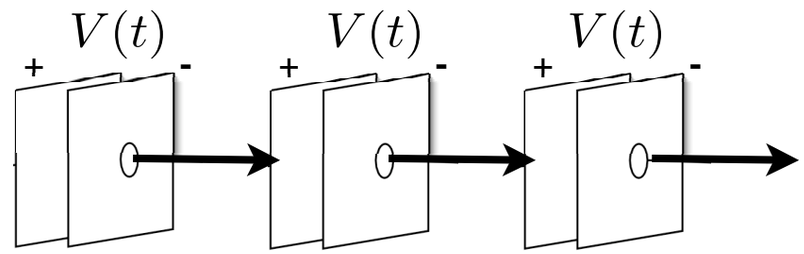
Linacs and Synchrotrons

- Essential difference:
  - pass *N* cavities 1 time each

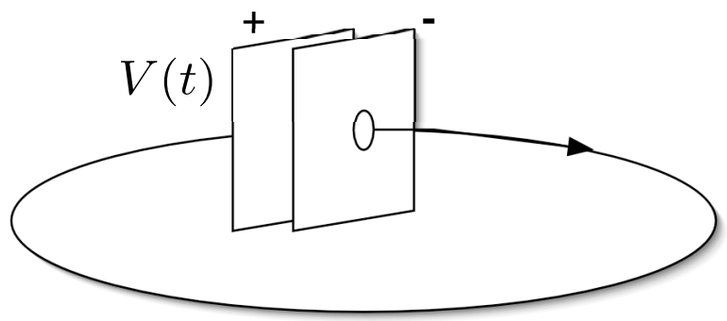
Linear Accelerator



— or —

  - pass 1 cavity *N* times

Circular Accelerator

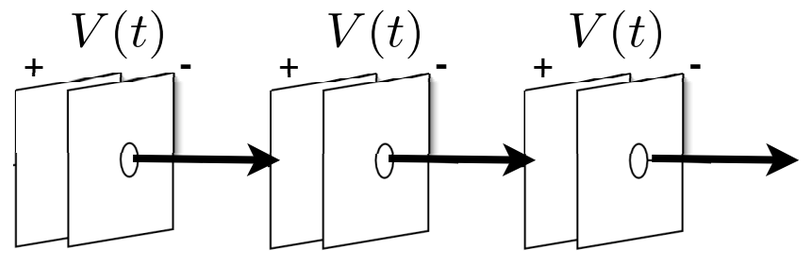


  - otherwise, essentially the same longitudinal dynamics

Linacs and Synchrotrons

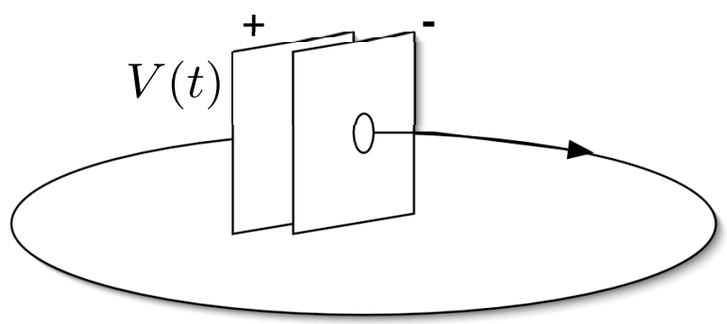
- Linac cavities can have different frequencies, each at different phases (e.g., FRIB); but typically one frequency, at least for major sections of the linac

Linear Accelerator



- Synchrotron — with only 1 cavity system, — inherently same frequency, though its value must change if particle speed changes during acceleration (protons, ions)
- Must consider time of flight between cavities / passages

Circular Accelerator



Repetitive Systems of Acceleration

- We will assume that particles are propagating through a system of accelerating cavities. Each cavity has oscillating fields with frequency f_{RF}, and maximum "applied" voltage V (i.e., this takes into account TTF's, etc.). The ideal particle would arrive at the cavity at phase ϕ_s.
- We will choose ϕ_s to be relative to the "positive zero-crossing" of the RF wave, such that the ideal particle acquires an energy gain of

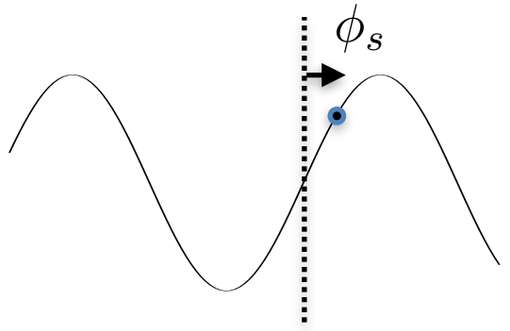
$$\Delta E_s = \Delta W_s = qV \sin \phi_s$$

  - this definition used for synchrotrons; linacs more often define ϕ_s relative to the "crest" of the RF wave
    - apologies for this possible *further* confusion…
      - the physics, of course, is the same

Acceleration of Ideal Particle

Wish to accelerate the ideal particle. As this particle exits the (*n+1)-th* RF cavity/station we would have

$$E_s^{(n+1)} = E_s^{(n)} + QeV \sin\phi_s$$



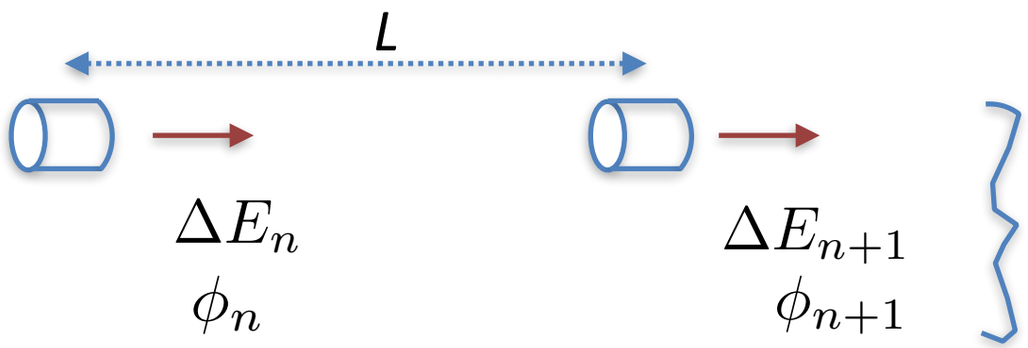
If we are considering a synchrotron, we can consider the above as the total energy gain on the (*n+1)-th* revolution. The ideal energy gain per second would be:

$$dE_s/dt = f_0 QeV \sin\phi_s \qquad f_0 = \text{revolution frequency}$$

Next, look at (longitudinal) motion of particles near the ideal particle: ϕ = phase w.r.t. RF system

$\Delta E \equiv E - E_s$ = energy difference from the ideal

- Assume accelerating system of cavities is set up such that ideal particle arrives at each cavity when the accelerating voltage *V* is at the same phase (called the "synchronous phase"); consider at "test" particle:



$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

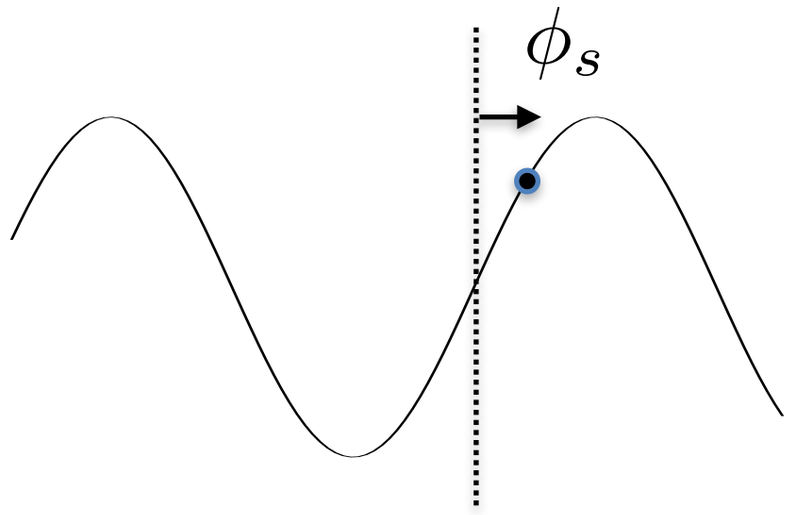
$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

(*difference equations*)

Notes:

$$h = L/\beta\lambda, \quad \lambda = c/f_{\rm rf} \quad \text{or,} \quad h = f_{\rm rf}L/v$$

Desire *h* to be an integer.
If *L* is circumference of a synchrotron then: $h = f_{\rm rf}/f_0$
where f_0 is the revolution frequency,
In this case, *h* is called the "harmonic number"



$$E = mc^2 + W; \qquad \Delta E \Leftrightarrow \Delta W$$

Applying the Difference Equations

```
while (i < Nturns+1) {
   phi = phi + k*dW
   dW  = dW + QonA*eV*(sin(phi)-sin(phis))
   points(phi*360/2/pi, dW, pch=21,col="red")
   i = i + 1
 }
```

Let's run a code…

v0_RFtrack.R

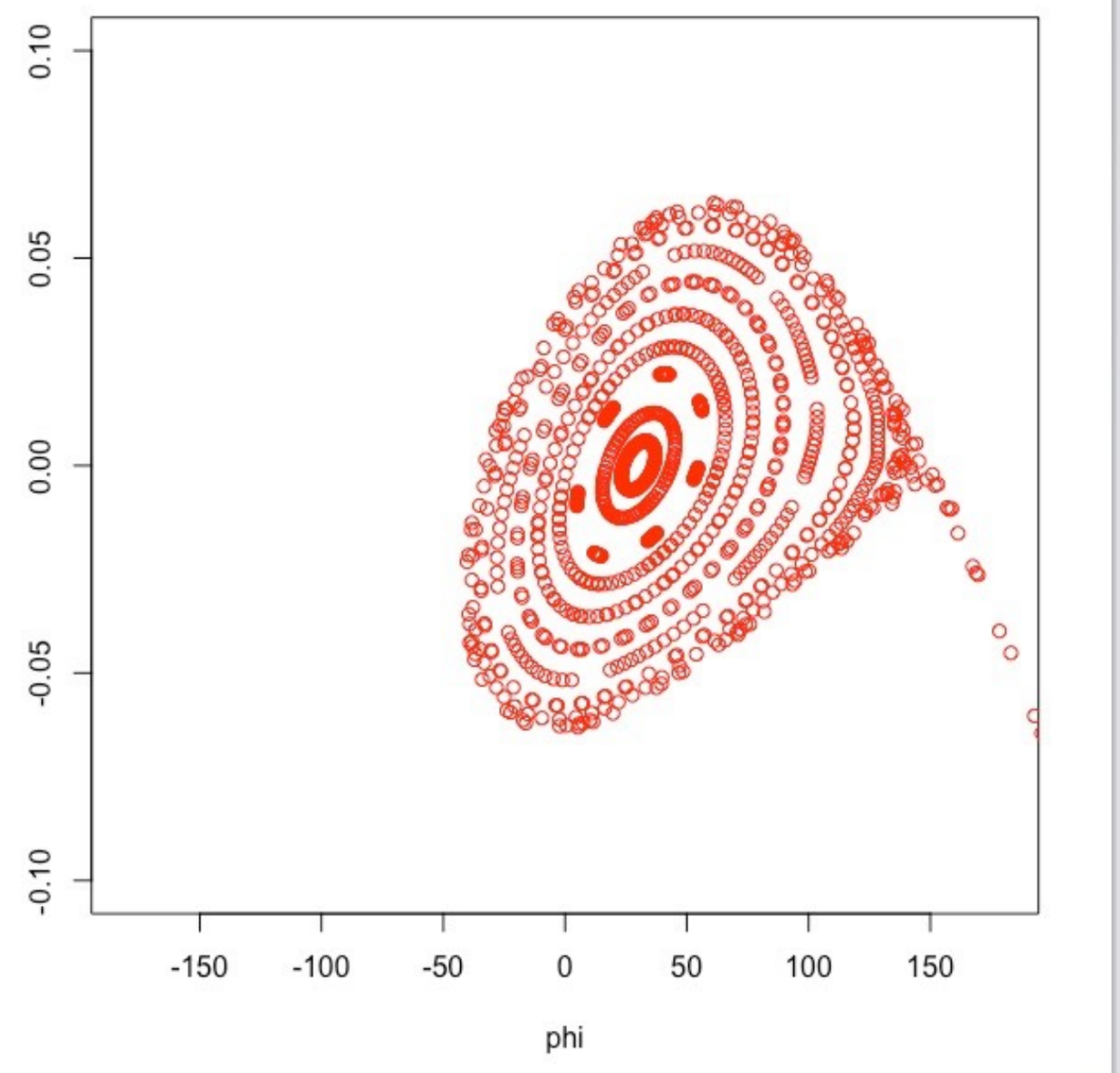
Source on Save | Run | Source

```
# Program to plot longitudinal phase space motion
#  through a system of cavities (just an example...)

Nturns = 100

# Some Parameters
Ws = 1.0 # MeV/u
phis = 30*pi/180  # synchronous phase angle
eV = 0.2 # MeV/u
QonA = 0.25
gamma = (931+Ws)/931
beta = sqrt(1-1/gamma^2)
eta = -1/gamma^2
h = 1/(beta*3e8/80.5e6)
k = 2*pi*h*eta/beta^2*(gamma-1)/gamma/Ws

# initialize the phase space plot
phi    = 0
dW     = 0
plot(phi, dW, xlim=c(-180,180), ylim=c(-0.1,0.1), typ="n")

trk = 1
while (trk < 16) {
# initialize particle positions in phase space
  u0 <- locator(1)
  phi    <- u0$x/180*pi
  dW     <- u0$y
# track the particle...
  i = 1
  while (i < Nturns+1) {
    phi = phi + k*dW
    dW  = dW + QonA*eV*(sin(phi)-sin(phis))
    points(phi*360/2/pi, dW, pch=21,col="red")
    i = i + 1
  }
trk = trk + 1
}
```

38:1 (Top Level) R Script

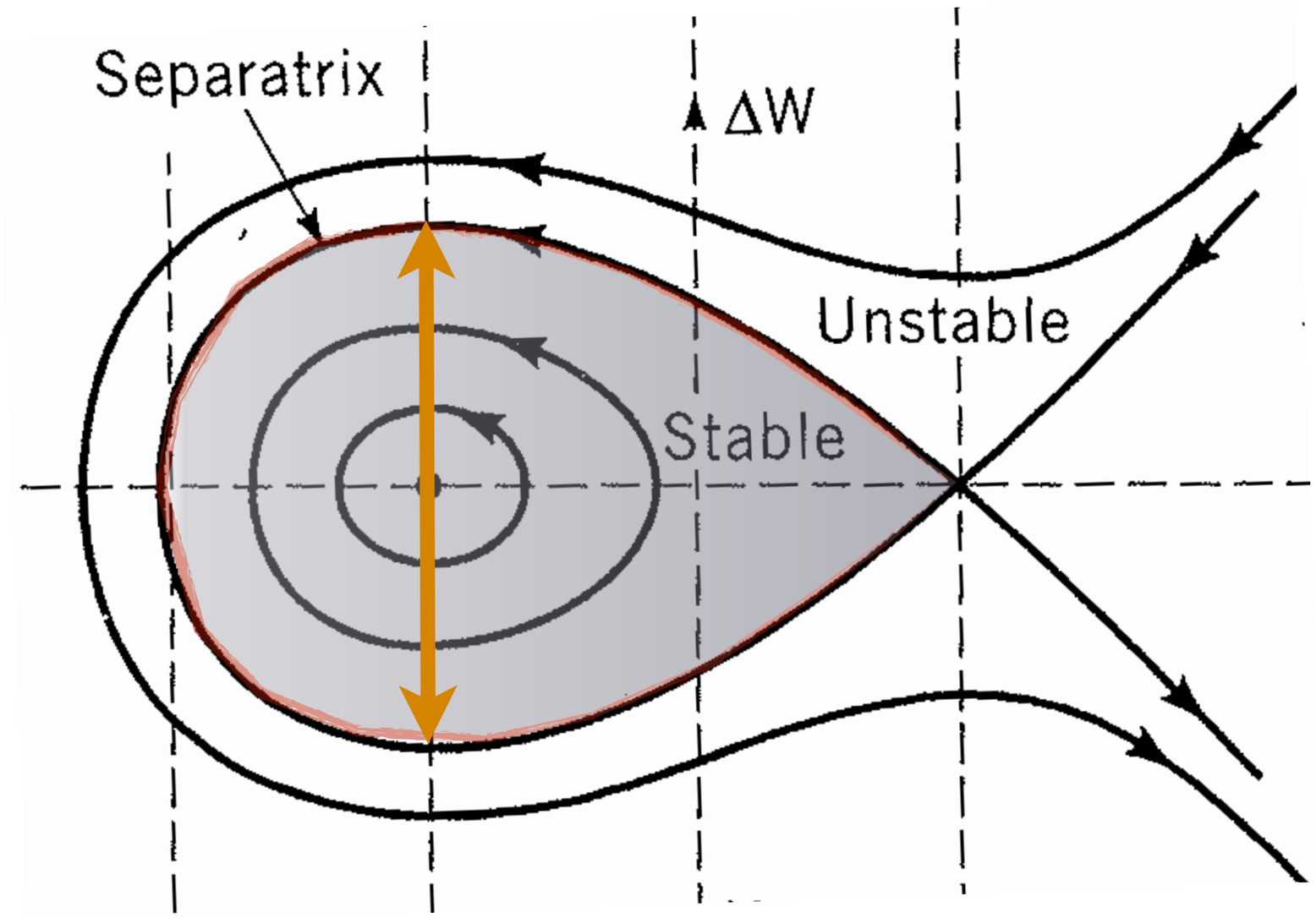
Files | Plots | Packages | Help | Viewer

Zoom | Export | Publish



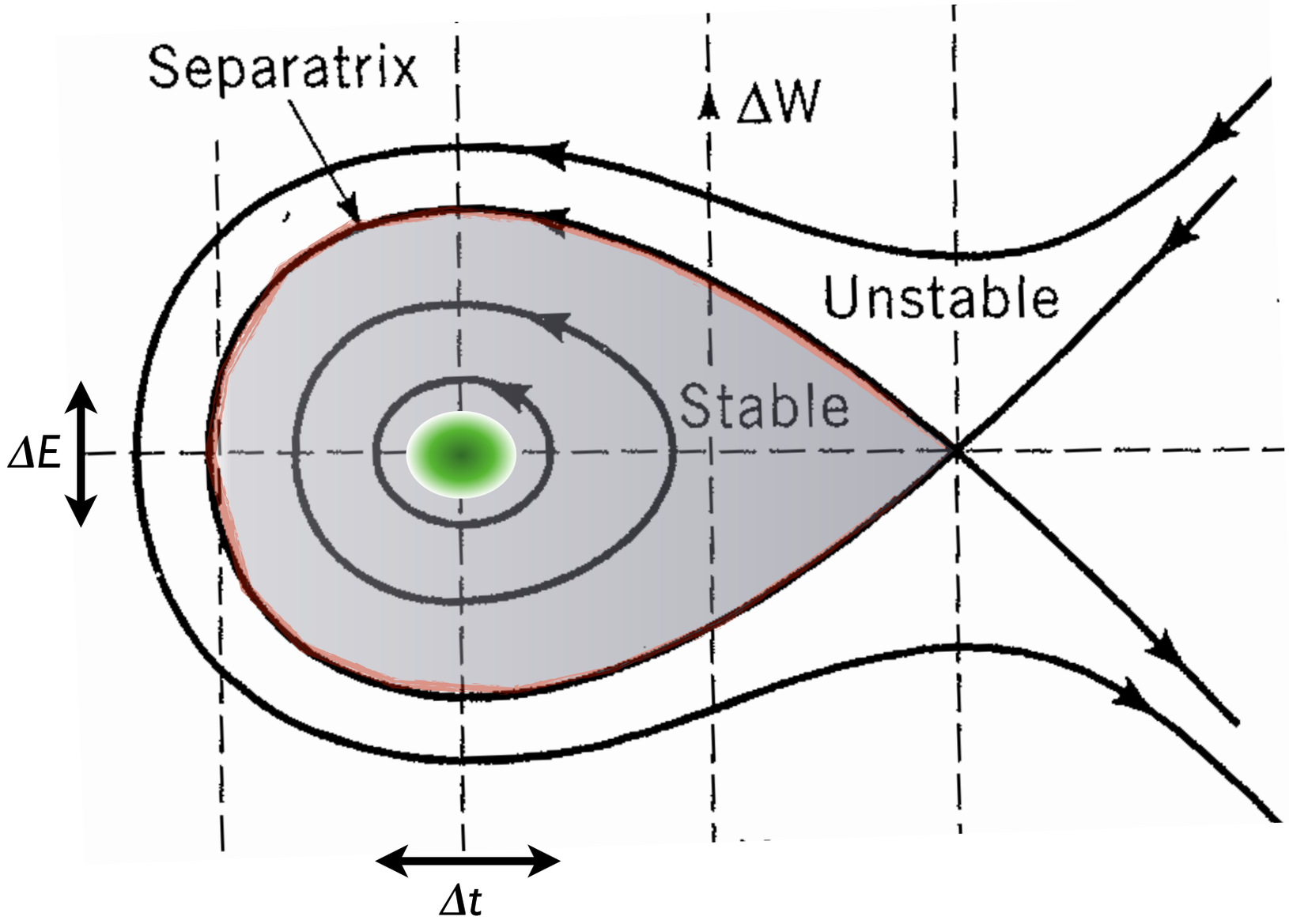
Acceptance and Emittance

- Stable region often called an RF “bucket”
  - “contains” the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system



Acceptance and Emittance

- Stable region often called an RF “bucket”
  - “contains” the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system
- Desire the beam particles to occupy much smaller area in the phase space



area: “eV-sec”
Note: *E, t canonical*

differential approach…

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

start with above difference eqs

$$\rightarrow \frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E}\,\Delta E, \qquad \frac{d\Delta E}{dn} = QeV(\sin\phi - \sin\phi_s)$$

$$\rightarrow \frac{d^2\phi}{dn^2} = \frac{2\pi h\eta}{\beta^2 E}\frac{d\Delta E}{dn} = \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) \qquad (1)$$

$$\Rightarrow \frac{d^2\phi}{dn^2} - \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) = 0$$

find 1st integral:

$$\int \left(\frac{d^2\phi}{dn^2}\right)\frac{d\phi}{dn}\,dn - \frac{2\pi h\eta}{\beta^2 E}QeV\int(\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn = 0$$

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{2\pi h\eta}{\beta^2 E}QeV(\cos\phi + \phi\sin\phi_s) = constant$$

or,

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta}QeV(\cos\phi + \phi\sin\phi_s) = constant \qquad (2)$$

The equation of the ***trajectories*** in phase space!

Synchrotron Oscillations

- Particles near the synchronous phase and ideal energy will oscillate about the synchronous particle with the "synchrotron frequency" (this is call synchrotron motion, even for a linac!) In a synchrotron, …
  - "synchrotron tune" == # of synch. osc.'s per revolution

compute small oscillation frequency:

in (1), let $\phi = \phi_s + \Delta\phi \quad \rightarrow \quad \sin\phi - \sin\phi_s = \sin\phi_s\cos\Delta\phi + \cos\phi_s\sin\Delta\phi - \sin\phi_s$

$$\approx \Delta\phi\ \cos\phi_s$$

$$\Rightarrow \frac{d^2\Delta\phi}{dn^2} - \left(\frac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s\right)\Delta\phi = 0 \qquad \Rightarrow \qquad \boxed{\nu_s = \sqrt{-\frac{h\eta QeV}{2\pi\beta^2 E}\cos\phi_s}}$$

$(2\pi\nu_s)^2$

if $\eta > 0$, choose $\cos\phi_s < 0$

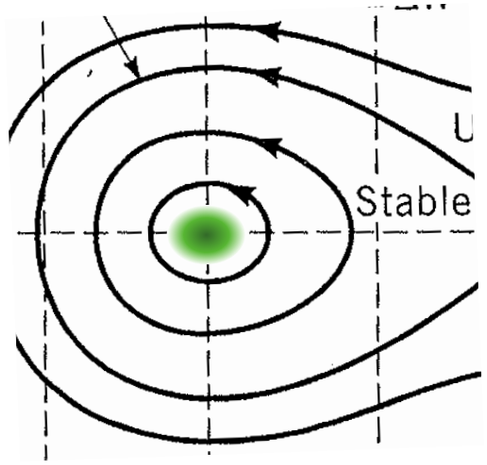
Comment on Frequencies of the Motion

- From what we've just seen, the synchrotron motion in a circular accelerator takes many (perhaps hundreds of) revolutions to complete one synchrotron period
- On the other hand, in the transverse plane, a particle will typically undergo many betatron oscillations during one revolution
- Thus, transverse/longitudinal dynamics typically occur on very different time scales — this actually justifies us studying them independently

Motion Near the Ideal Particle

Linearize the motion, and write in matrix form…

$$\phi = \phi_s + \Delta\phi$$



$$\begin{aligned}
\phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s) \\
&= \Delta E_n + QeV(\sin\phi_s\cos\Delta\phi_{n+1} + \sin\Delta\phi_{n+1}\cos\phi_s) - \sin\phi_s) \\
&= \Delta E_n + QeV\cos\phi_s\ \Delta\phi_{n+1} \\
&= \Delta E_n + QeV\cos\phi_s\left[\Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta\phi_{n+1} &= \Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= QeV\cos\phi_s\Delta\phi_n + \left(1 + \frac{2\pi h\eta}{\beta^2 E}QeV\cos\phi_s\right)\Delta E_n
\end{aligned}$$

or,

$$\begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_{n+1} = \begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ QeV\cos\phi_s & \left(1+\frac{2\pi h\eta}{\beta^2 E}QeV\cos\phi_s\right) \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_n$$

$$= \begin{pmatrix} 1 & 0 \\ QeV\cos\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_n$$

$$M = M_c \cdot M_d$$

"thin" cavity (*acts as longitudinal focusing element*) · *drift*

Note: for $\eta < 0$, M_d is a "backwards" drift; i.e., $\Delta\phi$ decreases for $\Delta E>0$

(when no bending)

$\eta = -1/\gamma^2$ in straight region (linac)

Remember from transverse motion, $x \propto \sqrt{\beta}\sin\Delta\psi$
and when *M* was periodic,

$$M = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \qquad \text{and} \quad trM = 2\cos\Delta\psi$$

$\Delta\psi$ = phase advance through periodic section

Can imagine "longitudinal" β, α, γ, $\Delta\psi$ parameters as well
Note: from *M* of previous page, if represents periodic structure (synchrotron or portion of linac), then

$$trM = 2 + \frac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s = 2\cos\Delta\psi_s$$

longitudinal phase advance

$$\Delta\psi_s = 2\pi\nu_s$$

oscillation frequency w.r.t. cavity number, "*n*" (e.g., synchrotron *tune*)

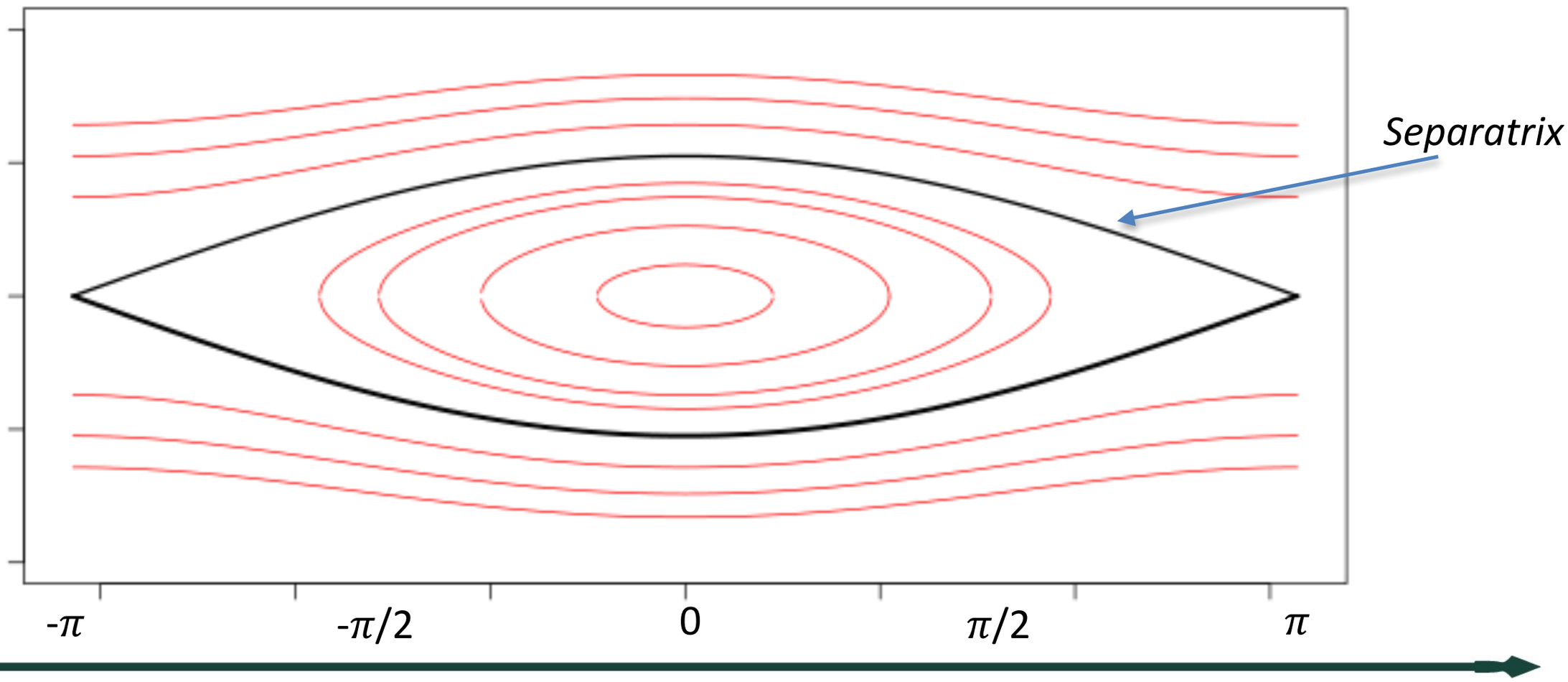
$$\cos\Delta\psi_s \approx 1 - \frac{1}{2}(\Delta\psi_s)^2 = 1 + \frac{\pi h\eta}{\beta^2 E} QeV\cos\phi_s \left[= \frac{1}{2}trM\right]$$

$$\Rightarrow \quad \nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV\cos\phi_s}$$

as found previously!

The Stationary Bucket

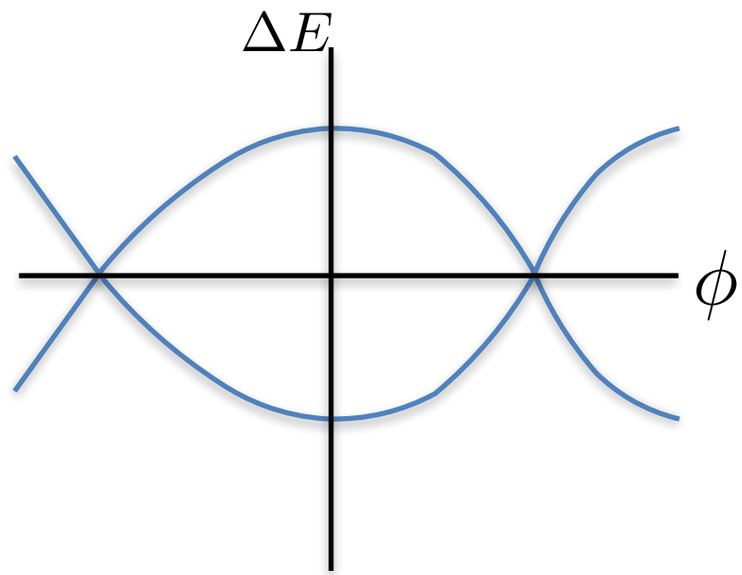
- Suppose do not wish to accelerate the ideal particle…
  - for lower energies, where the slip factor is negative, then need to choose $\phi_s = 0^o$



"stationary" bucket: $\phi_s = 0,\ 2\pi \quad (\sin\phi_s = 0)$ —> no average acceleration

anticipate stability: —> choose $\phi_s = 0, \quad \eta < 0$

then, $$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV\cos\phi = constant$$



on the separatrix: $\Delta E = 0 \quad \text{at} \quad \phi = \pm\pi$

$$0 - 2\frac{\beta^2 E}{2\pi h\eta} QeV = constant$$

thus, the Eq. of separatrix: $$\Delta E^2 + (1+\cos\phi)\frac{\beta^2 E}{\pi h\eta} QeV = 0$$

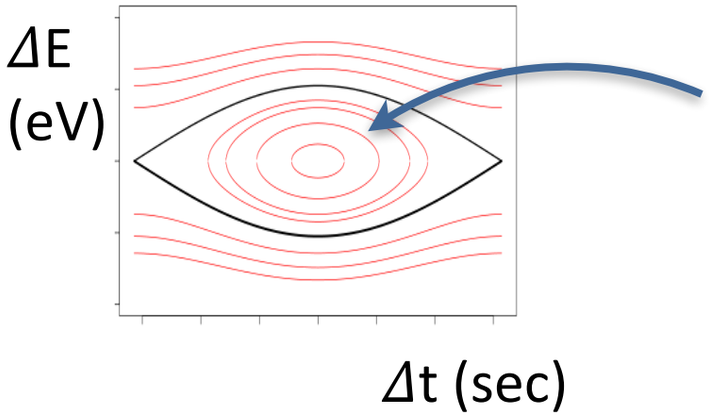
$$\Delta E^2 + \frac{2\beta^2 E}{\pi h\eta} QeV\cos^2(\phi/2) = 0$$

separatrix: $$\Delta E = \pm\sqrt{-\frac{2\beta^2 E}{\pi h\eta} QeV\cos(\phi/2)}$$ *(for "stationary bucket")*

thus, “bucket height”: $$a = \sqrt{\frac{2\beta^2 E}{\pi h|\eta|} QeV}$$

Phase space area of a stationary bucket: $$4\int_0^{\pi} a\ \cos(\phi/2)\ d\phi = 8a$$

and, if use *ΔE-Δt* coordinates rather than *ΔE-ϕ*, then area of a *stationary* bucket is…

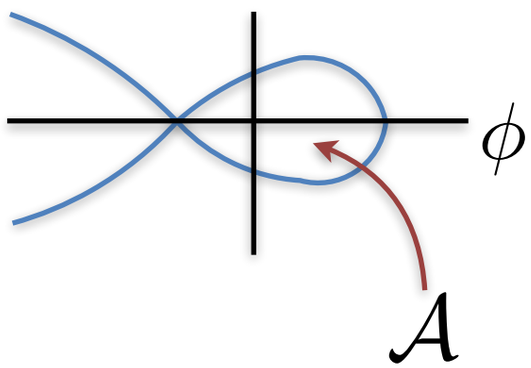


$$\mathcal{A}_0 \equiv \frac{8}{\pi f_{\rm rf}} \sqrt{\frac{\beta^2 EQeV}{2\pi h|\eta|}}$$

(*here, units of* eV-sec)

since $\phi = 2\pi f_{\rm rf} t$

Note: for $\sin\phi_s \neq 0$

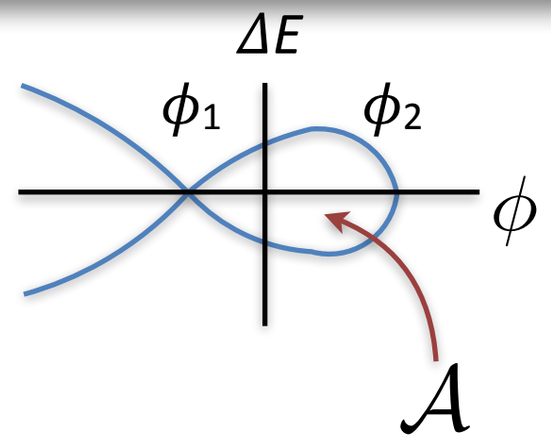


$$\mathcal{A} = \mathcal{A}_0 \cdot \mathcal{F}(\phi_s)$$

where $0 < \mathcal{F} < 1$

(determined numerically)

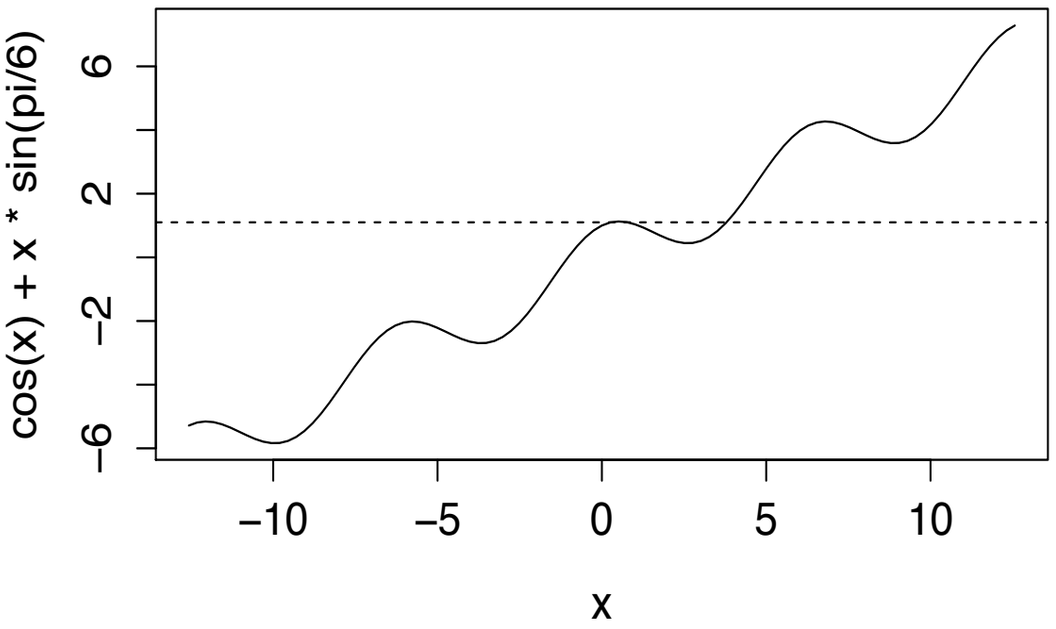
Area of a Moving Bucket



—> net average acceleration

curve: $$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi + \phi\sin\phi_s) = constant$$

"kinetic"-like *"potential"-like* *"total Energy"-like*



ϕ_1 is where "potential like" has derivative = 0: $\phi_1 = \pi - \phi_s$

Given $\phi_1 = \pi - \phi_s$, can now determine the "constant": $\Delta E = 0$ at ϕ_1, and so…

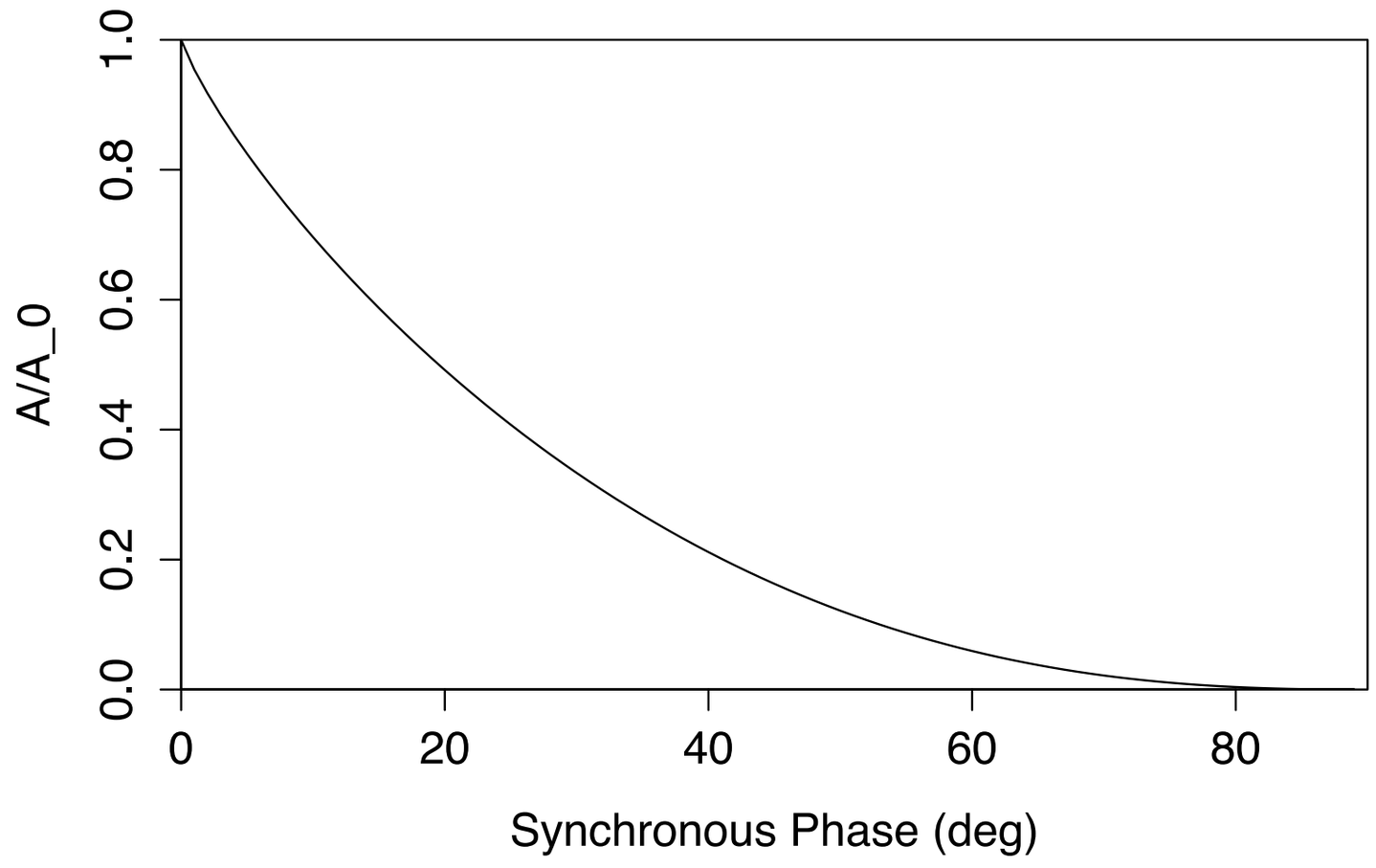
$$(0)^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi_1 + \phi_1\sin\phi_s) = constant$$

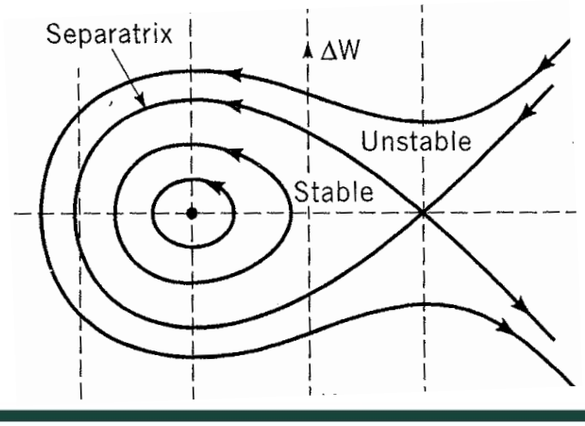
Then, find that ϕ_2 must satisfy:

$$\cos\phi_2 + \phi_2\sin\phi_s + \cos\phi_s + (\pi - \phi_s)\sin\phi_s = 0$$

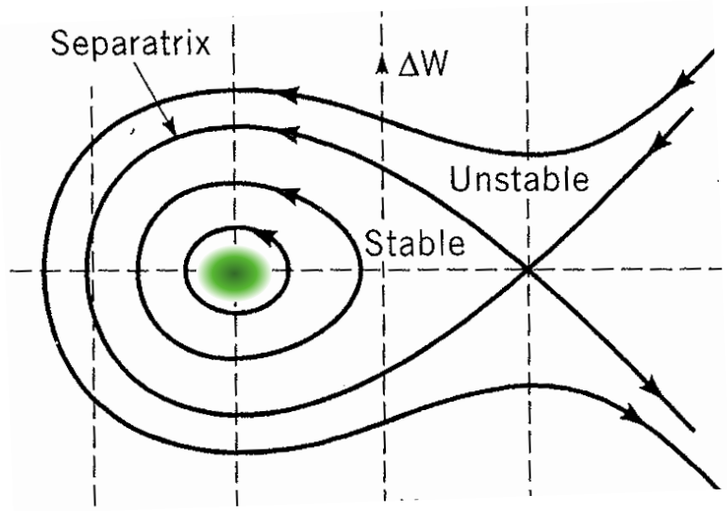
Numerical Solution for Bucket Area

```
# Solve for bucket area; phis = 0 is "stationary"

Xout <- array(0,dim=c(91,4))
phisDeg <- -1

for(i in (1:90)){
phisDeg <- phisDeg + 1
phis <- phisDeg*pi/180

f <- function(x){
   cos(x)+x*sin(phis)+cos(phis)-(pi-phis)*sin(phis) }
dE <- function(x){
   sqrt(cos(phis)-(pi-phis)*sin(phis)+cos(x)+x*sin(phis)) }

phi1 <- pi-phis
phi2 <- uniroot( f, c(-pi, 2*pi))$root
A <- -1/4/sqrt(2)*integrate(dE, phi1, phi2)$value

Xout[i,] = c(phis*180/pi, phi1*180/pi, phi2*180/pi, A)  }

plot(Xout[,1],Xout[,4],typ="l",
   xlab="Synchronous Phase (deg)", ylab="A/A_0",
   xaxs="i", yaxs="i",xlim=c(0,90))

Xout
```





Back to Small Oscillations…



from (2), $\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi + \phi\sin\phi_s) = constant$

if $\phi = \phi_s + \Delta\phi$ (*small*), then …

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi_s\cos\Delta\phi - \sin\phi_s\sin\Delta\phi + (\phi_s + \Delta\phi)\sin\phi_s) = constant$$

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi_s(1 - \frac{1}{2}\Delta\phi^2) - \sin\phi_s\Delta\phi + \phi_s\sin\phi_s + \Delta\phi\sin\phi_s) = constant$$

$$\Delta E^2 + \left(-\frac{\beta^2 E}{2\pi h\eta} QeV\cos\phi_s\right)\Delta\phi^2 = constant \qquad (3)$$

This Eqn. represents trajectories in longitudinal phase space of particles ***near*** the ideal particle.

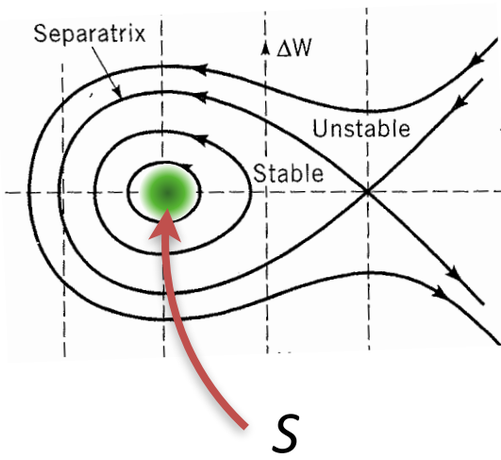
Beam Longitudinal Emittance

Suppose beam is well contained within an ellipse given by (3), and suppose we know either $\Delta\hat{E}$ or $\Delta\hat{\phi}$ (or, $\Delta\hat{t}$) of the distribution (i.e., maximum extent). Then, the *constant* is easily seen to be:

$$constant = \Delta\hat{E}^2 = -\frac{\beta^2 E}{2\pi h\eta} QeV \cos\phi_s \Delta\hat{\phi}^2$$

So, area of ellipse (the *longitudinal emittance*) is: $\pi\ \Delta E \Delta\phi$

or, in *E*-*t* coordinates, $S \equiv \pi\ \Delta\hat{E}\Delta\hat{t} = \pi\ \Delta\hat{E}\dfrac{\Delta\hat{\phi}}{2\pi f_{\rm rf}}$



$$\Rightarrow\quad S = \frac{1}{2f_{\rm rf}}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta} Q\cos\phi_S}\ \ \Delta\hat{\phi}^2$$

or,

$$S = 2\pi^2 f_{\rm rf}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta} Q\cos\phi_S}\ \ \Delta\hat{t}^2$$

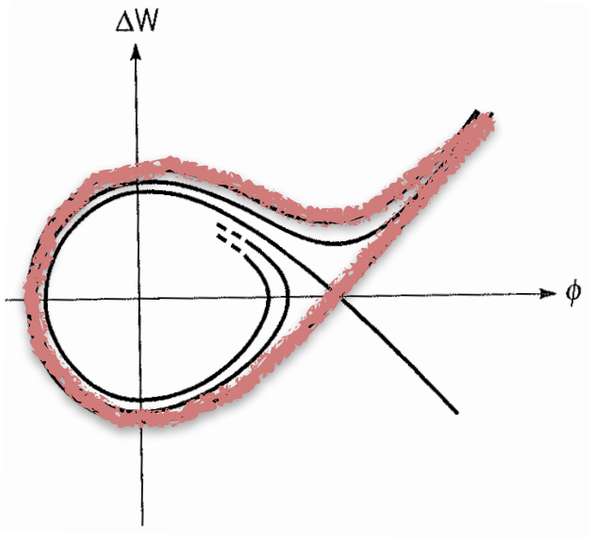
units: "eV-sec"

Golf Clubs *vs.* Fish

- Our analysis “assumes” slowly changing variables (including the energy gain!). Quite reasonable in many Alvarez-style linacs and in synchrotrons
- In linacs, fractional energy change can be large, and so this will distort the phase space
- Plots from Wangler’s book:

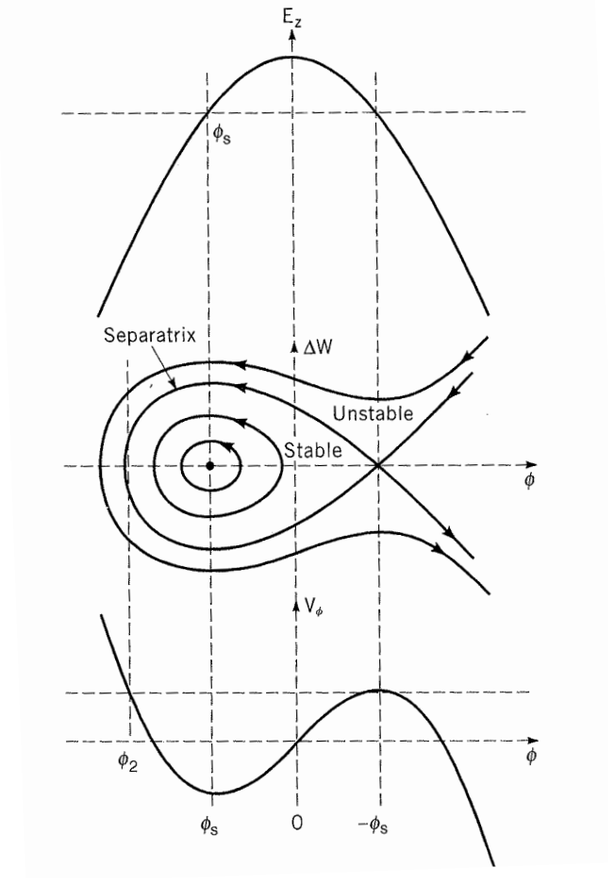
Here, a more rapid acceleration is included

(linac)



Here, assume that energy is “constant” or varying very slowly

(synchrotron)



Transition Energy

- In a synchrotron, there can be an energy at which the slip factor changes sign — this is call the “transition energy”

$$\eta = \alpha_p - \frac{1}{\gamma^2} = \left\langle \frac{D}{\rho} \right\rangle - \frac{1}{\gamma^2}$$

$$\eta = 0 = \alpha_p - \frac{1}{\gamma^2}$$

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

$$\gamma_t \equiv \frac{1}{\sqrt{\alpha_p}}$$

- In a typical FODO-style synchrotron, the transition gamma is roughly equal to the betatron tune

Transition

We had... $\Rightarrow \frac{d^2\Delta\phi}{dn^2} - \left(\frac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s\right)\ \Delta\phi = 0$

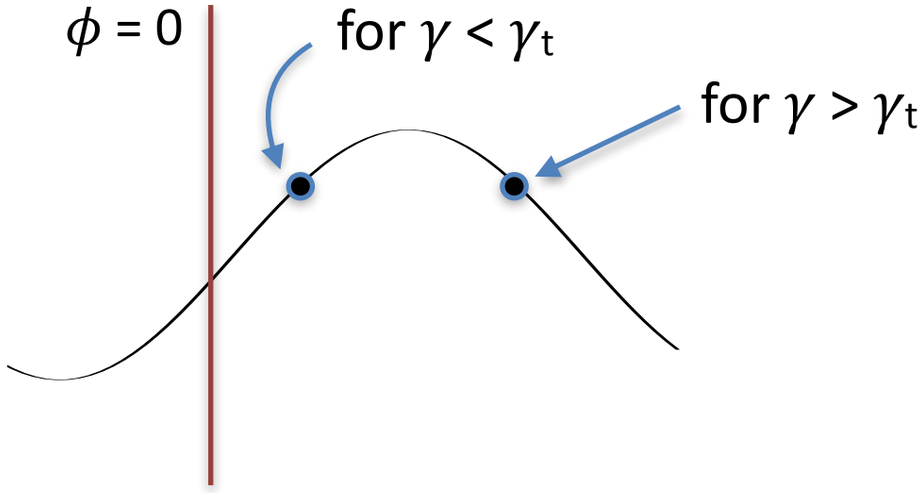
$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV\cos\phi_s}$$

if $\eta > 0,$ choose $\cos\phi_s < 0$

So,

when $\eta < 0$, we want cos $\phi_s > 0$

when $\eta > 0$, we want cos $\phi_s < 0$

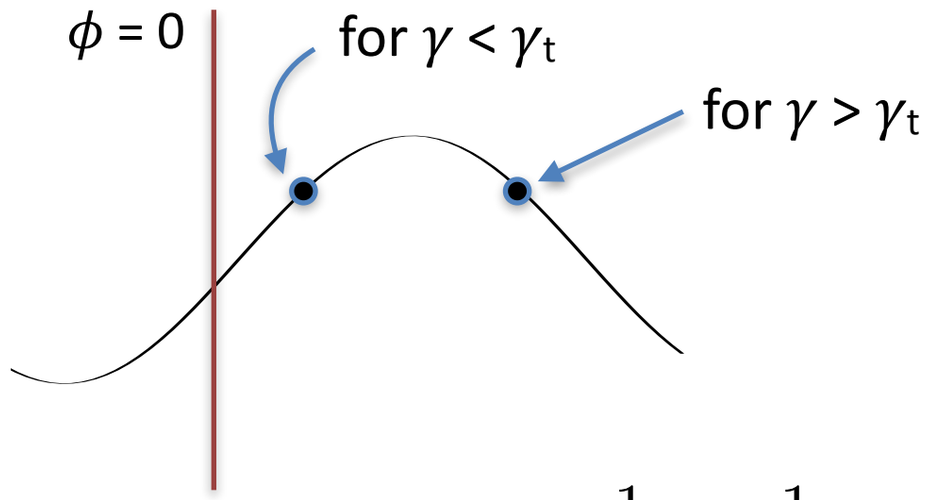


$\therefore$ if γ_t exists, need "phase jump" to occur at transition crossing

$$\gamma_t mc^2 = \text{transition energy}$$

Transition Crossing

- If the synchrotron accelerates through its transition energy, then the phase of the RF system has to be shifted at the time of transition crossing
- The synchrotron motion slows down as approach transition — it would stop if the slip factor were exactly zero!
  - loss of phase stability!
  - momentum spread also gets larger near transition
- So, best to accelerate quickly through this energy region!



$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV \cos\phi_s}$$

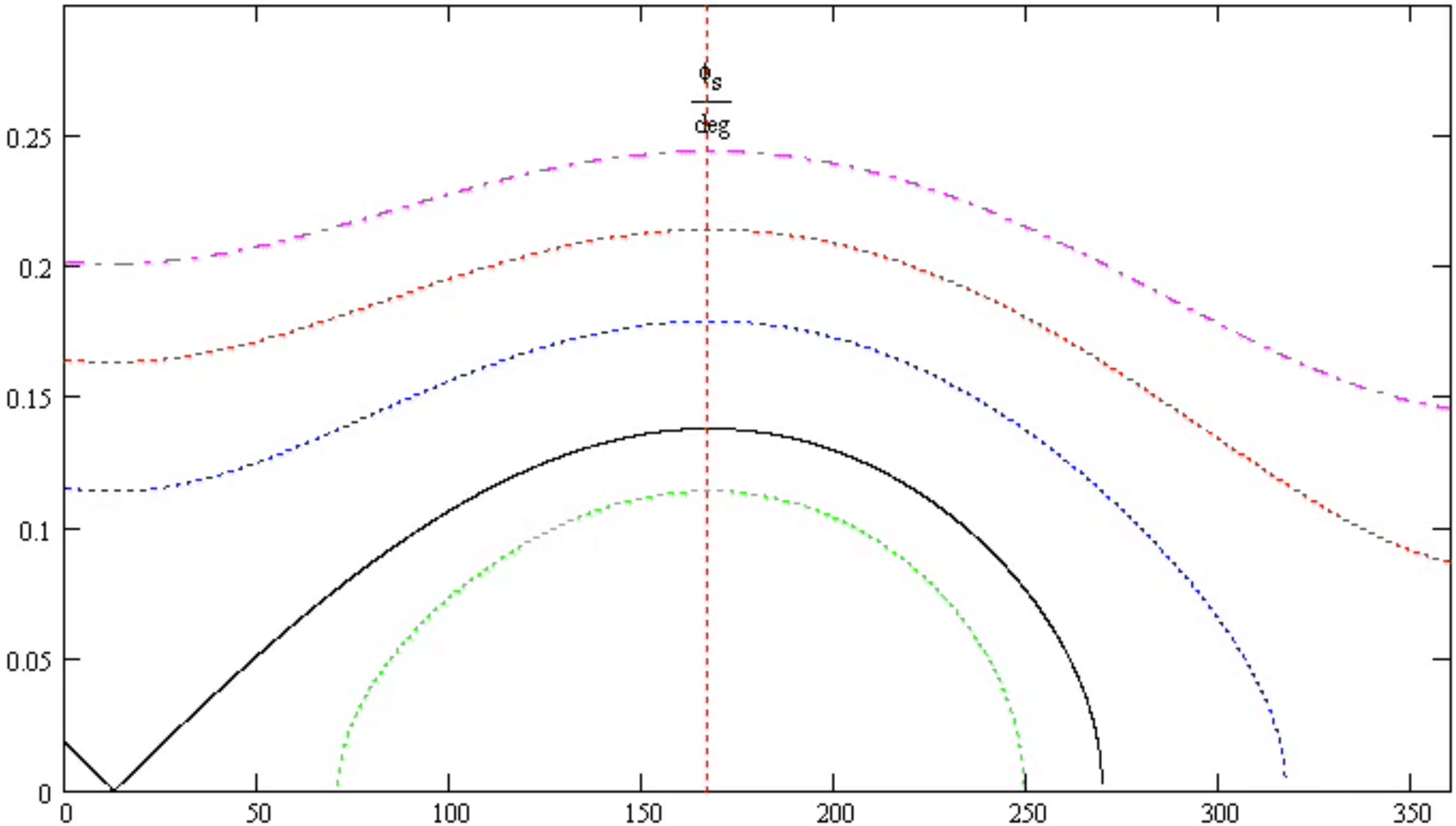
Buckets, Bunches, Batches, …

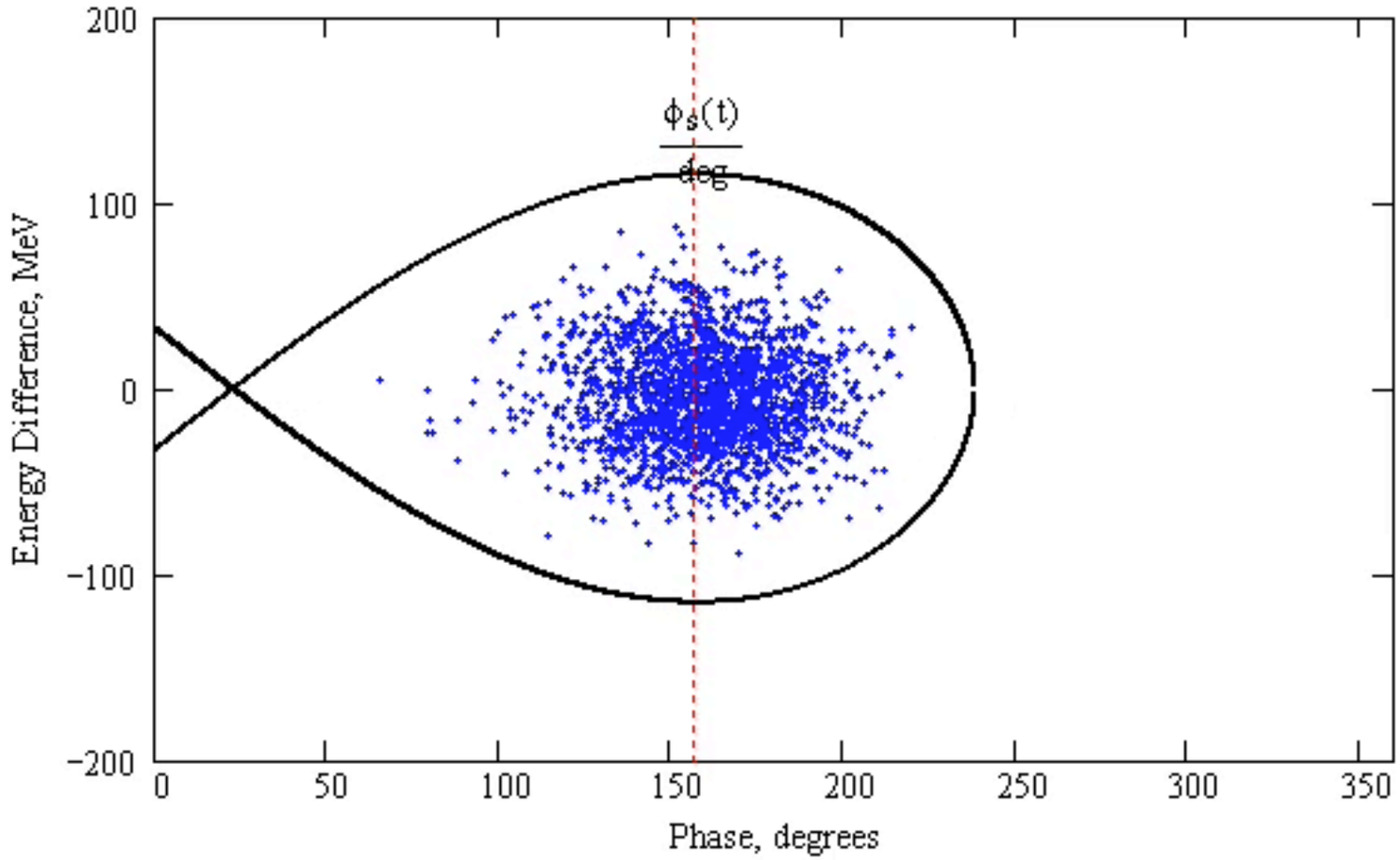
- Have seen definition of “buckets” — stable phase space area
- Buckets can be occupied by “bunches” of particles
  - note: need not be — can have “empty buckets”
  - thus, can (in principle) adjust bunch spacing, bunch arrangements, etc.
- A set of bunches that are created in an accelerator (pulsed) is often called a Batch (especially if from a synchrotron)
  - can also be called a Bunch Train as well (especially if from a linac)

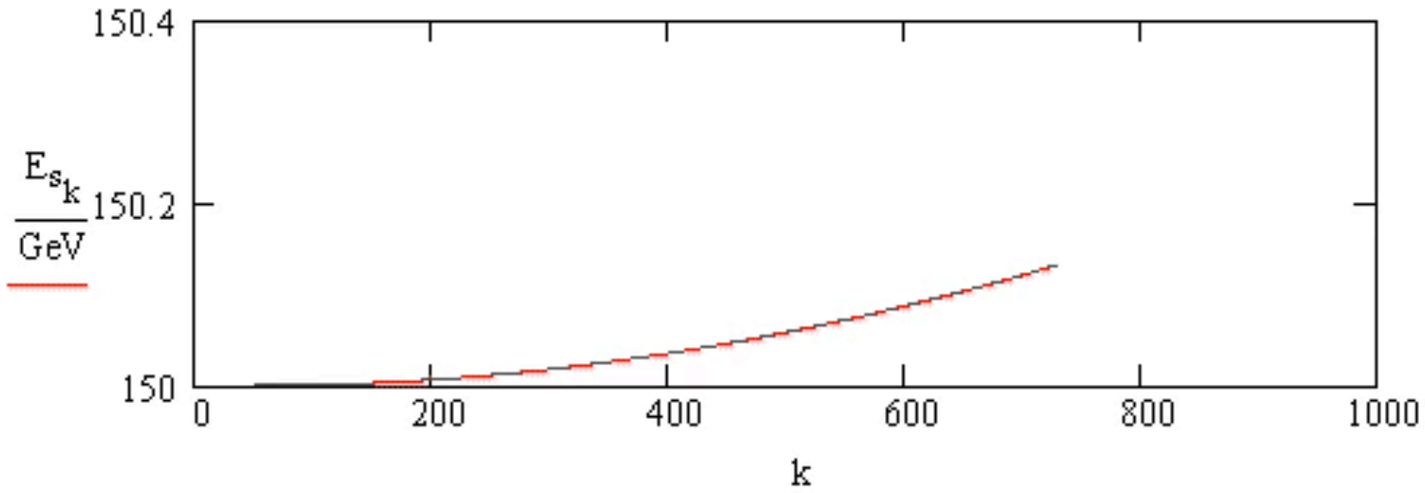
Some Movies…

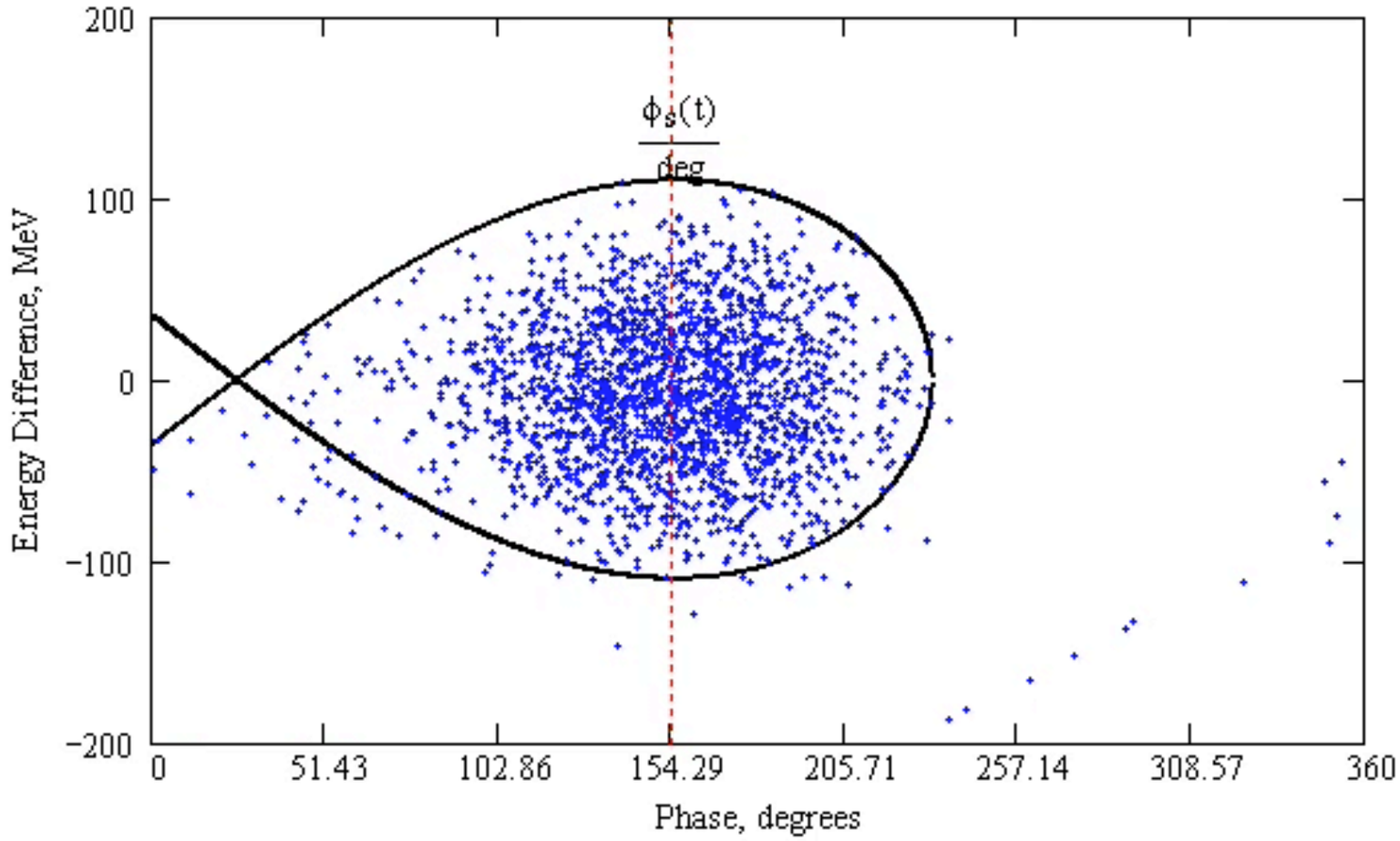
- Bucket Transformation
- Snap Capture
- Adiabatic Capture
- Parabolic acceleration
- Parabolic acceleration — full bucket
- Transition Crossing

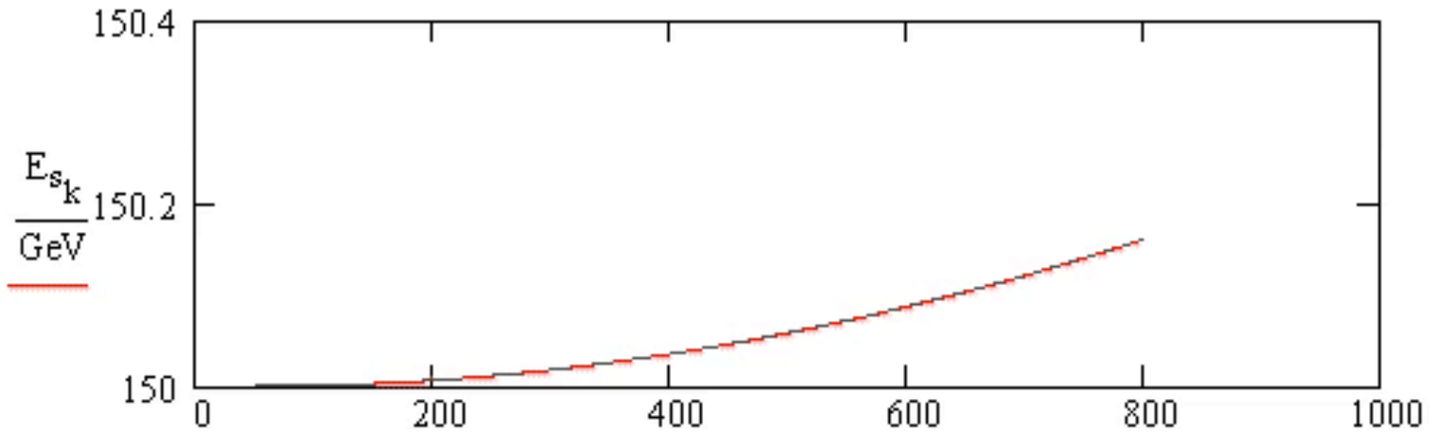
Phase space contours, for various values of k. Synchronous phase: $\phi_s = 167.25\,\text{deg}$

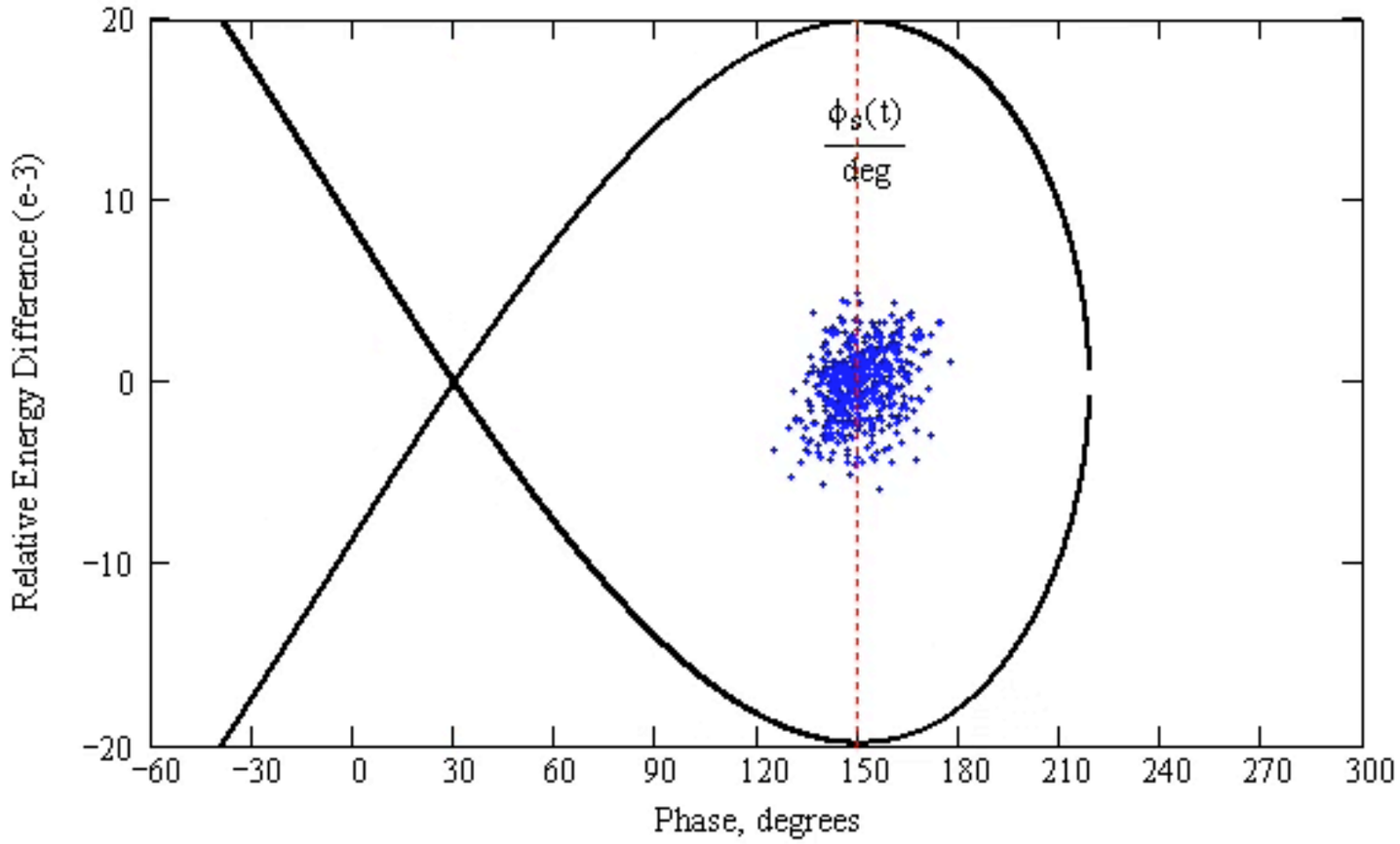



200
100
0
−100
−200
Energy Difference, MeV
$\phi_s(t)$
deg
0
50
100
150
200
250
300
350
Phase, degrees


150.4
150.2
150
E_{s_k}
GeV
0
200
400
600
800
1000
k

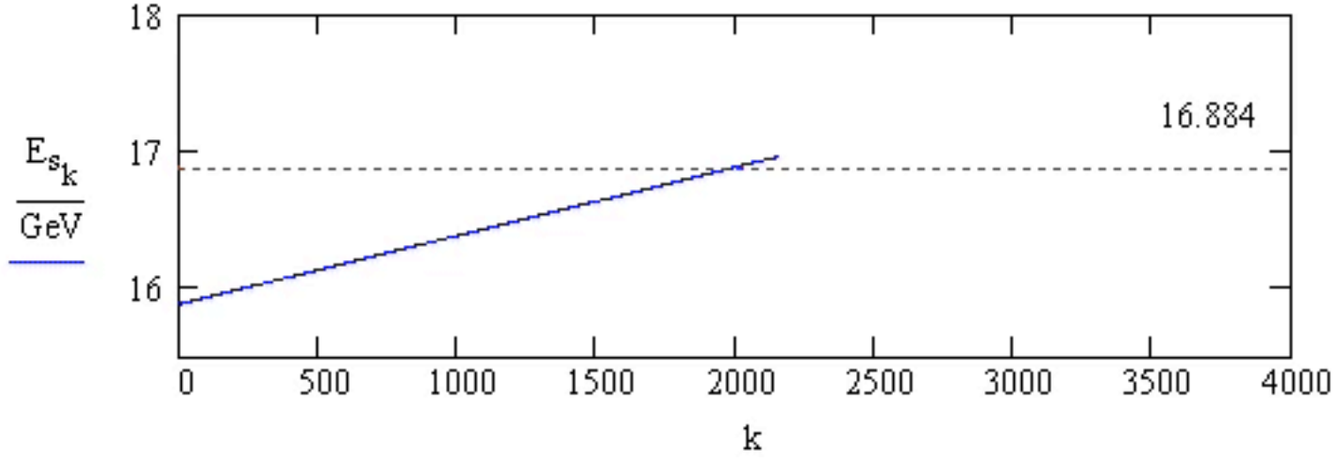

200
100
0
−100
−200
Energy Difference, MeV
$\phi_s(t)$
deg
0
51.43
102.86
154.29
205.71
257.14
308.57
360
Phase, degrees

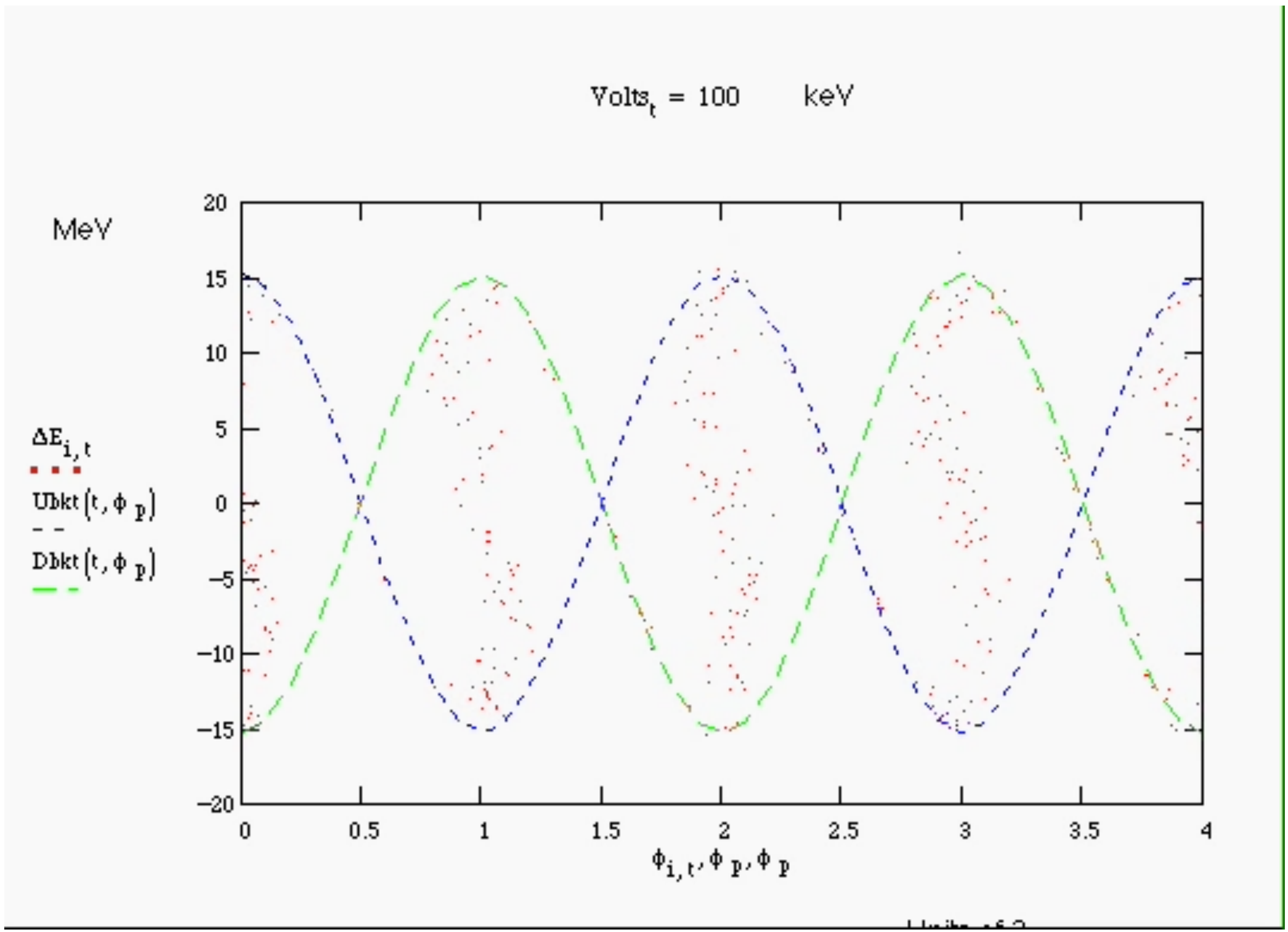

150.4
150.2
150
E_{s_k}
GeV
0
200
400
600
800
1000

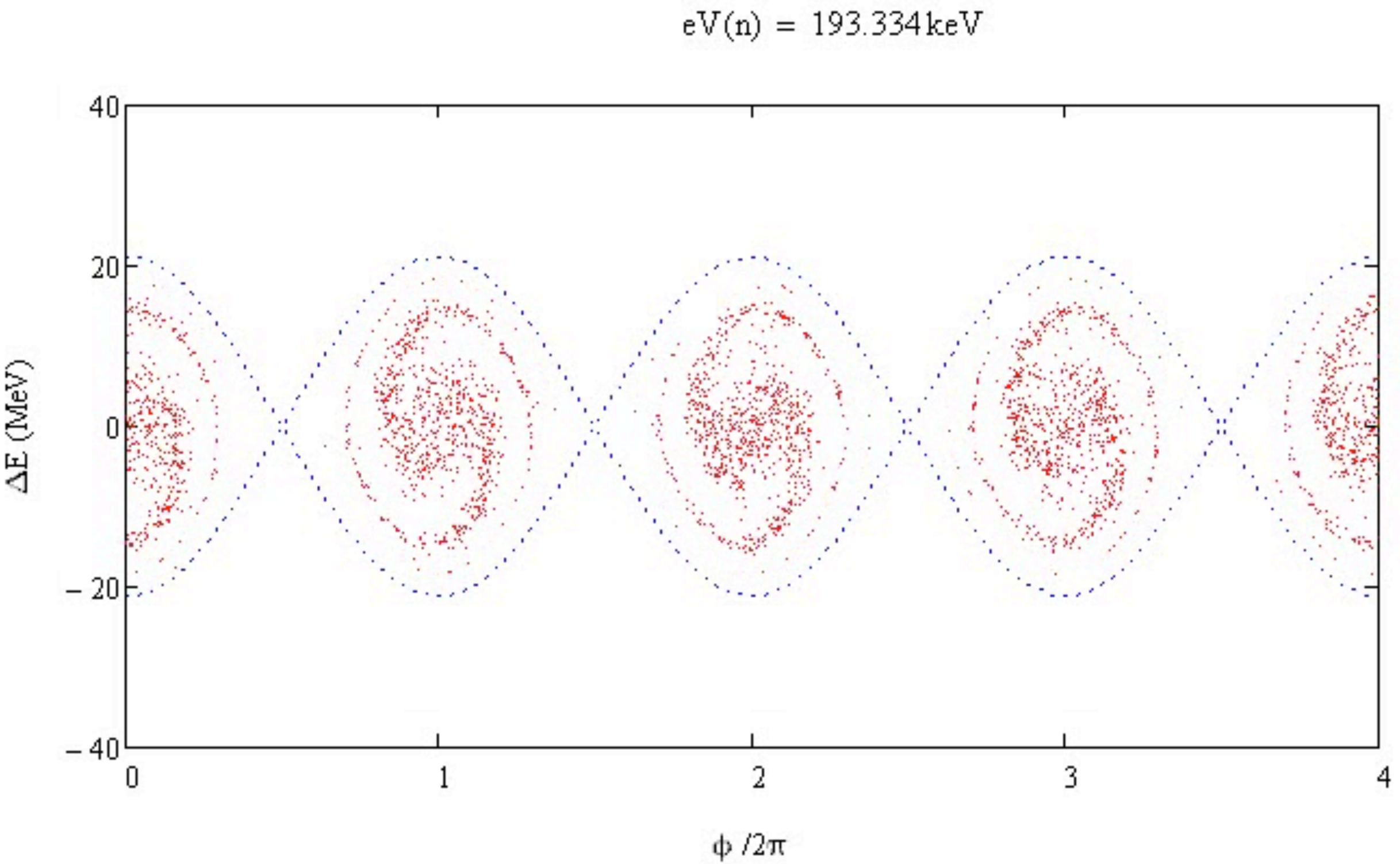


$\sigma_{EonE_t} = 1.958 \times 10^{-3}$

$t = 2.161 \times 10^{3}$




Volts$_t$ = 100 keV
MeV
20
15
10
5
0
−5
−10
−15
−20
0
0.5
1
1.5
2
2.5
3
3.5
4
$\Delta E_{i,t}$
Ubkt (t, ϕ_p)
Dbkt (t, ϕ_p)
$\phi_{i,t}, \phi_p, \phi_p$


eV(n) = 193.334keV
40
20
0
– 20
– 40
ΔE (MeV)
0
1
2
3
4
ϕ /2π