I\cos \pi\nu - J_0 \sin \pi\nu) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$
$$= \frac{1}{2\sin \pi\nu} (I\sin \pi\nu + J_0 \cos \pi\nu) \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$
$$= \frac{\Delta\theta}{2\sin \pi\nu} \begin{pmatrix} \beta_0 \cos \pi\nu \\ \sin \pi\nu - \alpha_0 \cos \pi\nu \end{pmatrix}$$

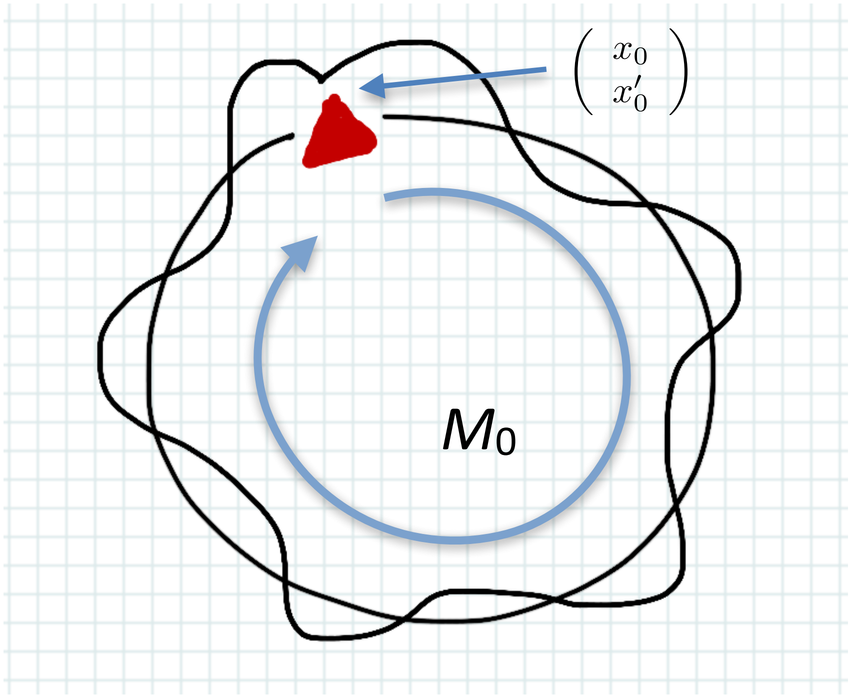
$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$M_0 = I\cos\mu + J_0 \sin\mu$$

$$M_0 = e^{J_0\mu} = e^{J_0 2\pi\nu}$$

$$J_0^2 = -I$$

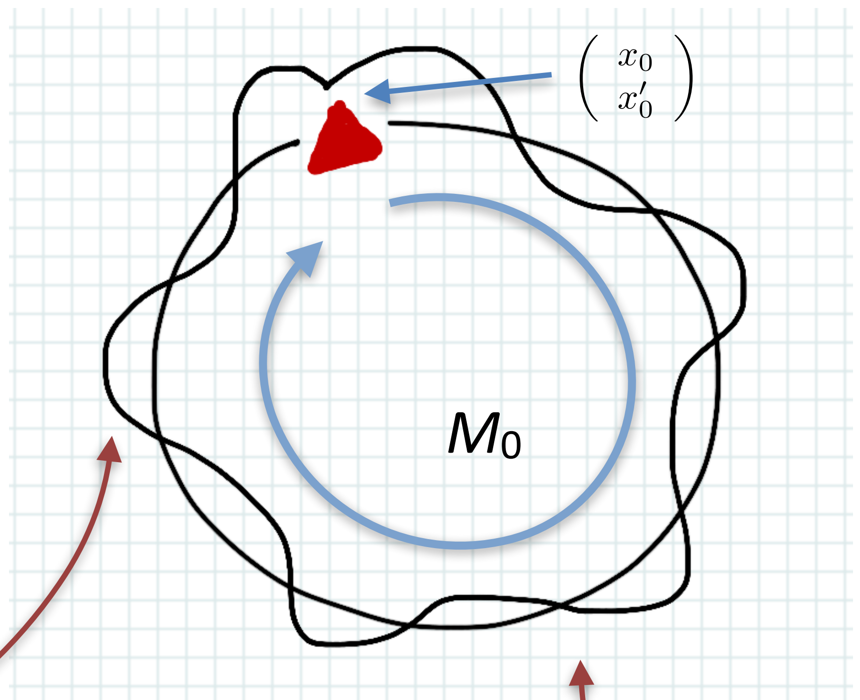
$$\mu = 2\pi\nu$$



Closed Orbit Distortion from Single Error

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{\Delta\theta}{2\sin\pi\nu} \begin{pmatrix} \beta_0 \cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$



$$\Delta x(s) = \frac{\Delta\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}\cos\left[|\psi(s)-\psi_0| - \pi\nu\right]$$

ν oscillations

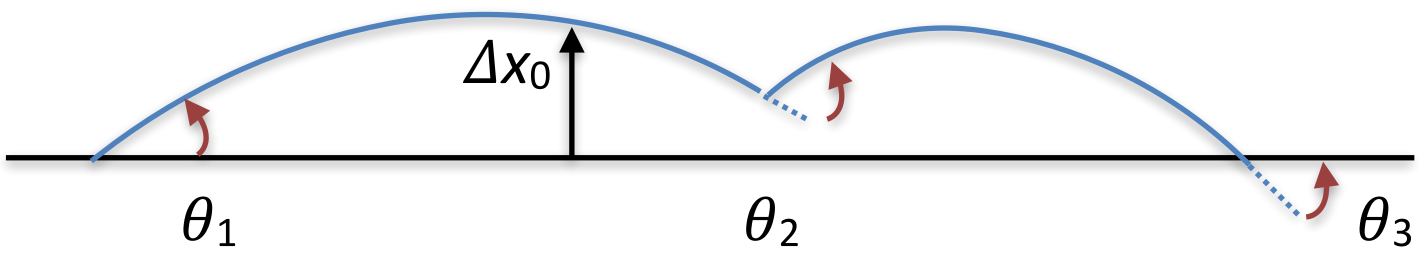
as ν —> integer, huge distortions
a resonance!

If have a collection of errors about the accelerator, then at any one point:

$$\Delta x(s) = \sum_i \frac{\Delta\theta_i\sqrt{\beta_i\beta(s)}}{2\sin\pi\nu}\cos\left[|\psi(s)-\psi_i| - \pi\nu\right]$$

Trajectory/Orbit Correction

- To make a local adjustment or correction of the position of the beam in a beam line or synchrotron, three correctors are required (in general):



$$\theta_1 = \frac{\Delta x_0}{\sqrt{\beta_0\beta_1}\sin\psi_{10}}$$

$$\theta_2 = -\theta_1\sqrt{\frac{\beta_1}{\beta_2}}\frac{\sin\psi_{13}}{\sin\psi_{23}}$$

The trajectory before θ_1 and after θ_3 is left undisturbed

$$\theta_3 = \theta_1\sqrt{\frac{\beta_1}{\beta_3}}\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

Orbit Corrections

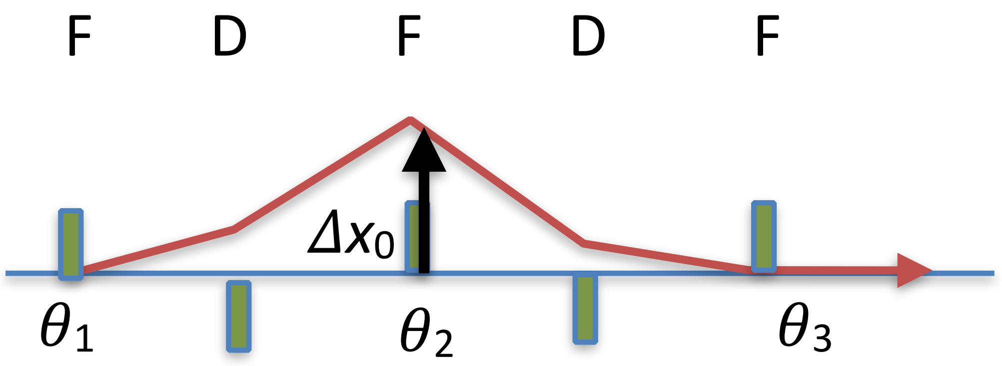
- As an example, in a "FODO" synchrotron, one would place correctors near the location of each quadrupole — at maximum beta locations, and at the source of likely steering errors (misaligned quads)

$$\psi_{13} = 2\psi_{12} = 2\psi_{23} = 2\mu$$

$$\theta_1 = \frac{\Delta x}{\hat{\beta} \sin \mu}$$

$$\theta_2 = -\theta_1 \frac{\sin 2\mu}{\sin \mu} \quad \Rightarrow \quad \theta_2 = -2\theta_1 \cos \mu$$

$$\theta_3 = \theta_1$$



Alignment Specifications Discussion

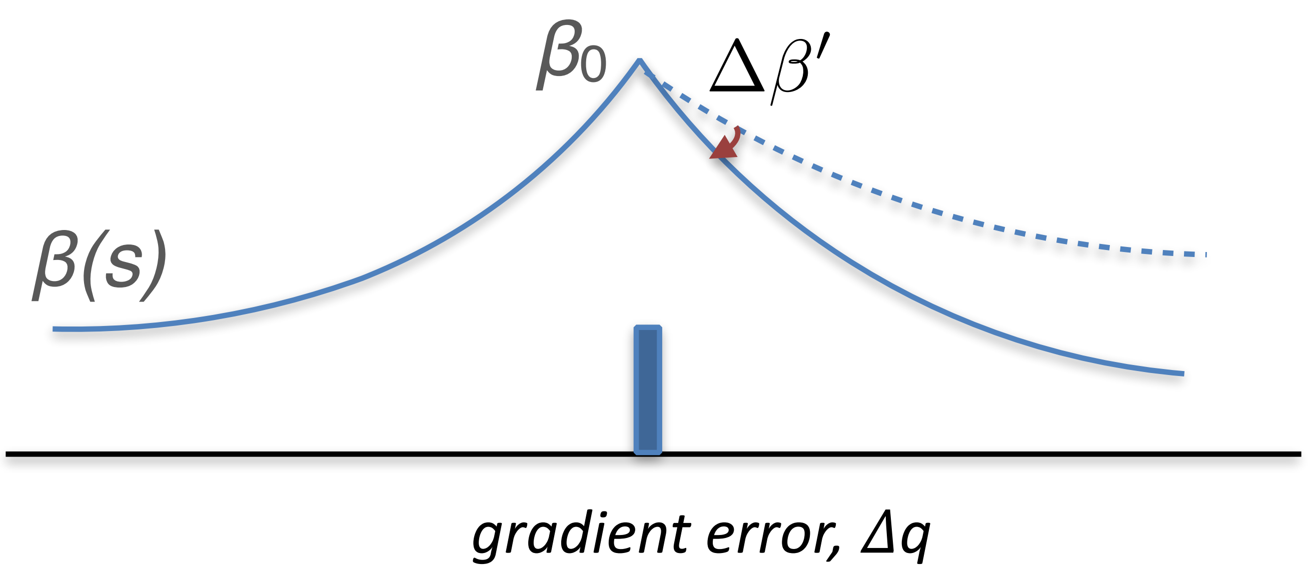
- see TrajTrace.R

Focusing (gradient) Errors

- Sources of gradient focusing errors
  - Quadrupole magnet field error
    - powering error; control error; manufacturing error
  - Dipole pole tip error (non-parallel poles)
  - etc.
- Impact of gradient errors
  - Look at Hill's Equation: $x'' + K(s)x = 0$
    - errors in the values of *K* will alter…
      - » phase advance (tune, or betatron frequency)
      - » amplitude function, β

Focusing (quadrupole) Errors

- β, α distortions and "beta-beat"



$$\Delta\alpha = -\frac{1}{2}\Delta\beta'$$

if ideal gradient produces strength $q = B'\ell/(B\rho)$,
then a gradient error will produce $\Delta q = \Delta B'\ell/(B\rho)$
and the slope of β will change according to

$$\Delta\alpha = \beta_0 \Delta q$$

Downstream the distortion will propagate:

$$\frac{\Delta\beta}{\beta}(s) \approx -\Delta q \beta_0 \sin 2\psi_0(s)$$

β Distortion in a Synchrotron

dipole error:

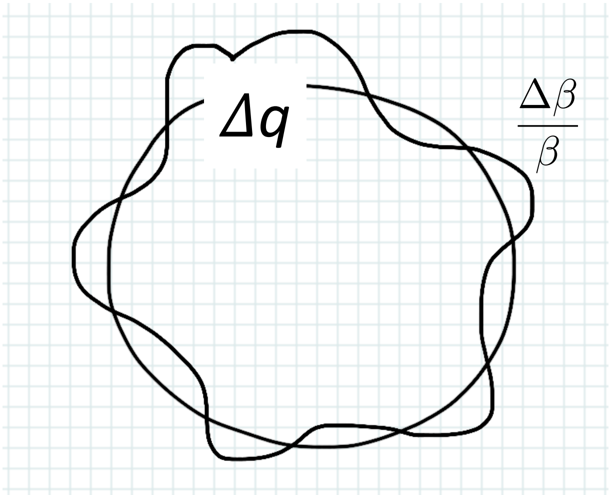
$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi$$

quad error:

$$\frac{\Delta\beta}{\beta}(s) \approx -\Delta q\beta_0\sin 2\psi_0(s)$$

- In a circular accelerator, the closed solution of the amplitude function(s) will be altered by the gradient error. With analysis similar to the situation for a closed *orbit* distortion, the gradient error will produce a closed *β*-distortion all around the ring according to (for small errors):

$$\frac{\Delta\beta}{\beta}(s) \approx -\frac{\Delta q\beta_0}{2\sin 2\pi\nu}\cos(2|\Delta\psi| - 2\pi\nu)$$



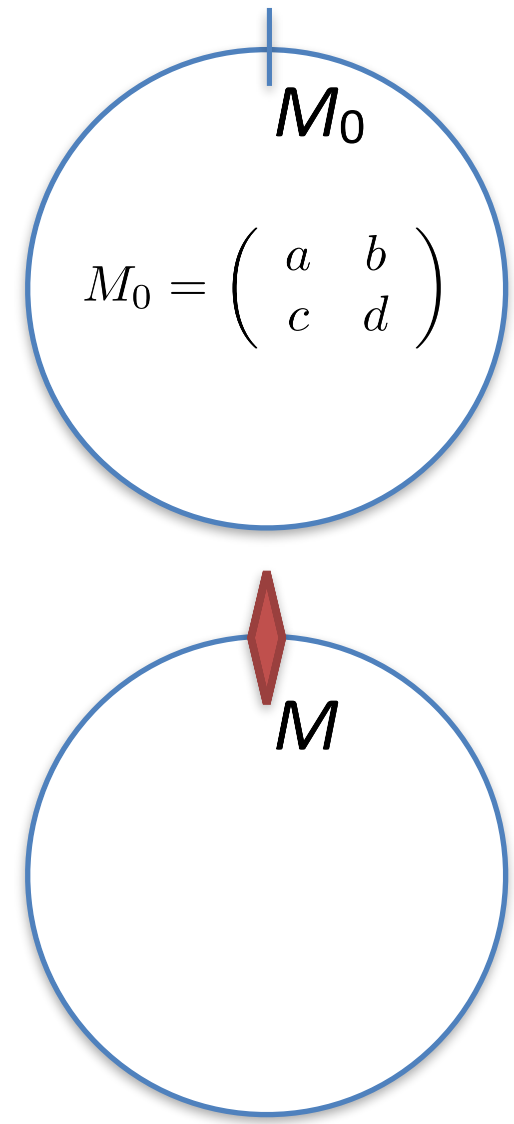
Focusing (quadrupole) Errors

- Phase/tune shift
  - a gradient error will distort the amplitude function, and therefore distort the development of the phase advance downstream. As the β distortion will oscillate about the ideal β function, the phase advance will slightly increase and decrease along the way. This is particularly important in a ring where the betatron tune, ν, might need fine control.
  - To see the change in tune for a synchrotron, we look at the effect on the matrix for one revolution…

The Tune Shift Formula

- M_0 is the one-turn matrix of ideal ring
- M is the one-turn matrix of the ideal ring followed by a small gradient error of strength q:

$$M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0$$



$$\text{then } M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c - aq & d - bq \end{pmatrix}$$

$$\mathrm{trace} M = 2\cos 2\pi\nu = a + d - bq = \mathrm{trace} M_0 - bq = 2\cos 2\pi\nu_0 - (\beta_0 \sin 2\pi\nu_0) q$$

$$\boxed{\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2} q\beta_0 \sin 2\pi\nu_0}$$

$$\begin{aligned} \cos 2\pi\nu &= \cos 2\pi(\nu_0 + \Delta\nu) \\ &= \cos 2\pi\nu_0 \cos 2\pi\Delta\nu - \sin 2\pi\nu_0 \sin 2\pi\Delta\nu \\ &\approx \cos 2\pi\nu_0 - 2\pi\Delta\nu \sin 2\pi\nu_0 \end{aligned}$$

$$\text{so,} \quad 2\pi\Delta\nu \sin 2\pi\nu_0 \approx \frac{1}{2} q\beta_0 \sin 2\pi\nu$$

$$\boxed{\Delta\nu \approx \frac{1}{4\pi} \beta_0 q}$$

Focusing (quadrupole) Errors

- What happens if the gradient error is ***too*** big?

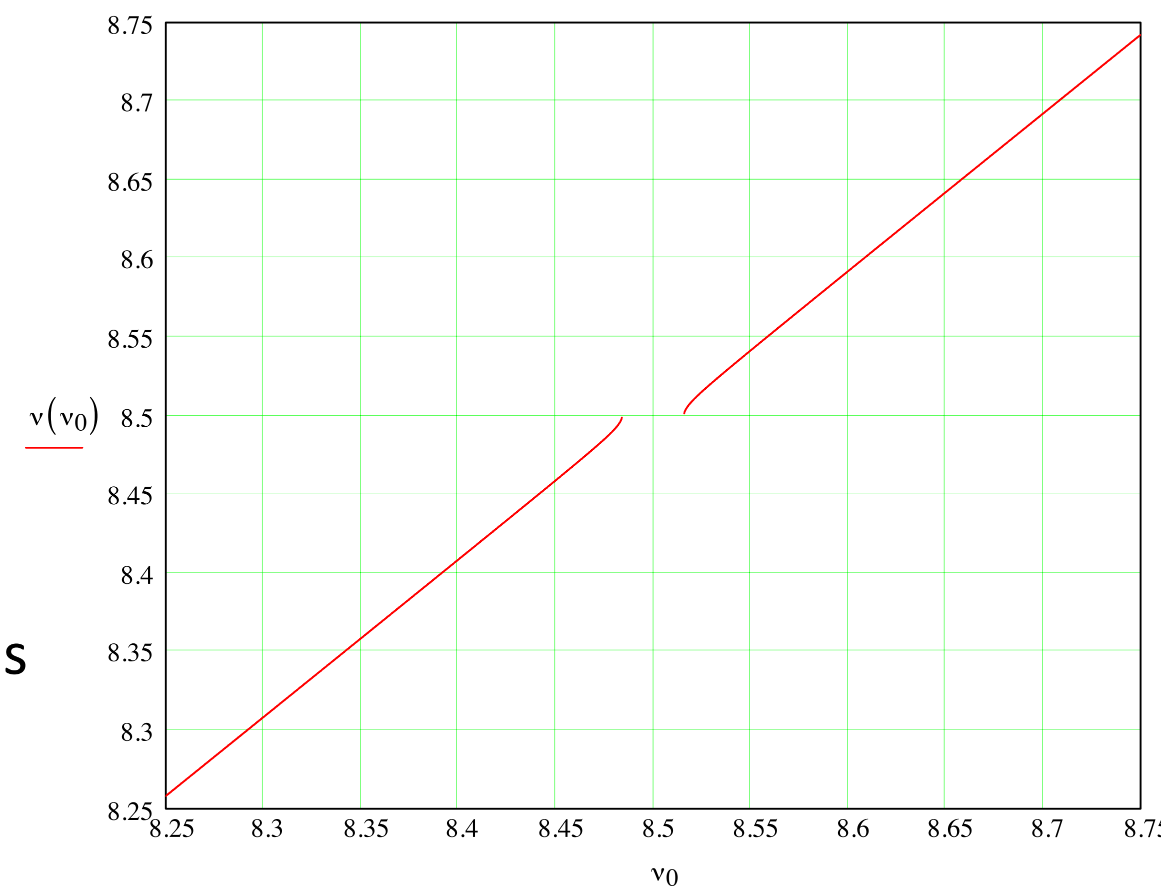
$$\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2} q\beta_0 \sin 2\pi\nu_0$$

if too large, $|\cos 2\pi\nu|$ can become >1, thus unstable!

- Half-integer stop band

$$\frac{\Delta\beta}{\beta}(s) \approx -\frac{\Delta q \beta_0}{2\sin 2\pi\nu} \cos(2|\Delta\psi| - 2\pi\nu)$$

as ν —> integer/2, huge distortions
a resonance!



Beta-Mismatch Invariant

- We noted that a local gradient error will produce an unintended distortion in the amplitude function (in its slope, in particular):

$$\Delta\alpha = \beta_0 \Delta q$$

- In the absence of further gradient errors,
  - $| \Delta\beta\Delta\gamma - \Delta\alpha^2 |$ is an invariant, and thus will have the same value further down the beam line
- proof:

$$J = MJ_0M^{-1} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$J + \Delta J = M(J_0 + \Delta J_0)M^{-1}$$

$$J + \Delta J = MJ_0M^{-1} + M\Delta J_0 M^{-1}$$

$$\Delta J = M\Delta J_0 M^{-1}$$

$$\det(\Delta J) = \det M \det(\Delta J_0) \det M^{-1}$$

$$\det(\Delta J) = \det(\Delta J_0)$$

$$|\Delta\beta\Delta\gamma - \Delta\alpha^2| = invariant$$

Gradient Specifications Discussion

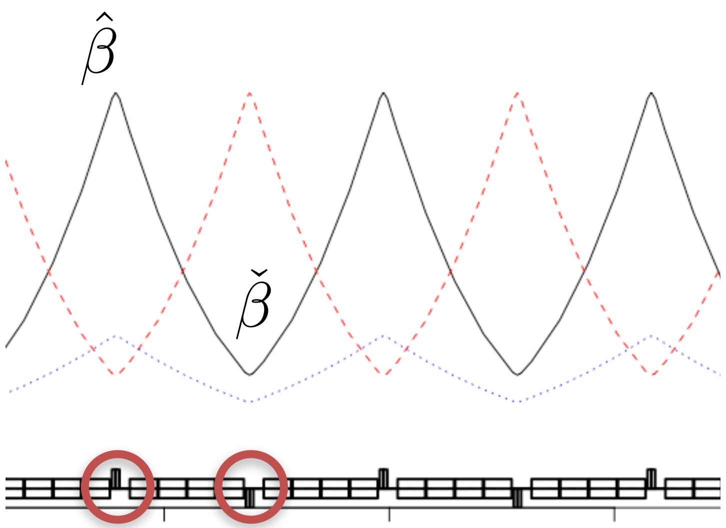
Tune correction/adjustment

- In the same way that an *error* will change the tune of a synchrotron, so can a quadrupole field *adjustment* be made to implement a desired change in the tune
- Note, however, that a quad change will alter the horizontal tune in one direction, but will alter the vertical tune in the other direction. Also, since the amplitude functions, β_x and β_y, may be different, the actual shifts in the two tunes will also be different in magnitude.
- Thus, to exercise independent control of ν_x and ν_y, there needs to be two quadrupoles (or 2 circuits)

Tune correction/adjustment

- Suppose we have a FODO arrangement, and we put adjustable quadrupoles near every "main" quadrupole (N = # quads):

$$\Delta\nu_x = \frac{N}{4\pi}\left[\hat{\beta}\Delta q_1 + \check{\beta}\Delta q_2\right]$$
$$\Delta\nu_y = -\frac{N}{4\pi}\left[\check{\beta}\Delta q_1 + \hat{\beta}\Delta q_2\right]$$



- The quadrupoles can be wired in two separate circuits, and thus the two tunes can be independently adjusted by any (reasonable) amount desired.

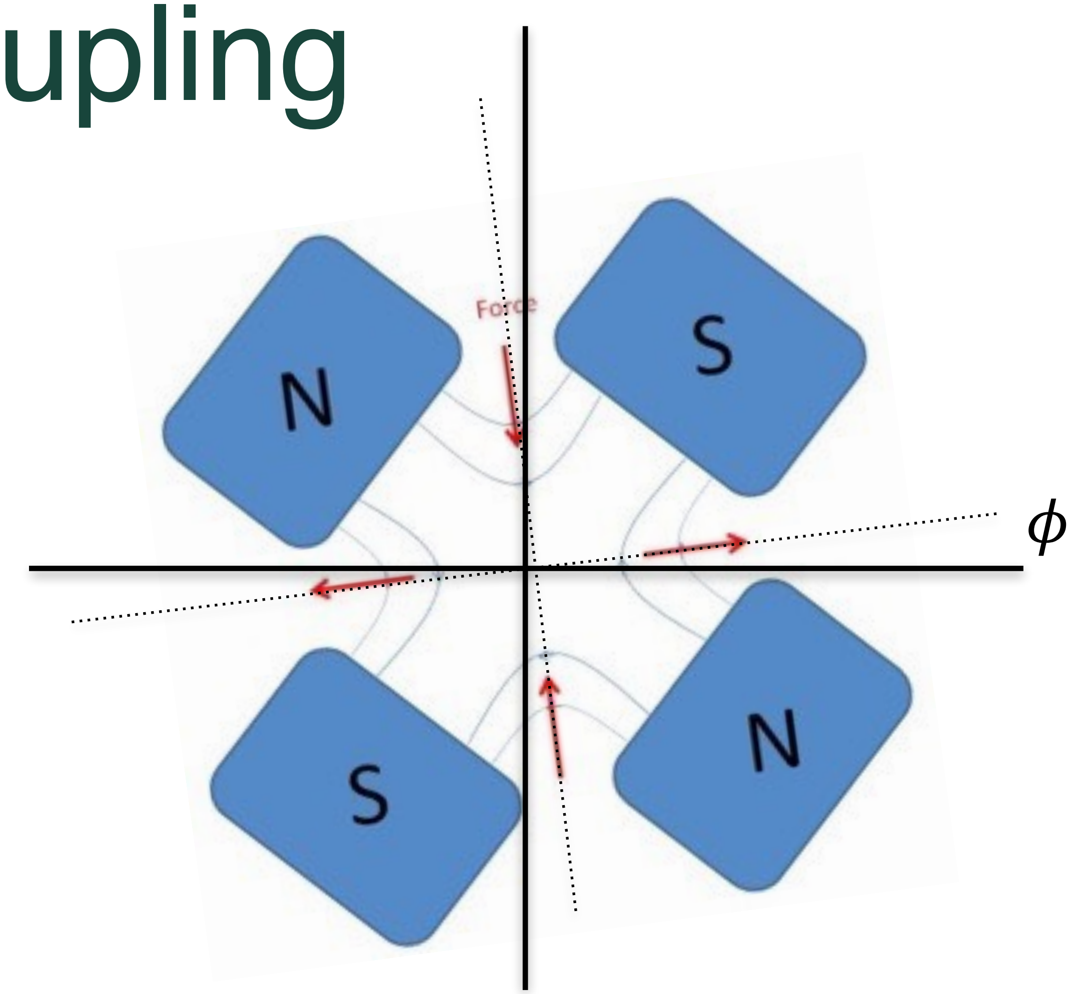
Errors creating Linear Coupling

- So far, have discussed systems where the horizontal and vertical motion are distinct. This occurs naturally when using dipole and quadrupole fields:
  - $B_y = B_0 + B'x \qquad B_x = B'y$
  - vertical fields cause motion in *x*, horizontal fields cause motion in *y*
- We've seen that a rotated (about its axis) dipole magnet will create a field component in the other plane, causing steering effects. A rotated quadrupole magnet will produce focusing fields that depend on both *x* and *y* — *coupled motion.*

Errors creating Linear Coupling

- Rotated quadrupole magnet

$$B_x = B' \cos 2\phi\, x + B' \sin 2\phi\, y$$
$$B_y = -B' \sin 2\phi\, x + B' \cos 2\phi\, y$$



a normal quad, rotated by a small angle:

$$B_y = B'\, x + 2\phi\, B'\, y$$
$$B_x = B'\, y - 2\phi\, B'\, x$$

normal *skew quad, strength:* $\frac{\Delta B'\ell}{B\rho} = 2\phi \frac{B'\ell}{B\rho} \equiv k$

Clearly, skew quad field couples the horizontal and vertical motion: $\Delta x' = \frac{B_y \ell}{B\rho} = \frac{\Delta B'\ell}{B\rho} y$

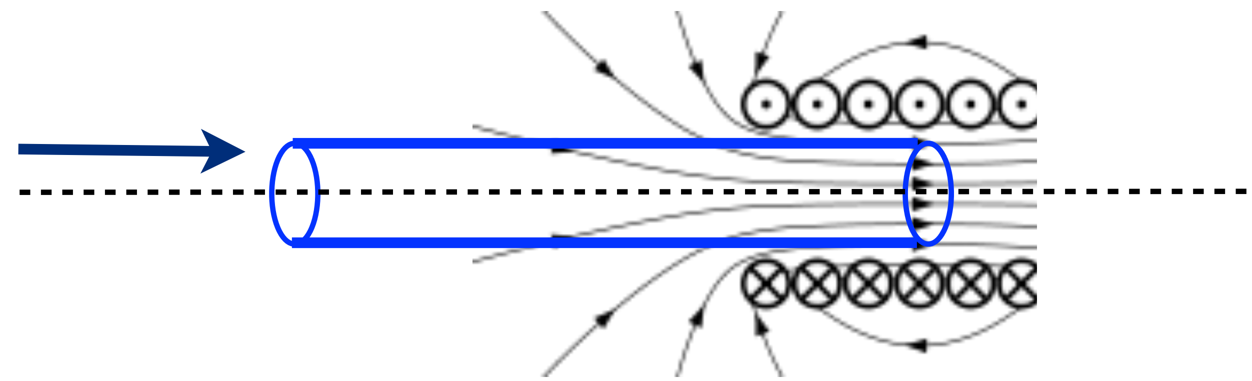
Linear Coupling From Solenoid Fields

- Have seen previously that solenoid magnets can be used to focus "round" beam distributions
- Solenoid fields can still be present in quad focusing accelerators, from beam instrumentation that use solenoids, or from particle detectors/experiments in a collider, etc.
    - in these situations, the effects from these fields are usually small, but often noticeable
    - can then treat as a small perturbation on the normal betatron motion

Linear Coupling From Solenoid Fields

- We saw earlier…

$$\Delta p_\theta \approx q \int_{-\infty}^{0} (\vec{v} \times \vec{B})_\theta dt \quad = -\frac{qB_0}{2}\, r$$



upon entrance:

(opposite signs upon exit)

$$\Delta x' = \frac{B_0}{2B\rho} y$$
$$\Delta y' = -\frac{B_0}{2B\rho} x$$

through central region:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \rho\sin\theta & 0 & \rho(1-\cos\theta) \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & -\rho(1-\cos\theta) & 1 & \rho\sin\theta \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_0$$

$$\theta = \frac{B_0 \ell}{B\rho}$$

$$\rho = \frac{B\rho}{B_0}$$

The motion in each plane depends upon the trajectory in *both* planes

Effects of coupling on betatron tunes

- Coupling moves the frequencies about — moves the betatron oscillation tunes, in the case of an synchrotron — and so can defeat the precise tune control needed to avoid resonances in devices such as colliders and other storage rings.
- At an even more elementary level, coupling is an irritant in diagnosing beam behavior, for the eigenfrequencies and eigenmodes are no longer associated with the degrees of freedom specified in the design.

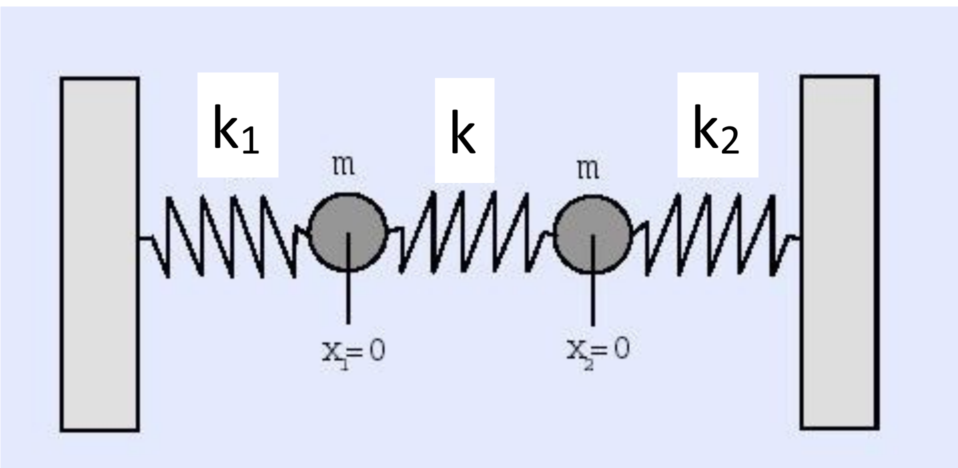
Eigen-frequencies of Coupled Oscillator

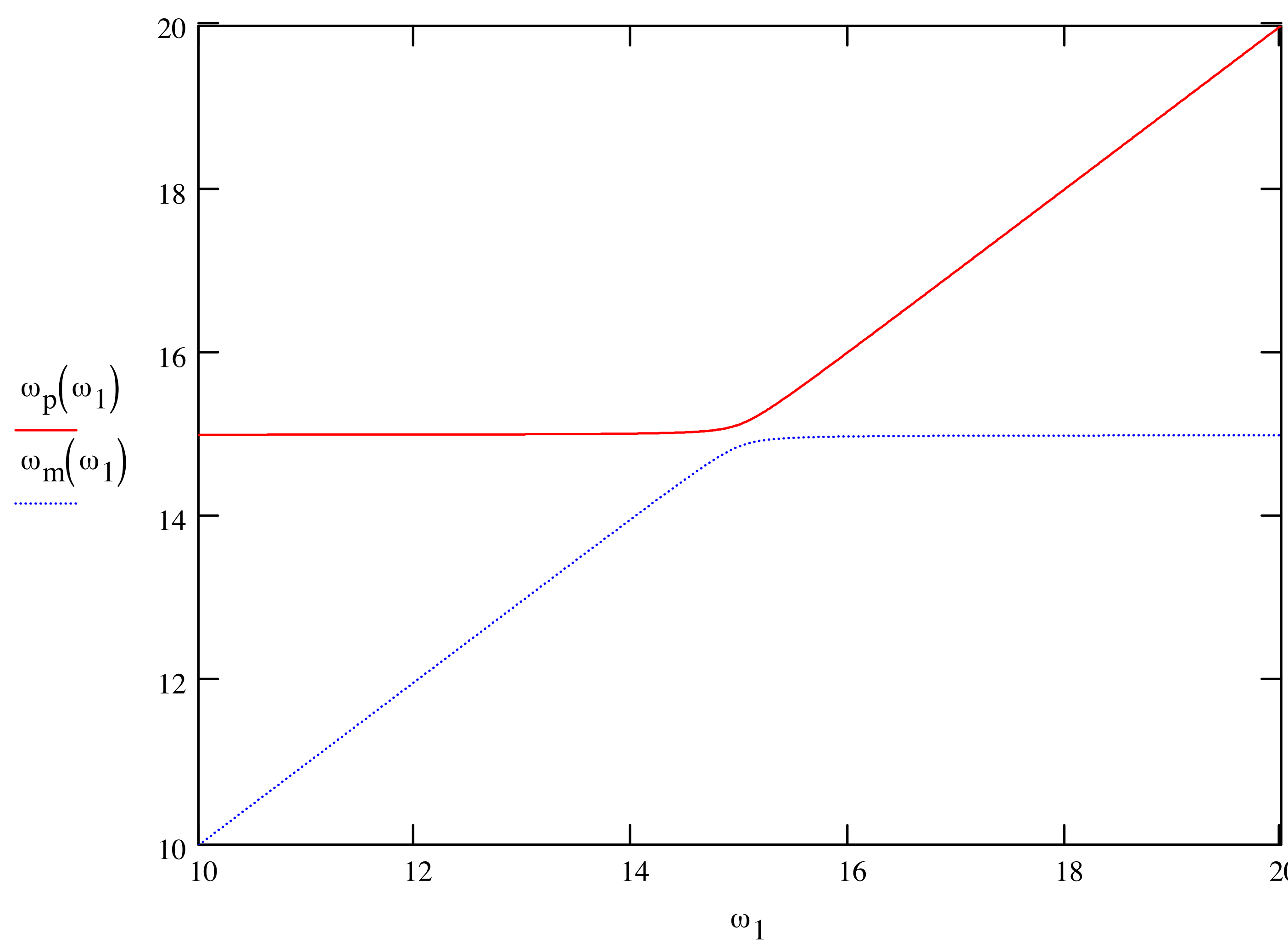
$q := 2$ $\omega_2 := 15$

$$\omega_m(\omega_1) := \sqrt{\frac{\omega_1^2 + \omega_2^2 - \sqrt{\left(\omega_2^2 - \omega_1^2\right)^2 + 4 \cdot q^4}}{2}}$$

$\omega_1 := 10, 10.01 .. 20$

$$\omega_p(\omega_1) := \sqrt{\frac{\omega_1^2 + \omega_2^2 + \sqrt{\left(\omega_2^2 - \omega_1^2\right)^2 + 4 \cdot q^4}}{2}}$$





In a synchrotron, find that the minimum separation that can be obtained in the presence of a skew quadrupole field is:

$$\Delta\nu_{min} = \frac{|k|}{2\pi}\sqrt{\beta_x\beta_y} = \frac{|\phi q|}{\pi}\sqrt{\beta_x\beta_y}$$

(if due to a rotated quad)

Beam Transport through Coupled Systems

- We've just seen the possible introduction of a "4x4" matrix approach to analyzing coupled motion
- If we look at 4x4 transport matrices that operate on (*x,x',y,y'*) vectors, then the transport of covariance matrices works just as before:

$$\Sigma = M\Sigma_0 M^T$$

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle & \langle xy \rangle & \langle xy' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle & \langle x'y \rangle & \langle x'y' \rangle \\ \langle yx \rangle & \langle yx' \rangle & \langle y^2 \rangle & \langle yy' \rangle \\ \langle y'x \rangle & \langle y'x' \rangle & \langle y'y \rangle & \langle y'^2 \rangle \end{pmatrix}$$

4x4 matrices now

can also extend to 6x6, which includes *W-t* (or *z-z'* or *z-dp/p*, or…)

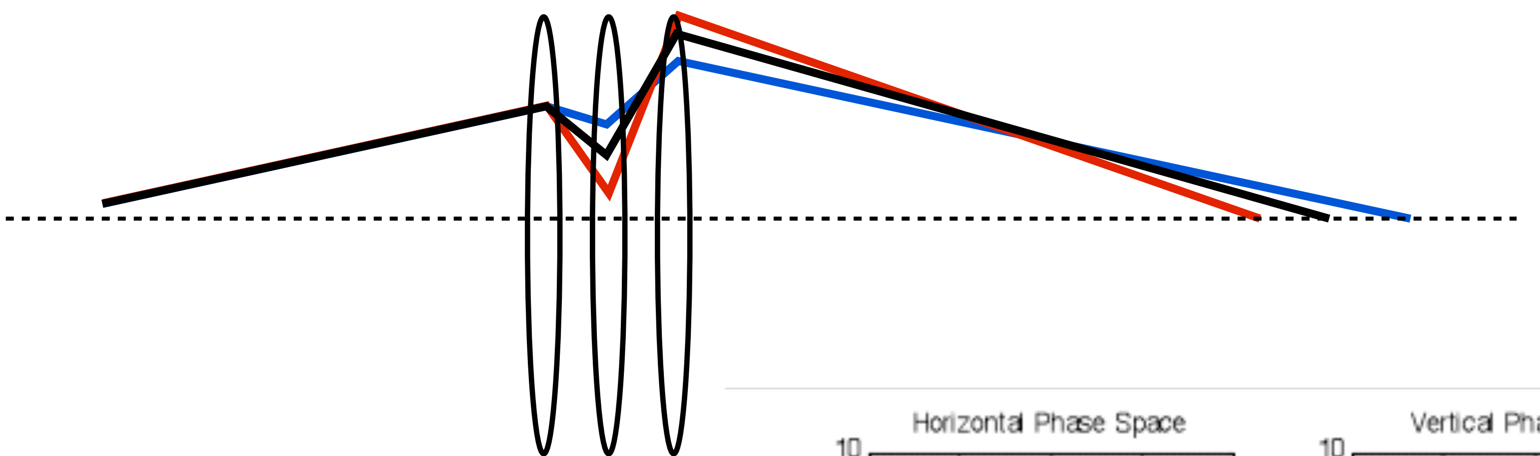
Chromatic Effects

- We may think of dispersion (and the Dispersion function) as being the propagation of a steering error, where the *error* was introduced due to $\Delta p/p$.
- $\Delta p/p$ will similarly introduce gradient "errors"
  - thus, expect the tune to depend upon $\Delta p/p$
  - and, expect the amplitude function $\beta = \beta(\Delta p/p)$
- Some Examples
  - Chromatic Aberration in a final focus (FRIB)
  - Tune spread in a synchrotron due to momentum — chromaticity

Chromatic Aberration in Final Focus

$$\frac{1}{f} = \frac{B'\ell}{B\rho}$$

$$\Delta\alpha = \beta\Delta(1/f) = -(\beta/f)\frac{\Delta p}{p}$$

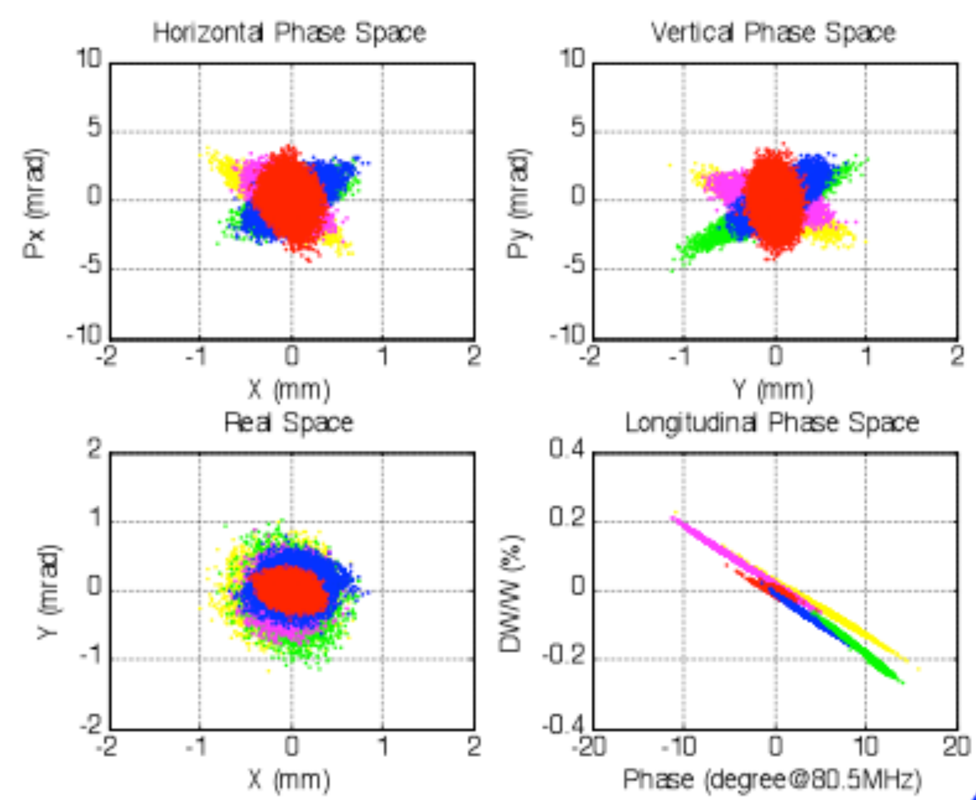


FODO cell: $\frac{\beta}{f} \sim 1 - 5$

Final Focus Triplet: $\frac{\beta}{f} \sim 10^2 - 10^3$

Right: Phase space plots of expected particle distributions at FRIB target. Note the different orientations of the different charge states of the Uranium beam.

$$\Delta(B\rho)/(B\rho) = \Delta p/p - \Delta Q/Q$$



In regions where the amplitude function, β, is large, AND where the quadrupoles are very strong (short *f*), then the chromatic aberrations become very important

Chromaticity of a Circular Accelerator

- Chromaticity -- change in the betatron tune, ν, with respect to relative momentum deviation ($\Delta p/p$):

$$x'' + K(s)x = x'' + \frac{qB'(s)}{p}x = 0$$

$$\xi \equiv \frac{\Delta\nu}{\Delta p/p}$$

- There will be a different chromaticity value for each degree of freedom:

$$\xi_x = \frac{\Delta\nu_x}{\Delta p/p}$$

$$\xi_y = \frac{\Delta\nu_y}{\Delta p/p}$$

How to estimate the scale of the effect?

The Natural Chromaticity

- While there may be error fields that contribute to chromatic effects (sextupole fields — later), there will be a "natural" chromaticity due to the ideal magnets of the synchrotron lattice
- Starting from $\Delta\nu = \frac{1}{4\pi}\beta\Delta q$ for a single gradient error,

$$\Delta q \equiv \frac{\Delta B'\ell}{B\rho}$$

$$\Delta\nu = \int \frac{1}{4\pi}\beta(s)\left[-\frac{B'(s)}{B\rho}\frac{\Delta p}{p}\right]ds$$

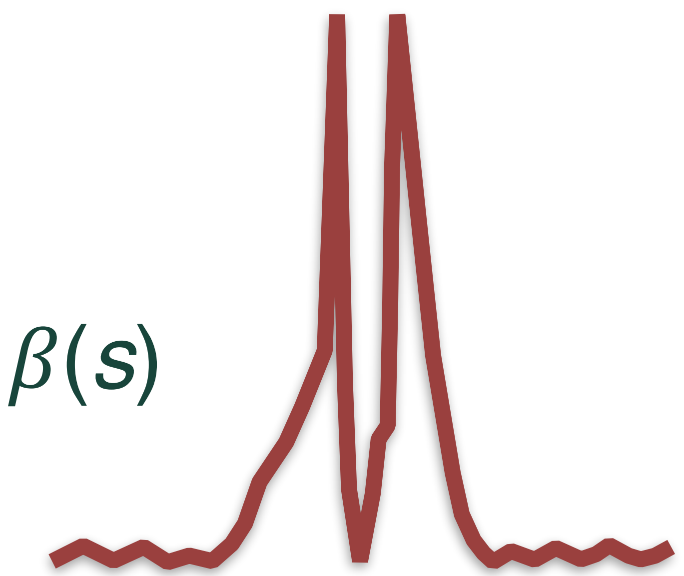
$$\xi = -\frac{1}{4\pi}\int \beta(s)K(s)ds$$

$$\xi \equiv \frac{\Delta\nu}{\Delta p/p}$$

Can show that for a FODO-style lattice, $\xi \approx -\nu$

Natural Chromaticity of a Low-β Insertion

- We saw in our LHC example that the beta function has values:
  - 180 m in cells
  - ~4500 m in final focus triplet
  - 0.5 m at the Interaction Point
- Estimate ξ_{nat} due to IP:



$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

$$\Leftrightarrow \quad K\beta = \gamma + \alpha'$$

$$\int K(s)\beta(s)ds = \int \gamma(s)ds + \int \alpha'(s)ds$$

$$\int K(s)\beta(s)ds = \int \gamma(s)ds \approx \gamma^* \cdot 2L^* \cdot 2 = 4L^*/\beta^*$$

$$\xi_{IP} \approx -\frac{4L^*}{4\pi\beta^*} = -23 \text{ m}/(\pi\ 0.5\ ) = -15$$

Chromatic Corrections

- Example: suppose synchrotron has ξ = -20, and the beam has a momentum spread of ±0.1%; then the particle distribution will have a *spread* in tunes between $\nu_0 \pm 0.02\, \nu_0$.
- In order to ensure that all particles have the same tunes (hor/ver), within tolerable levels, need to be able to adjust the overall chromaticity of the ring.
- Desire focusing element with a focusing strength that depends on momentum (linearly, preferably).
- This can be accomplished using sextupole fields in regions with horizontal dispersion.

Chromatic Corrections

- Sextupole Field:

$$B_y = \frac{1}{2}B''(x^2 - y^2)$$
$$B_x = B''xy$$

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = B''x$$ *gradient*

So here, if "x" is due to Dispersion: $x = D\frac{\Delta p}{p}$

$$\Delta\nu = \frac{1}{4\pi}\beta/f$$

then, $$\frac{1}{f} = \frac{(\partial B_y/\partial x)\ell}{B\rho} = \frac{B''\ell}{B\rho}\cdot D\frac{\Delta p}{p}$$

$$\Delta\xi = \frac{1}{4\pi}\beta\, D\,\frac{B''\ell}{B\rho}$$

ℓ = length of the sextupole field

Note: since $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \propto D\cdot\frac{\Delta p}{p}$, provides focusing in one plane, defocusing in the other plane

Thus, need 2 sextuples (or 2 families of sextuples) for optimal independent corrections/adjustment of ξ_x, ξ_y.

Also Note: introduces (intentionally!) a non-linear field!!

Correction/Adjustment of Chromaticity

- Suppose we have a FODO arrangement, and we put adjustable sextuple magnets near every "main" quadrupole (N = # sextupole magnets):

$$\Delta\xi_x = \frac{N}{4\pi}\left[\hat{\beta}\hat{D}\Delta S_1 + \check{\beta}\check{D}\Delta S_2\right]$$

$$\Delta\xi_y = -\frac{N}{4\pi}\left[\check{\beta}\hat{D}\Delta S_1 + \hat{\beta}\check{D}\Delta S_2\right]$$

$$S \equiv \frac{B''\ell}{B\rho}$$

- The sextupoles can be wired in two separate circuits, and thus the two chromaticities can be independently adjusted by any (reasonable) amount desired.

The Introduction of a Non-Linear Element

- For the first time in our discussion, have introduced a "non-linear" transverse magnetic field for use in the accelerator system — sextuples for chromatic and/or chromaticity correction
- This opens the door to new and interesting phenomena:
  - phase space distortions
  - tune variation with amplitude
  - dynamic aperture