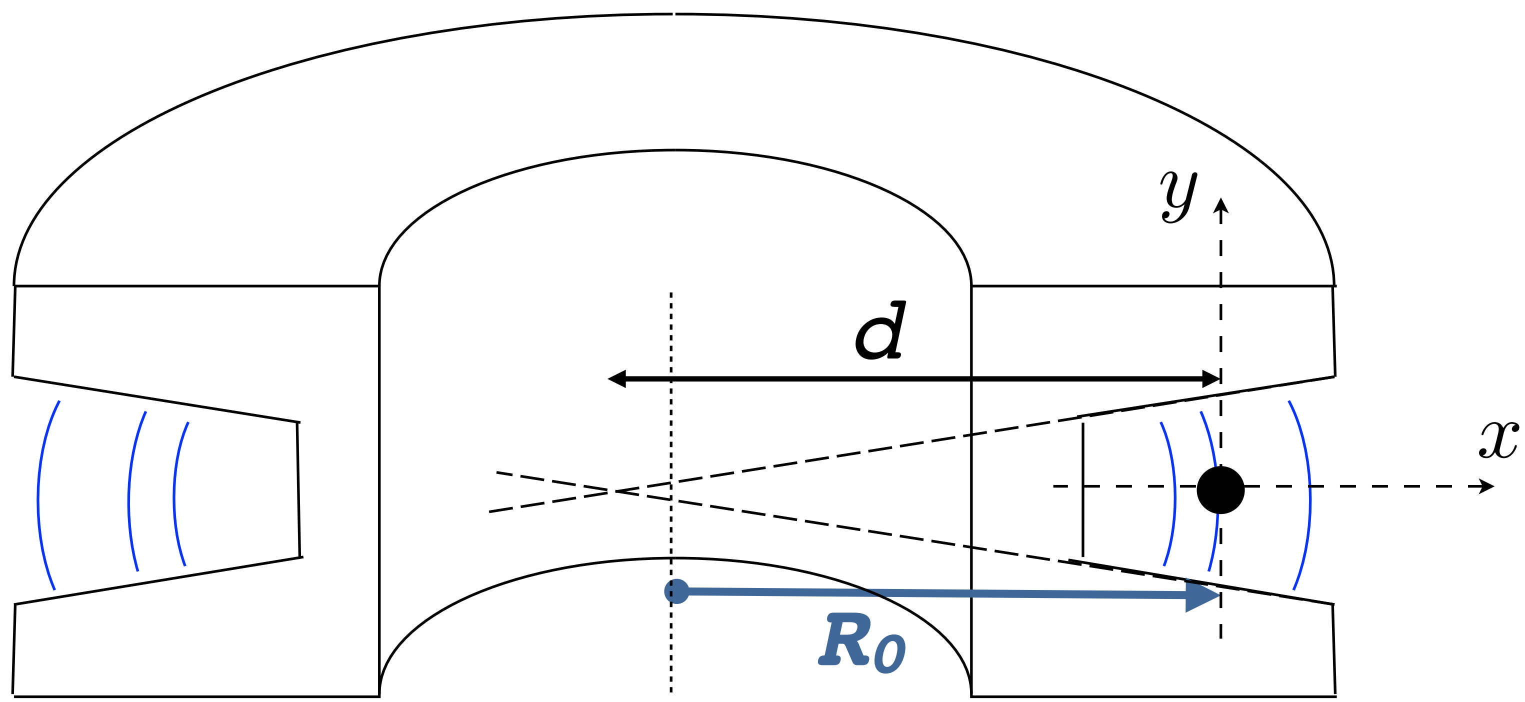
Repetitive Systems of Transverse Focusing

- Constant or Adiabatically Changing Particle Energy
  - As the beam energy changes, can keep the "optics" of the system the same by scaling the fields with the particle momentum
  - Adiabatically changing parameters of the system will maintain certain quantities invariant
    - in particular, properly normalized emittances
- In what follows, will assume that the energy is either constant or that the energy gain per passage through an RF cavity is small relative to the overall kinetic energy of the particle, such that energy changes can be considered adiabatic; adjust transverse fields accordingly

The Weak Focusing Synchrotron/Betatron

- Early accelerators employed what is now called "weak focusing"



$$B = B_0 \left(\frac{R_0}{r} \right)^n$$

n is determined by adjusting the opening angle between the poles

n = "field index"

$$n \approx \frac{R_0}{d}$$

Let's look at the stability of transverse motion in this system…

Stability within a Weak Focusing Betatron

- Start with: $$B = B_y(y=0) = B_0 \left(\frac{R_0}{r}\right)^n$$ *n = "field index"*

- Expand about the desired radius:

$$r = R_0 + x \qquad B = B_0 \left(\frac{1}{1 + x/R_0}\right)^n \approx B_0 (1 - \frac{n}{R_0} x)$$

- Thus: $$B_y = B_0 - \frac{nB_0}{R_0} x \qquad B_x = -\frac{nB_0}{R_0} y \qquad (\nabla \times \vec{B} = 0)$$

- So, $$x'' + K_x x = x'' + \left(-\frac{nB_0/R_0}{B_0 R_0} + \frac{1}{R_0^2}\right) x = 0$$

- or, $$\boxed{x'' + \frac{1-n}{R_0^2} x = 0}$$

$$y'' + K_y\, y = \boxed{y'' + \frac{n}{R_0^2}\, y = 0}$$

must have
$0 \leq n \leq 1$
for stability

Aperture of Weak Focusing System

- The solutions of the equations of motion are:

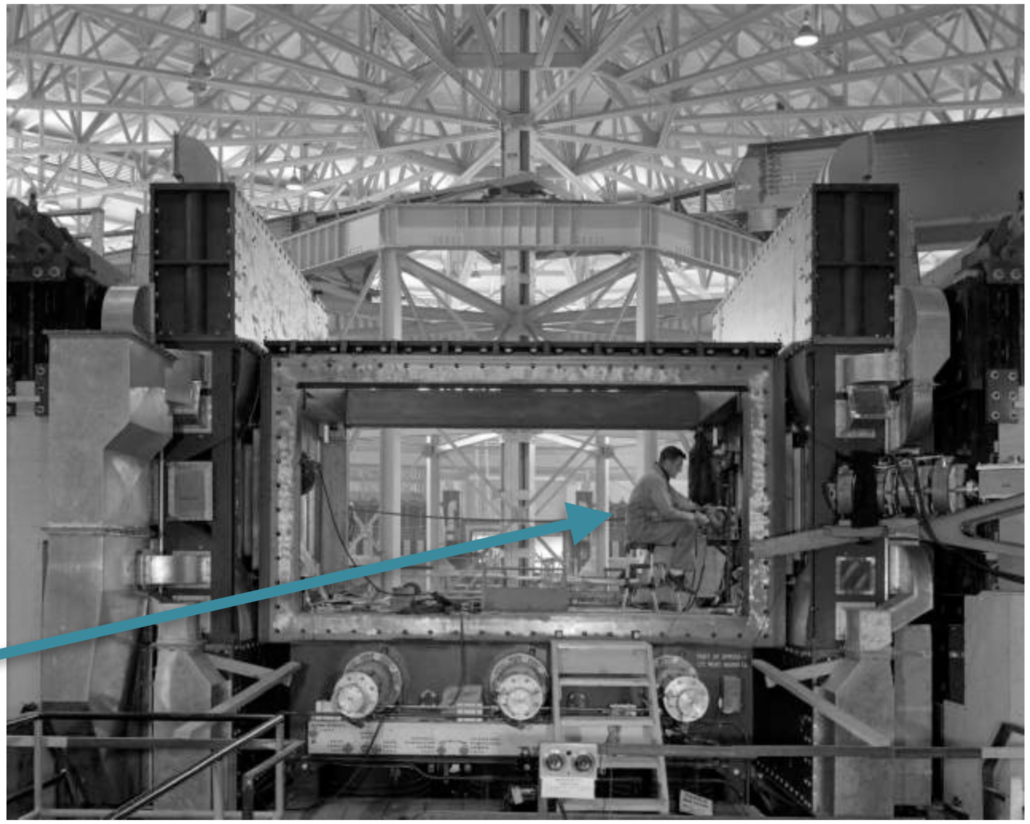
$$x'' + \frac{1-n}{R_0^2} x = 0 \qquad x = x_0 \cos\left(\frac{\sqrt{1-n}}{R_0} s\right) + x_0' \frac{R_0}{\sqrt{1-n}} \sin\left(\frac{\sqrt{1-n}}{R_0} s\right)$$

$$y'' + \frac{n}{R_0^2} y = 0 \qquad y = y_0 \cos\left(\frac{\sqrt{n}}{R_0} s\right) + y_0' \frac{R_0}{\sqrt{n}} \sin\left(\frac{\sqrt{n}}{R_0} s\right)$$

SO, maxima in *x, y* grow with the RADIUS of the accelerator, for a given set of initial beam conditions

Higher energies required larger radii (for ~ constant *B),* and hence the *apertures* had to grow as well

sitting inside the ***beam chamber*** of the Bevatron (LBNL)



Check: The Weak Focusing Synchrotron

- We had, for example: $y = y_0 \cos(\frac{\sqrt{n}}{R_0} s) + y_0' \frac{R_0}{\sqrt{n}} \sin(\frac{\sqrt{n}}{R_0} s)$
  - from which follows: $y' = -y_0 \frac{\sqrt{n}}{R_0} \sin(\frac{\sqrt{n}}{R_0} s) + y_0' \cos(\frac{\sqrt{n}}{R_0} s)$
  - or, in matrix form:

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\frac{\sqrt{n}}{R_0} s) & \frac{R_0}{\sqrt{n}} \sin(\frac{R_0}{\sqrt{n}} s) \\ -\frac{\sqrt{n}}{R_0} \sin(\frac{R_0}{\sqrt{n}} s) & \cos(\frac{\sqrt{n}}{R_0} s) \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

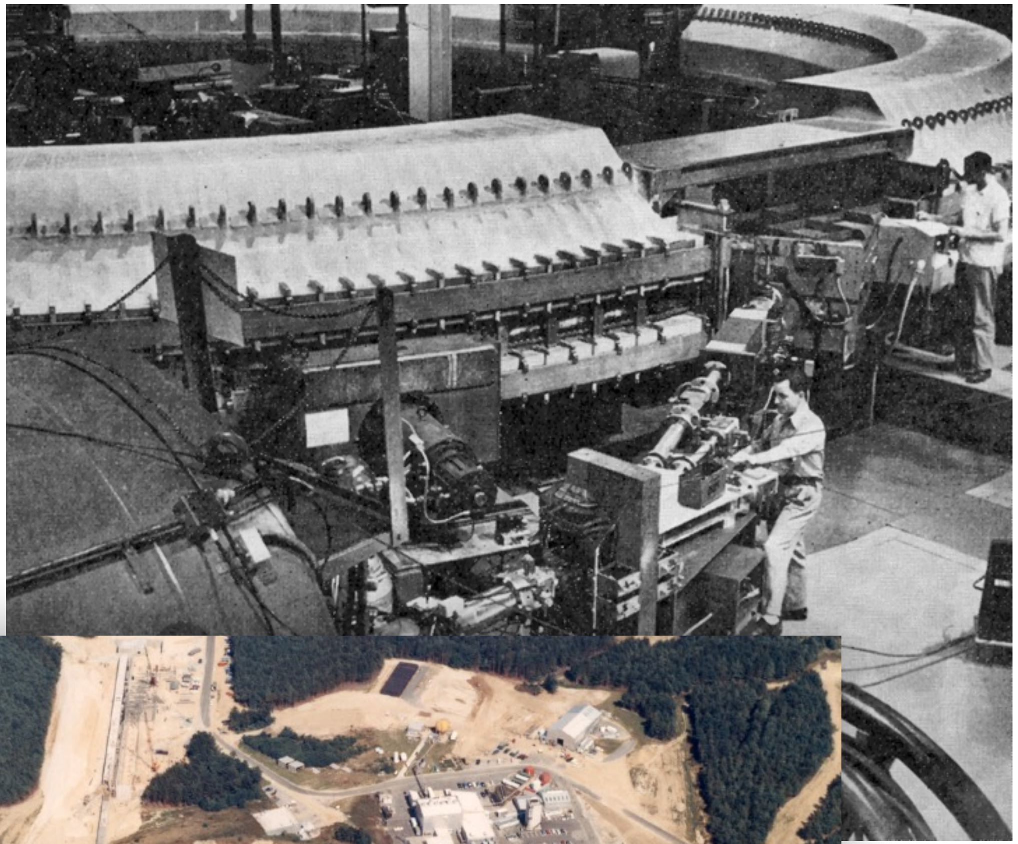
- For 1 revolution, $s = 2\pi R_0$ and the trace of *M* is …

$$|trM| = |2\cos(2\pi\sqrt{n})| \leq 2 \qquad (|2\cos(2\pi\sqrt{1-n})| \leq 2, \text{ for horizontal})$$

$$0 \leq n \leq 1$$
$$\text{for stability}$$

Discovery of Strong Focusing

- The Cosmotron (BNL)
  - (weak focusing)



- Through looking at upgrade options, strong focusing was discovered and the decision was made to go for a new, much larger synchrotron
  - The Alternating Gradient Synchrotron (AGS)

Discovery of Strong Focusing*

- Consider the "weak-focusing" magnet system just discussed. Suppose the ring is made up of 2*N* identical magnets, each with field index *n*
- Take every other magnet and have the magnet *open* to the inside, instead of the outside: $n \rightarrow -n$
  - All have the same central field value, B_0, but the field "gradients" will alternate *n*, -*n*, *n*, -*n*, …
- Analyze the resulting system using a matrix approach, and applying the stability criterion

Courant, Livingston, and Snyder, 1952.
Christofolis, c. 1950

Discovery of Strong Focusing [2]

- Consider one degree of freedom, say the vertical
  - for one of the N cells, the matrix would be…

(B'>0) (B'<0) $K = |B'|/B\rho$

$$M_c = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\sqrt{K}L)\cosh(\sqrt{K}L) + \sin(\sqrt{K}L)\sinh(\sqrt{K}L) & \cdots \\ \cdots & \cos(\sqrt{K}L)\cosh(\sqrt{K}L) - \sin(\sqrt{K}L)\sinh(\sqrt{K}L) \end{pmatrix}$$

from which

$$trM = 2\cos(\sqrt{K}L)\cosh(\sqrt{K}L)$$

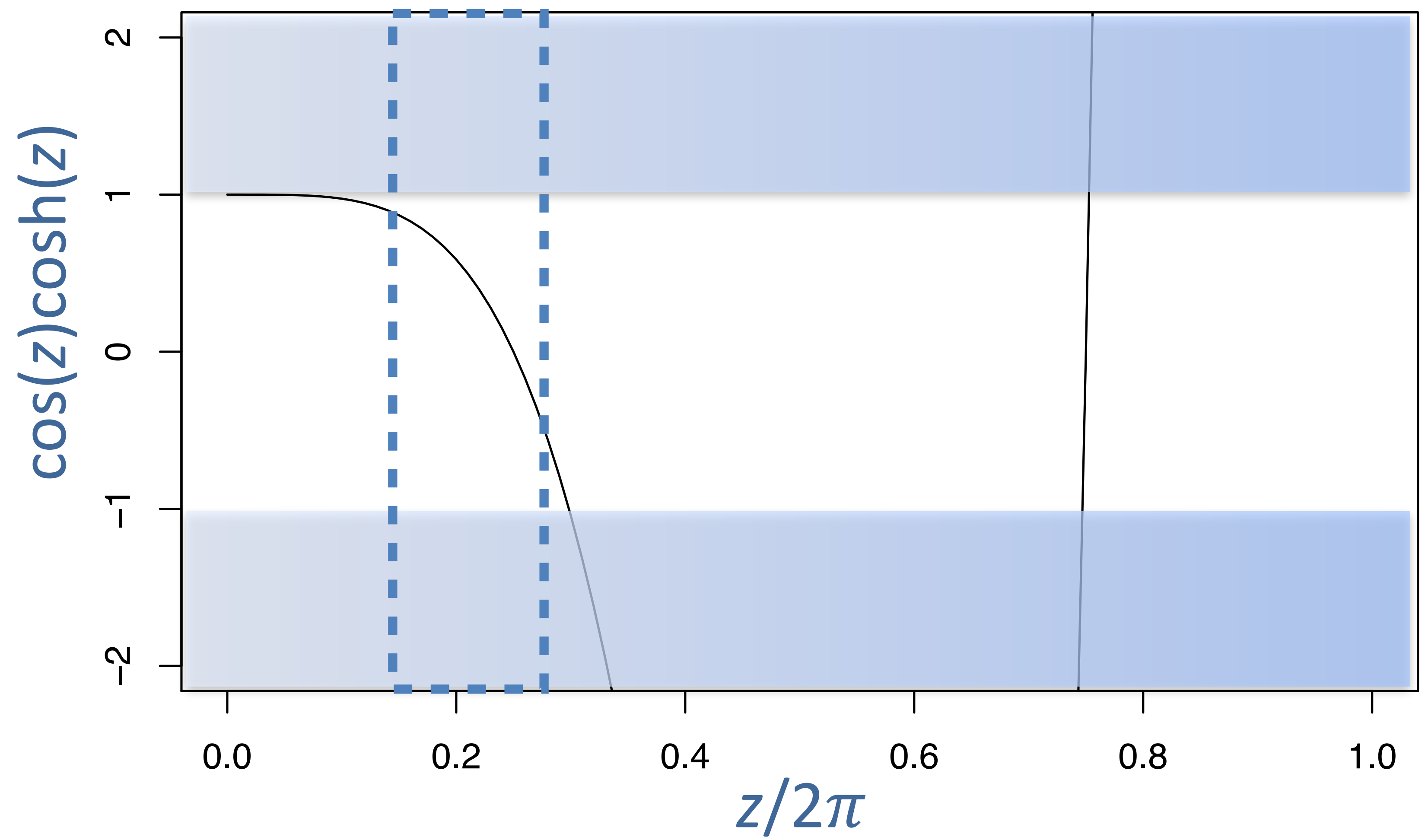
- So, for stability, we would need:

$$\boxed{|\cos(\sqrt{K}L) \cdot \cosh(\sqrt{K}L)| < 1}$$

Courant, Livingston, and Snyder, 1952.
Christofolis, c. 1950

Discovery of Strong Focusing [3]

- We see a range in which the system would be stable



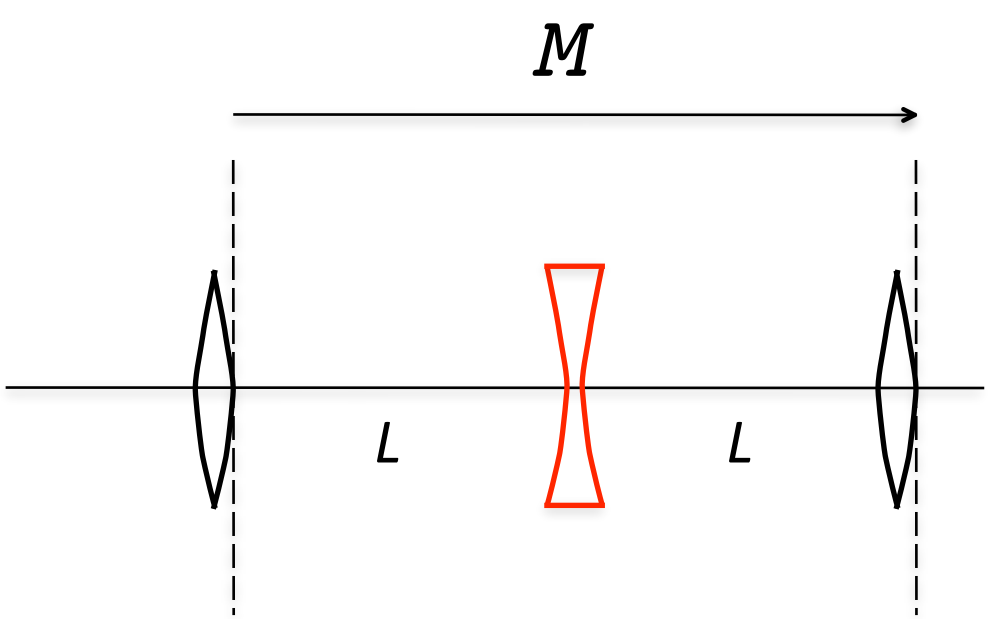
- Choose $z = \sqrt{K}\,L$

  $K = (z/L)^2$
- Also, $L = 2\pi R_0 \,/\, 2N$
- In the weak focusing case, $K_0 = n/R_0^2$. So,… $$\frac{K}{K_0} = 4\left(\frac{z}{2\pi}\right)^2 \frac{N^2}{n}$$
- Let's pick $z/2\pi = 0.2$, $n = 0.5$, and $N = 25$:

$$K/K_0 = 200!$$

Another Example: FODO system

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1-L/F \end{pmatrix}\begin{pmatrix} 1 & L \\ 1/F & 1+L/F \end{pmatrix} \\
&= \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix}
\end{aligned}
$$



and repeat…

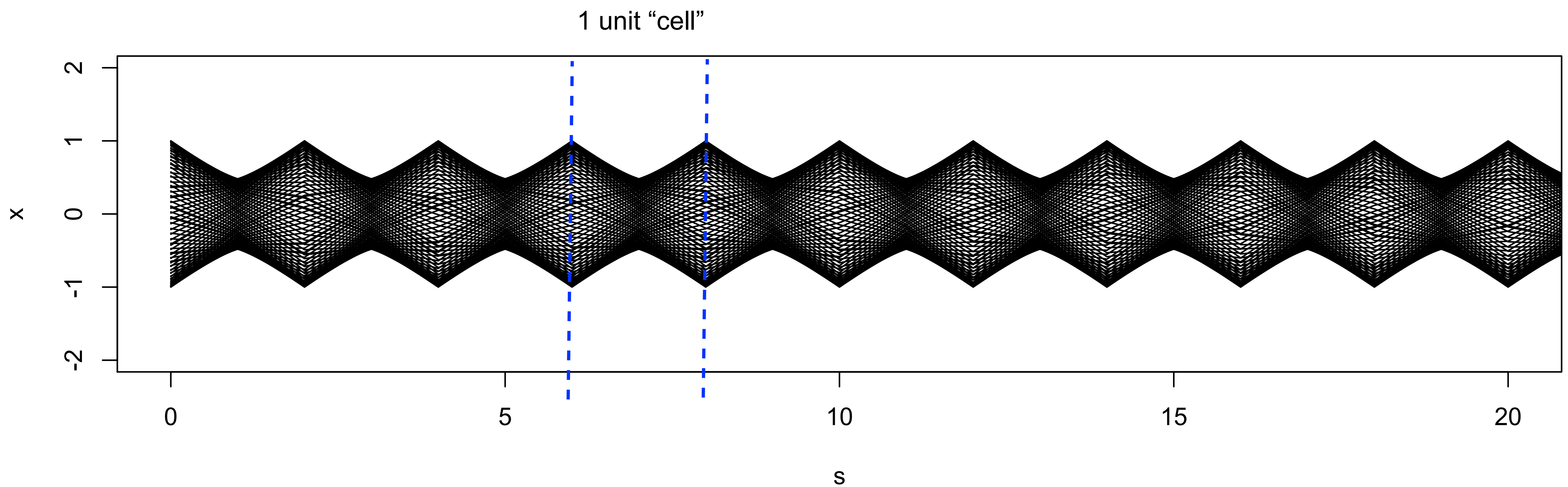
So, $tr\, M = 2 - L^2/F^2$ and thus, for stability,

$$-2 < 2 - L^2/F^2 < 2$$

$$-4 < -L^2/F^2 < 0$$

$$F > L/2$$

Particle Trajectories in a Periodic Lattice



$$\frac{dx'}{ds} = \frac{d^2x}{ds^2} = -\frac{eB'(s)}{p}x$$

$$x'' + K(s)x = 0$$

(Hill's Equation)

$$K(s) = \frac{e}{p}\frac{\partial B_y}{\partial x}(s)$$

$$\boxed{x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]}$$

The Periodic Amplitude Function

- Previously, …
  - Transport matrix, in terms of amplitude function at end points, and phase advance between:

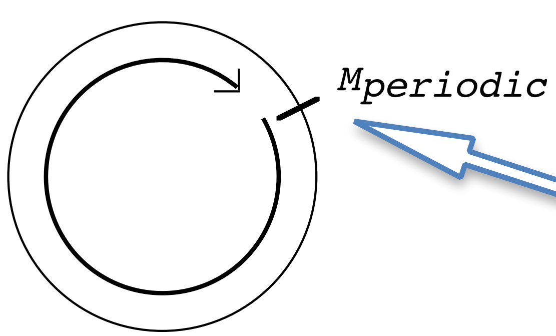
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

Periodic Solutions

- Within a system made up of periodic sections it is natural to want the beam envelope to have the same periodicity.
- Taking the previous matrix to be that of a periodic section, and demanding the C-S parameters be periodic yields...

$$M_{periodic} = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$



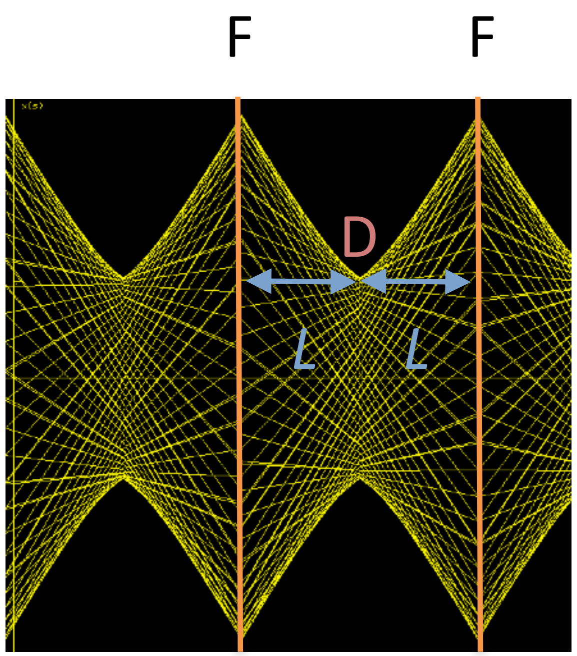
Natural choice in a circular accelerator, when values of β, α above correspond to one particular point in the ring

Choice of Initial Conditions

- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a "ring," then natural to choose the **periodic** solutions for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- In a system like a linac, wish to "match" to desired initial conditions at the input to the system (somewhere after the source, say) using an arrangement of focusing elements

Computation of Courant-Snyder Parameters

- As an example, consider again the FODO system



$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1-L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1+L/F \end{pmatrix} \\
&= \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix}
\end{aligned}
$$

- Thus, use above matrix of the periodic section to compute functions at exit of the F quad..

FODO Cell Courant-Snyder Parameters

$$M_{periodic} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

here, μ = phase advance through one period

- From the matrix: $M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *4 numbers*

$$trace M = a + d = 2 - L^2/F^2 = 2\cos\mu \Rightarrow \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1 + L/2F}{1 - L/2F}} \qquad \alpha = \frac{a - d}{2\sin\mu} = \sqrt{\frac{1 + L/2F}{1 - L/2F}}$$

- If go from D quad to D quad, simply replace *F* --> -*F* in matrix *M*
  - So, at exit of the D quad:

$$\beta = 2F\sqrt{\frac{1 - L/2F}{1 + L/2F}} \qquad \alpha = -\sqrt{\frac{1 - L/2F}{1 + L/2F}}$$

for completeness,

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

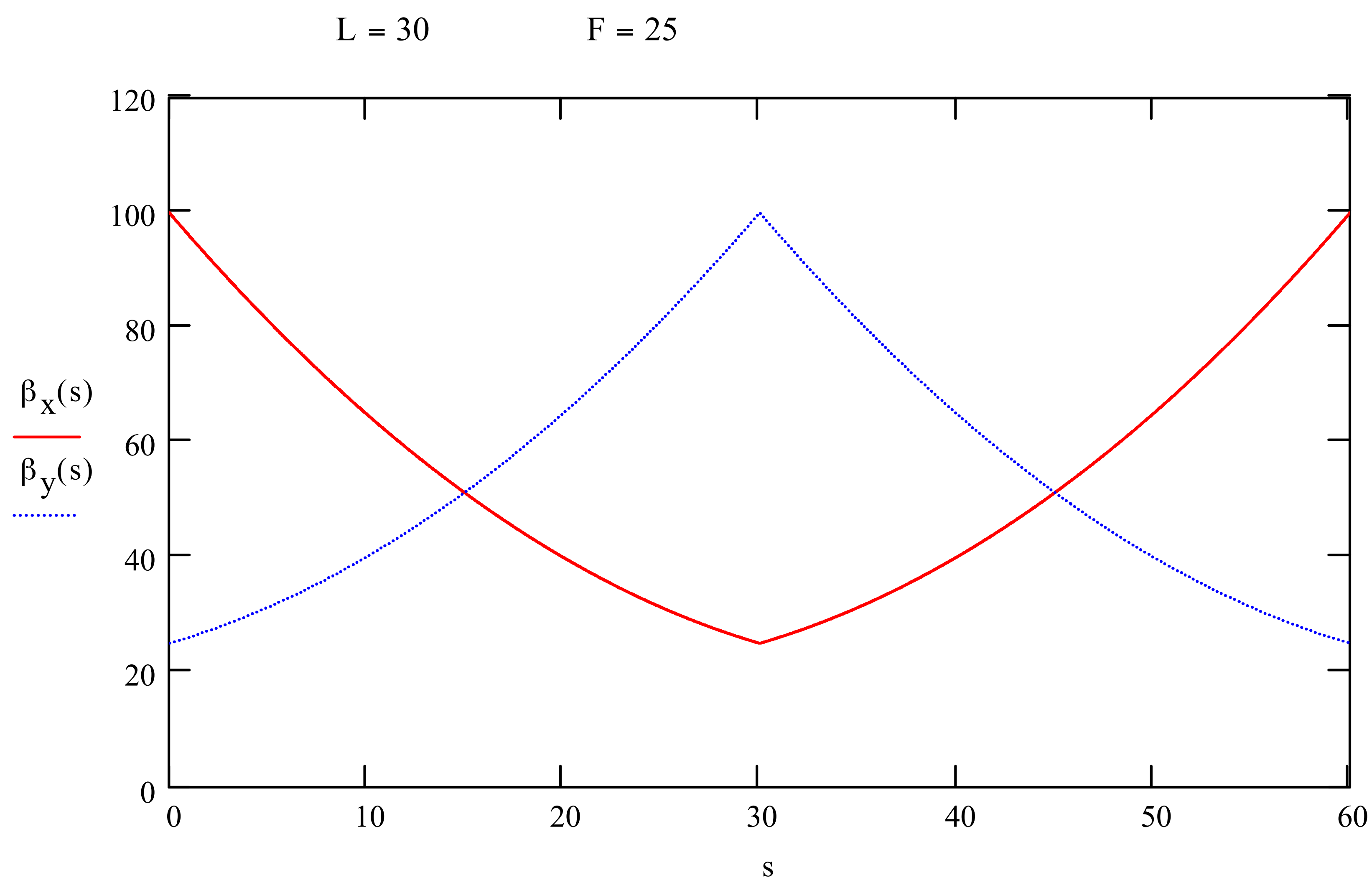
Periodic FODO Cell Functions

- Numerical Example: Standard FODO Cell of the old Fermilab Tevatron (~100 of these made up the ring)

$$\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2 (69^\circ)$$
$$\beta_{max} = 2(25\ \mathrm{m})\sqrt{1.6/0.4} = 100\ \mathrm{m}$$
$$\beta_{min} = 2(25\ \mathrm{m})\sqrt{0.4/1.6} = 25\ \mathrm{m}$$
$$\nu \approx 100 \times 1.2/2\pi \sim 20$$

Note: this thin lens example is actually accurate to a few percent!

$$(F/\ell = 25/2 >> 1)$$



Periodic Courant-Snyder Parameters

- We can write the matrix of a periodic section as:

$$
\begin{aligned}
M_0 &= \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\Delta\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin\Delta\psi \\
&= I\cos\Delta\psi + J\sin\Delta\psi \qquad = e^{J\Delta\psi}
\end{aligned}
$$

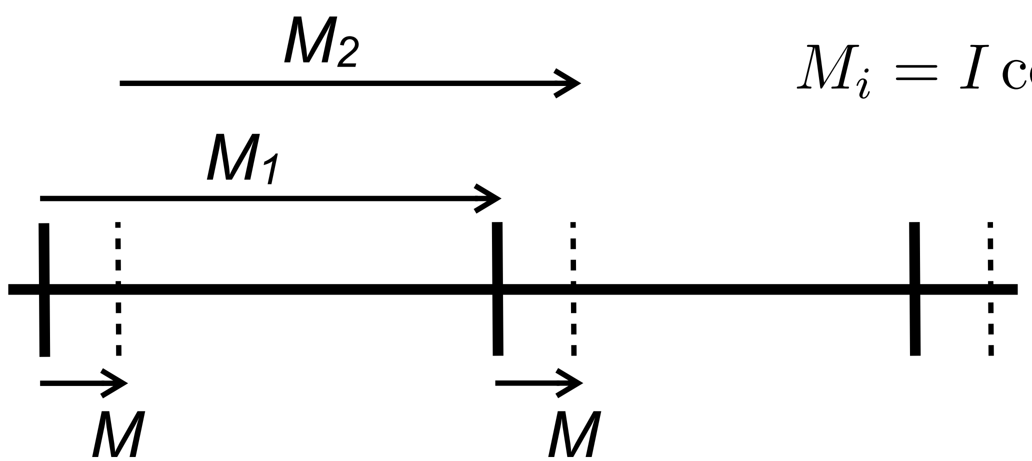
- where

$$
J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \qquad det J = 1, \quad trace(J) = 0; \quad J^2 = -I
$$

α, β are values at the beginning/end of the periodic section described by matrix M

Tracking β, α, γ ...

- Let M_1 and M_2 be the "periodic" matrices as calculated at two points, and M propagates the motion between them. Then,



$$M_i = I\cos\Delta\psi + J_i \sin\Delta\psi \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$M_2 = M\ M_1\ M^{-1}$$

- Or, equivalently,
  - if know C-S parameters (i.e., J) at one point, can find them at another point downstream if given the matrix for motion in between:

$$J_2 = M\ J_1\ M^{-1}$$

- Doesn't have to be part of a periodic section; valid between any two points of an arbitrary arrangement of elements

For comparison, remember $K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, and $K = M\ K_0\ M^T$; actually equivalent

Evolution of the Phase Advance

- Again, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan\Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase and the Courant-Snyder parameters along a beam line from one point to another

The Betatron Tune

- In a cyclic accelerator (synchrotron), the particles will oscillate (betatron oscillations) with a certain oscillation frequency — the betatron frequency.
- The betatron frequency is determined by the total phase advance once around the ring:

$$\Delta\psi_{total} = \oint \frac{ds}{\beta(s)}$$

$$\mathrm{tr}M = 2\cos(2\pi\nu)$$

$$\nu \equiv \Delta\psi_{total}/2\pi$$

$$f_{betatron} = \nu f_{rev}$$

Betatron Tune: # of oscillations per revolution

Ex: Tune of a FODO synchrotron

- Suppose a ring is made up of N FODO cells
- Each cell has phase advance given by the lens spacing and lens focal length: $\sin\frac{\mu}{2} = \frac{L}{2F}$
- So, the tune of this simple synchrotron would be:

$$\nu = N\mu/2\pi \approx N\frac{L}{2\pi F} = \frac{2LN}{4\pi F} = \frac{C}{2\pi}\frac{1}{2F} = \frac{R}{2F}$$

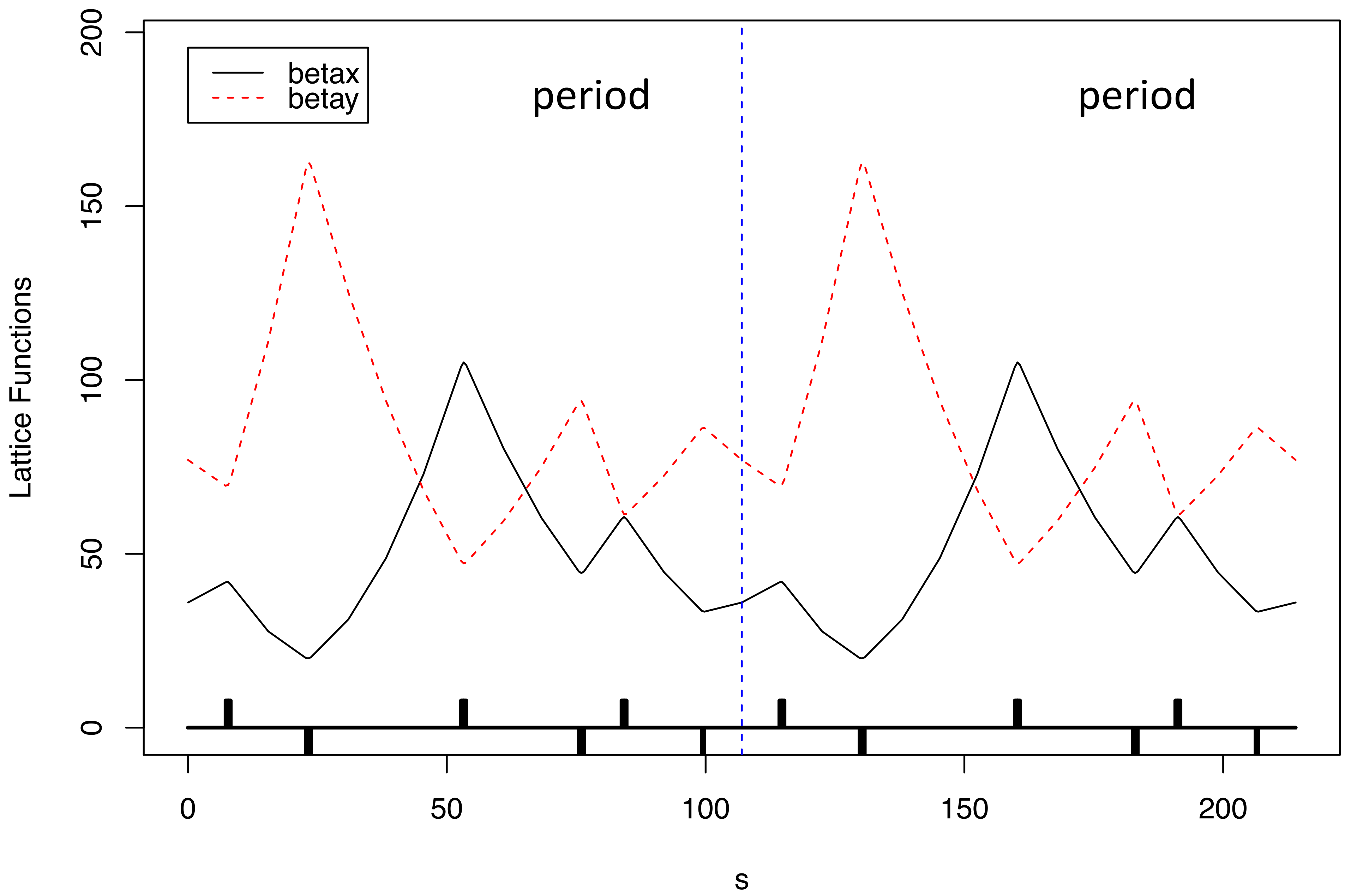
- Ex: Main Injector at Fermilab: $R \sim 500$ m; $F \sim 13$ m
  - so, $\nu \sim 20$
  - thus, if initiate a betatron oscillation in this synchrotron it will oscillate ~20 times per revolution around the ring

Computing Courant-Snyder Parameters

reminder

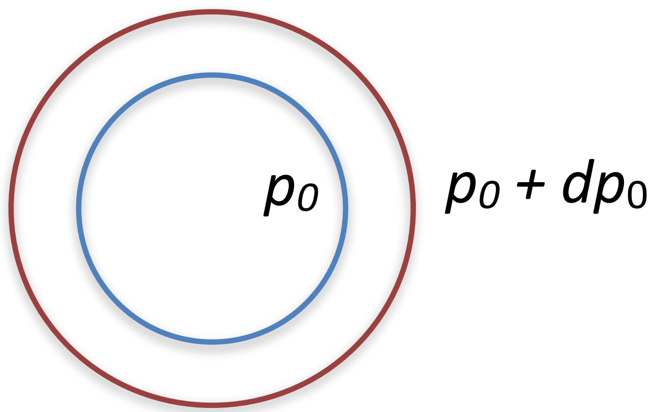
- If linac or beam line:
  - use initial conditions of beam at entrance to compute β, α from phase space distribution
- If synchrotron:
  - use periodicity to determine THE parameters of the ring at one point
- THEN,
  - propagate β, α from the chosen starting point downstream, using equations on our earlier slides

Arbitrary Distribution of Quadrupoles

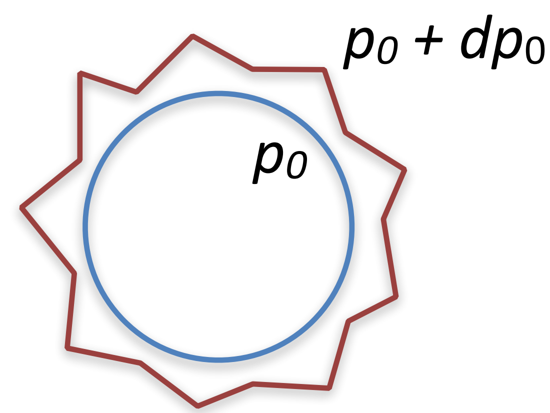


Periodic Dispersion Function

uniform bend field:



add gradients…

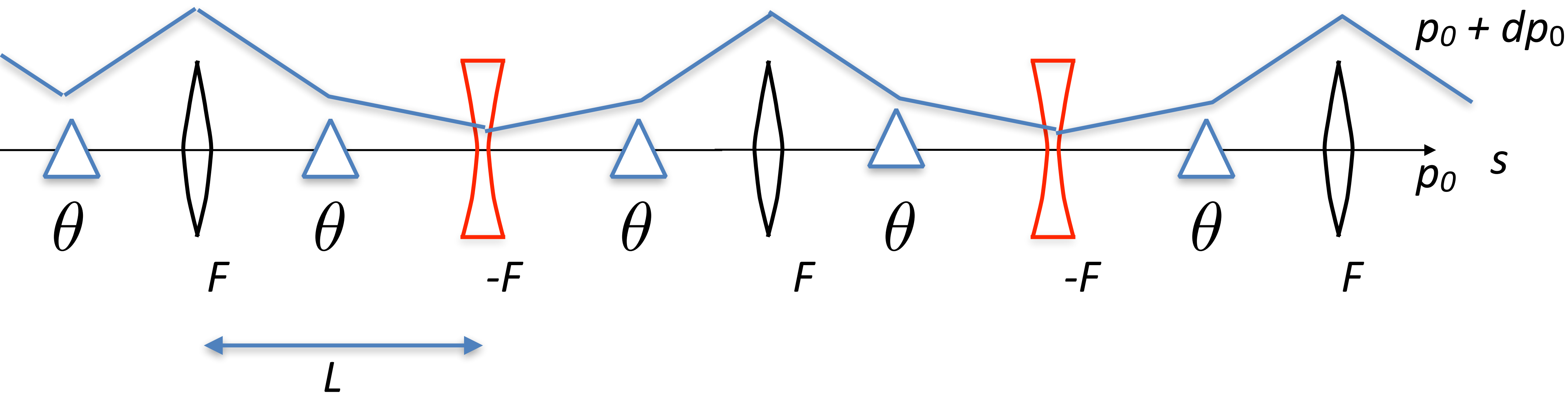


the trajectory "closed" orbit for momentum p+Δp

$$D(s,p) = \frac{\Delta x(s,p)}{\Delta p/p}$$

the orbit of an off-momentum particle which closes on itself is described by the *periodic* dispersion function

Ex: FODO Cells with Bending Magnets



$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

$$D_{max,min} = \frac{L\theta}{2\sin^2(\mu/2)}\left(1 \pm \frac{1}{2}\sin(\mu/2)\right)$$

Values of dispersion function are typically ~ few meters

Note: in a weak-focusing synchrotron, would have $D = R_0$!