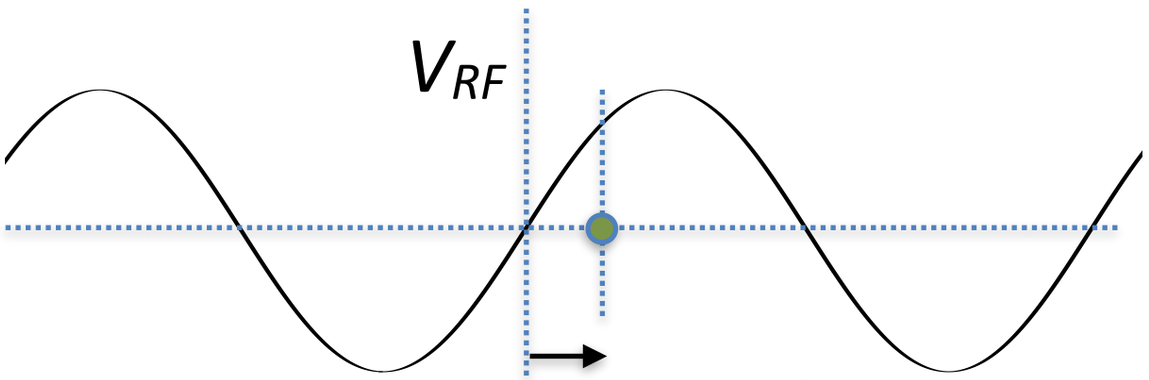
Day Four

- Discussion of Repetitive Systems
- Long-Term Transverse Stability
- Long-Term Longitudinal Stability

Quick Review [1] …

- linearized longitudinal motion

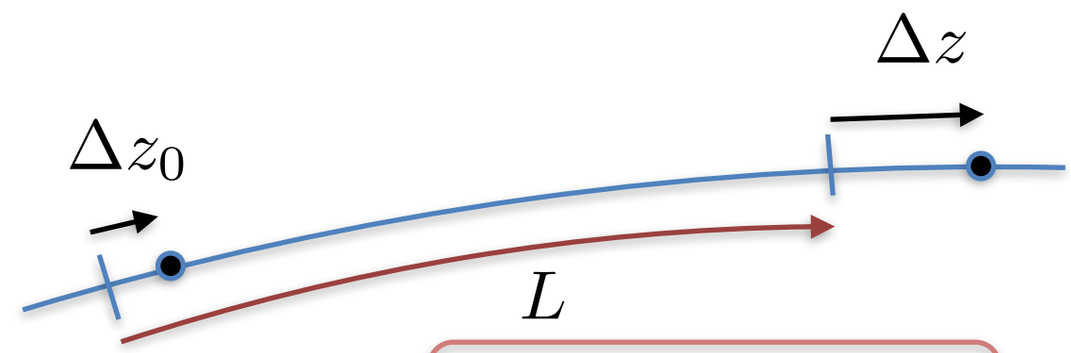


$$\phi = 2\pi f_{RF} t$$

$$W_s = W_{s,0} + qV\sin(\phi_S)$$

$$W_s = \text{“ideal” energy}$$

$$\Delta W \equiv W - W_s$$



$$\Delta z = \Delta z_0 - \eta L \frac{\Delta p}{p}$$

$$\Delta t = \Delta t_0 + \eta \frac{L}{\beta c}\frac{\Delta p}{p}$$

drift:

$$\begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix} = \begin{pmatrix} 1 & \eta \frac{L}{c}\frac{1}{\beta^3\gamma}\frac{1}{mc^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix}_0$$

through cavity:

$$\begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (2\pi f_{RF}) qV\cos\phi_s & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix}_0$$

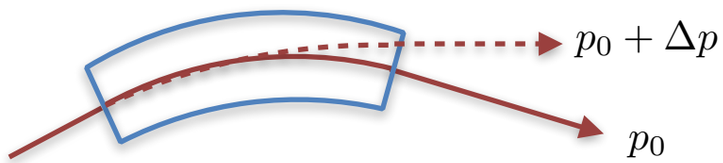
- adiabatic damping:

$$\epsilon \propto \frac{1}{p} \qquad x_{rms} \propto \frac{1}{\sqrt{p}}$$

Quick Review [2] …

Dispersion and the Dispersion Function:

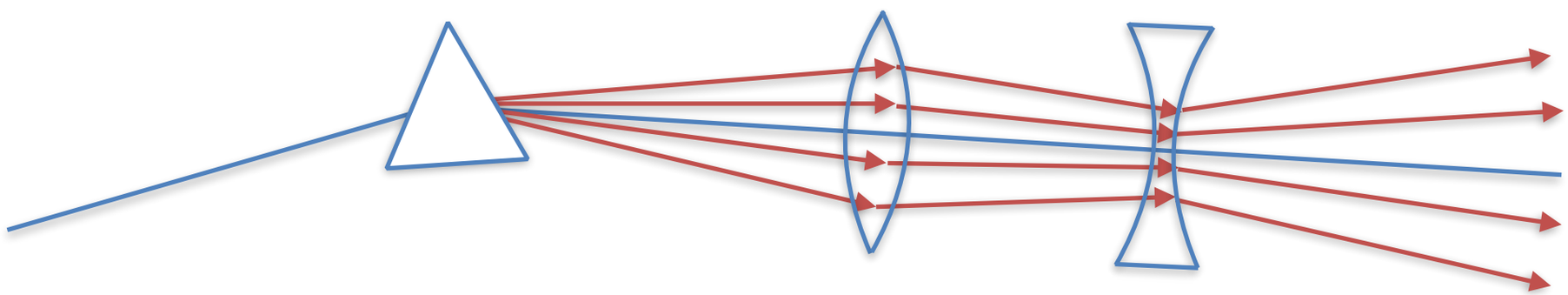
$$D'' + K\ D = \frac{1}{\rho_0}$$



$$\frac{\Delta\theta}{\theta_0} = -\frac{\Delta p}{p}$$

through bending magnet,

$$\Delta D' = \theta$$



$$x_{rms}(s) = \sqrt{\langle x^2(s)\rangle} = \sqrt{D^2(s)\langle(\Delta p/p)^2\rangle + \epsilon\beta(s)/\pi}$$

Momentum Compaction Factor:

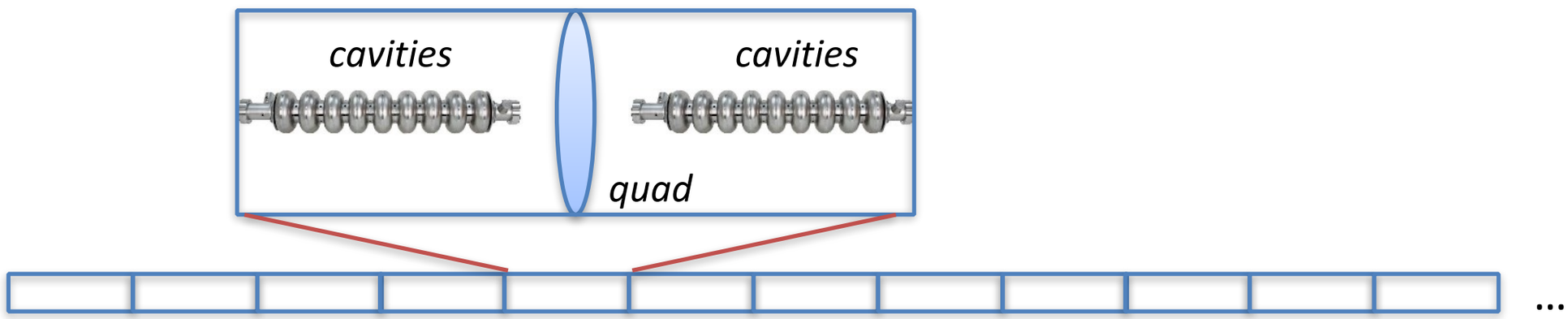
$$\alpha_p \equiv \left(\frac{dL/L}{dp/p}\right) = \frac{\int[D(s)/\rho(s)]ds}{\int ds} = \langle D/\rho\rangle \qquad \eta = \alpha_p - \frac{1}{\gamma^2} = \left\langle\frac{D}{\rho}\right\rangle - \frac{1}{\gamma^2}$$

Long-Term Stability

- Have discussed linear beam optics and transport through simple “lens” systems
  - similar techniques for both transverse and longitudinal degrees of freedom
    - dipole magnets to steer, quadrupole magnets (sometimes solenoids) to focus transversely
    - cavities to accelerate, also provide longitudinal focusing (time focusing)
- Transporting across a large room or building is one thing; will such systems be stable if transport many kilometers? millions (billions?) revolutions in a ring?

Stability

- Example: Linear Collider — can we make a system made up of hundreds of “modules” that accelerate and focus repeatedly?



- Example: Hadron Collider — can we make a system that can be traversed stably billions of times? Tevatron:
  - N_{rev} = (24 hr)(3600 s/hr)(3e8 m/s)/(6e3 m) = 4e9

The Stability Criterion

- For a system made up of multiple elements, we can perform "ray tracing" as we've seen so far — this may be all that's needed to understand a relatively small system. But, suppose the "system" is very long and made of many repetitions of the same type of elements (or, perhaps the "repetition" is a complete circular accelerator, for instance) -- how do we ensure that the motion is stable for many (infinite?) passages?
- Look at matrix describing motion for one passage through a repetitive period:

$$M = M_N M_{N-1} \cdots M_2 M_1$$

- Now suppose repeat this operation *k* times. We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \to \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

The Stability Criterion

- From the discipline of "linear algebra", we know that any vector within a vector space (i.e., that is operated on, say, by a matrix *M*) can be written in terms of the eigenvectors of the matrix *M*
  - Eigenvector: V $\quad MV = \lambda V$
    - *where* λ is an eigenvalue of *M* *(real or imaginary)*
- A 2x2 matrix *M* will have two eigenvalues, λ_1 and λ_2 and two corresponding eigenvectors, V_1 and V_2; so any vector that *M* operates on can be written as

$$X = c_1 V_1 + c_2 V_2$$

The Stability Criterion

- So, if the matrix *M* is applied to vector X_0 a number of times, the resulting vector X_k after the *k*-th iteration will be

$$X_k = M^k X_0 = M^k (c_1 V_1 + c_2 V_2) = c_1 \lambda_1^k V_1 + c_2 \lambda_2^k V_2$$

V = eigenvector
λ = eigenvalue

- Now, also from linear algebra, the determinant of the matrix *M* will be the product of the eigenvalues. So,

$$\det M = 1 = \lambda_1 \lambda_2 \rightarrow \lambda_2 = 1/\lambda_1 \rightarrow \lambda = e^{\pm i\mu}$$

The Stability Criterion

- Since for our case the eigenvalues are reciprocals of each other, and since we can write $\lambda = e^{\pm i\mu}$, then

$$X_k = M^k X_0 = c_1 \lambda_1^k V_1 + c_2 \lambda_2^k V_2 = c_1 e^{ik\mu} V_1 + c_2 e^{-ik\mu} V_2$$

If μ is imaginary, then repeated application of *M* gives exponential growth; if μ is real, gives oscillatory solutions…

- To find the eigenvalues, we solve the "characteristic equation": from $MV = \lambda V$,

$$\text{characteristic equation: } \det(M - \lambda I) = 0$$

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } (a-\lambda)(d-\lambda) - bc = 0$$

The Stability Criterion

- Solving for the eigenvalues, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\text{then } \det(M - I\lambda) = \left| \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \right| = (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$
$$\lambda^2 - trM\lambda + 1 = 0$$
$$\lambda + 1/\lambda = trM$$
$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = trM$$

$$ad - bc = \det M = 1$$

$$a + d = trM = \text{“}trace\text{”} \ of \ M$$

$$\text{So, } \mu \text{ real (stability)}$$
$$\rightarrow |trM| < 2$$

The Stability Criterion