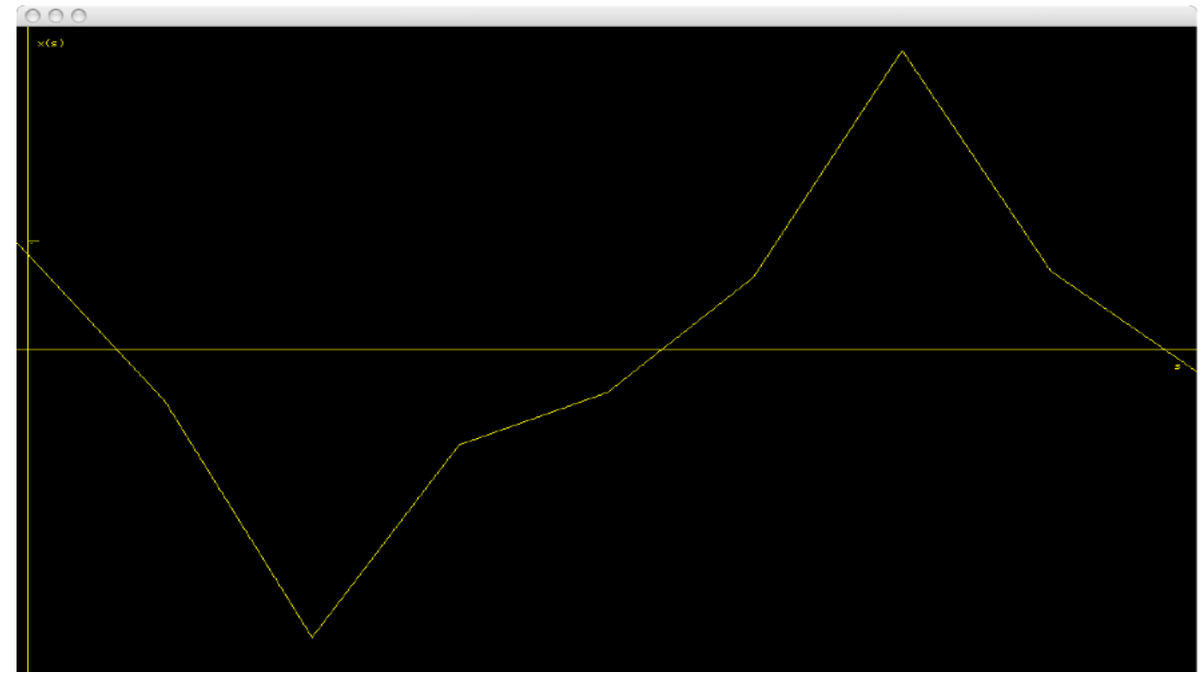
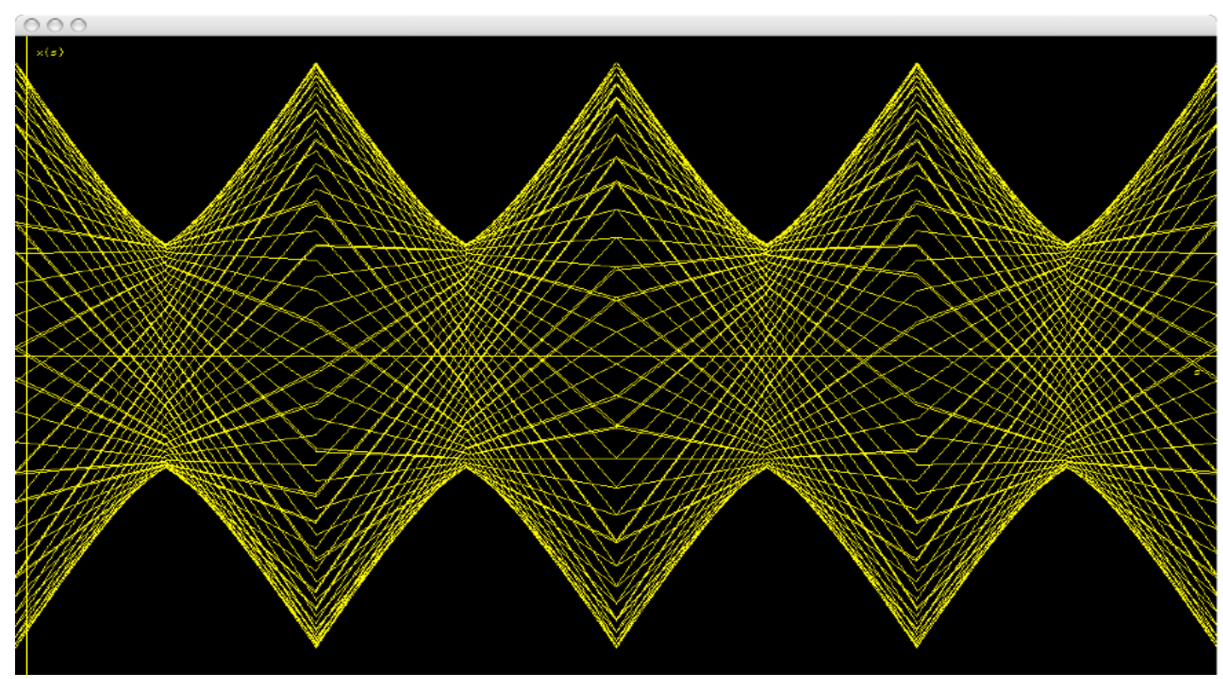
The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line





Can we describe the maximum amplitude of particle excursions in analytical form?

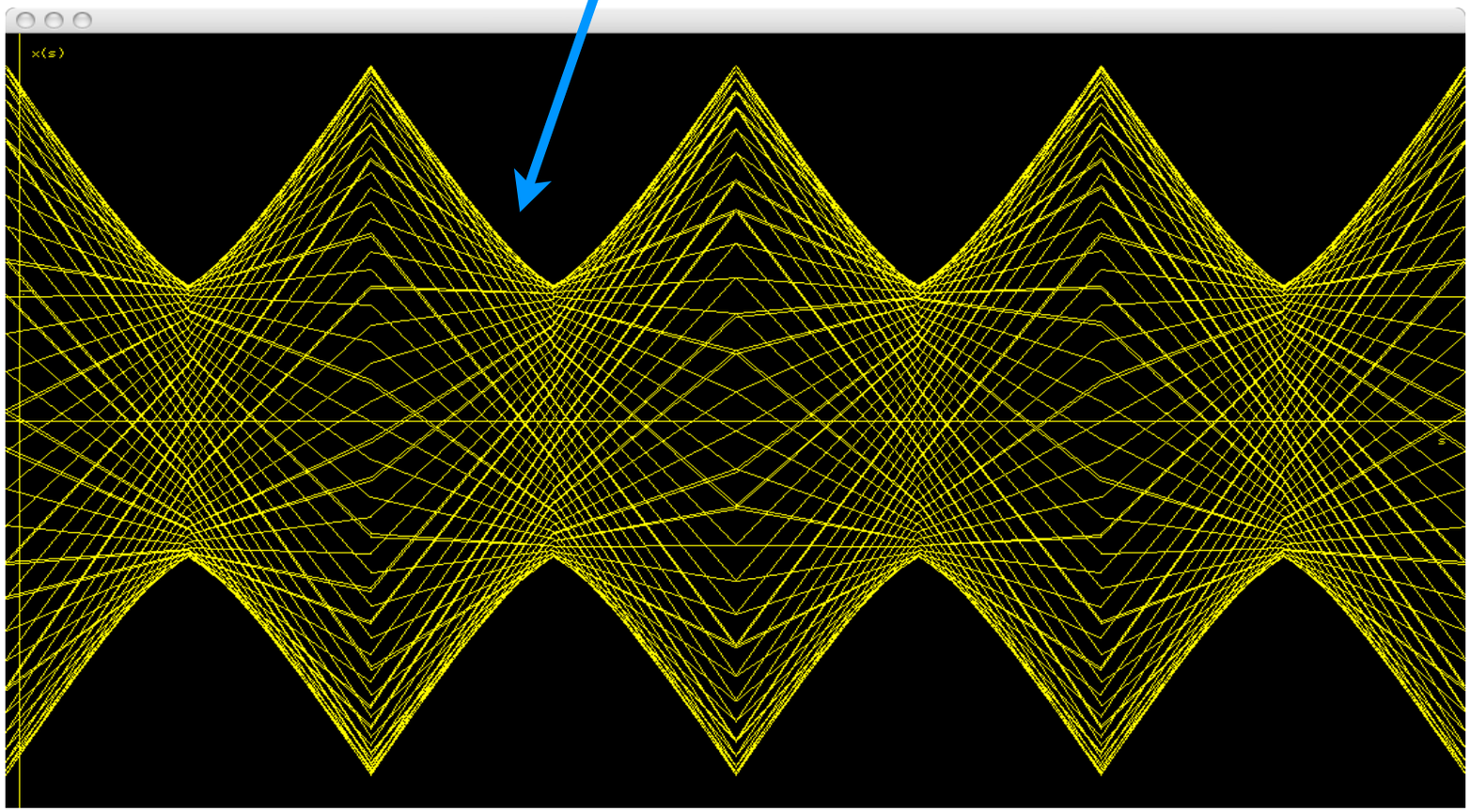
of course! *coming up next* ...

Pushing the “Envelope”

- Wish to look for a functional form of the outer envelope of particle motion, and the rate at which the phase of the oscillatory motion develops within that envelope

- This will enable us to decouple the motion of individual particle from intrinsic properties of the accelerator design

Envelope described by an “amplitude function”



Hill's Equation — Analytical Solution

- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, s.
- So, assume solution is sinusoidal, with a phase which advances as a function of location s; also assume amplitude is modulated by a function which also depends upon s:

$$\boxed{x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]}$$

- Then, plug into Hill's Equation ...

Analytical Solution (cont'd)

$$
\begin{aligned}
x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\
x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\
x'' &= \ldots
\end{aligned}
$$

Plugging into Hill's Equation, and collecting terms...

$$
\begin{aligned}
x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\
&\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0
\end{aligned}
$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, must have contents of each [ ] = 0 simultaneously for *sum* = 0.

Analytical Solution (cont'd)

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

thus, we need

$$\boxed{\psi'' + \frac{\beta'}{\beta}\psi' = 0}$$

and

$$\boxed{-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0}$$

$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

$$2\beta\beta'' - (\beta')^2 - 4\beta^2(\psi')^2 + 4K\beta^2 = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

Differential equation that the amplitude function must obey

The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Note: the phase advance is an observable quantity. So, while could choose different value of `const`, then β would just scale accordingly; thus, valid to choose `const` = 1.

Some Comments

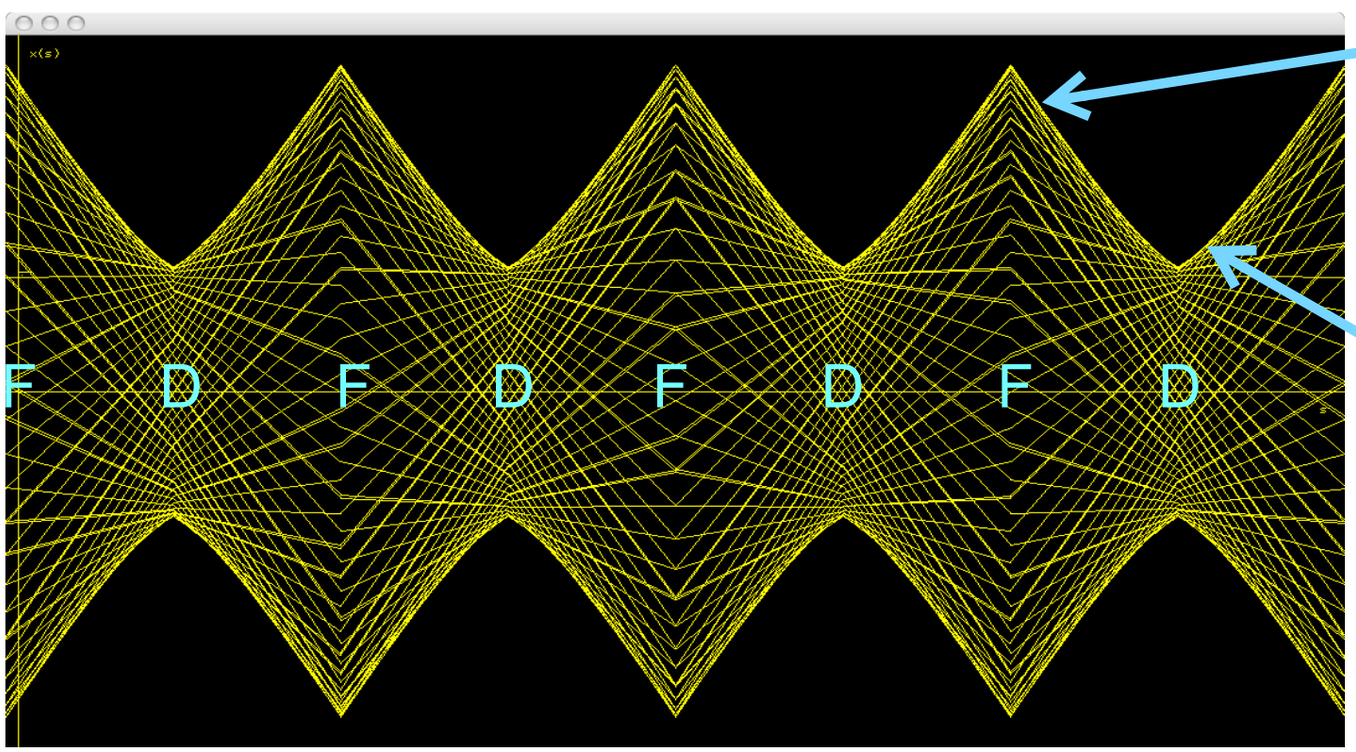
$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But the amplitude function is also a local wavelength of the motion.
- This seems strange at first, but ...
  - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
  - Thus, the spacing and/or strengths (i.e., *K(s)*) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes must be tied together.

The Amplitude Function, β

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- Since the amplitude function is a wavelength, it might have numerical values of many meters, say. However, typical particle transverse motion is on the scale of mm. So, this means that the constant *A* must have units of $m^{1/2}$, and it must be numerically small. More on this subject coming up...



Higher β --
smaller phase advance
larger beam size

Lower β --
greater phase advance
smaller beam size

Equation of Motion of Amplitude Function

From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\boxed{\beta'' + 4K\beta = const.}$$

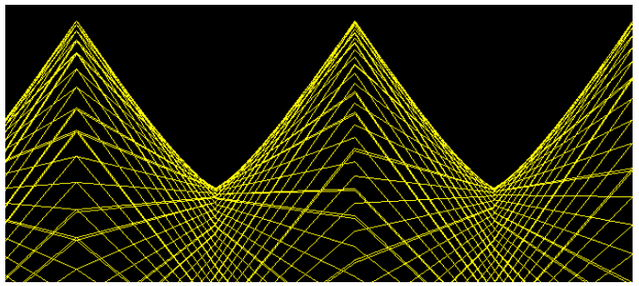
is the general equation of motion for the amplitude function, β.

(in regions where K is either zero or constant)

Piecewise Solutions

- $K = 0$: $\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$ Parabola!
  - since $\beta > 0$, then from original diff. eq. …

$$2\beta\beta'' - (\beta')^2 = 4 \qquad \beta'' = \frac{4 + (\beta')^2}{2\beta} > 0$$

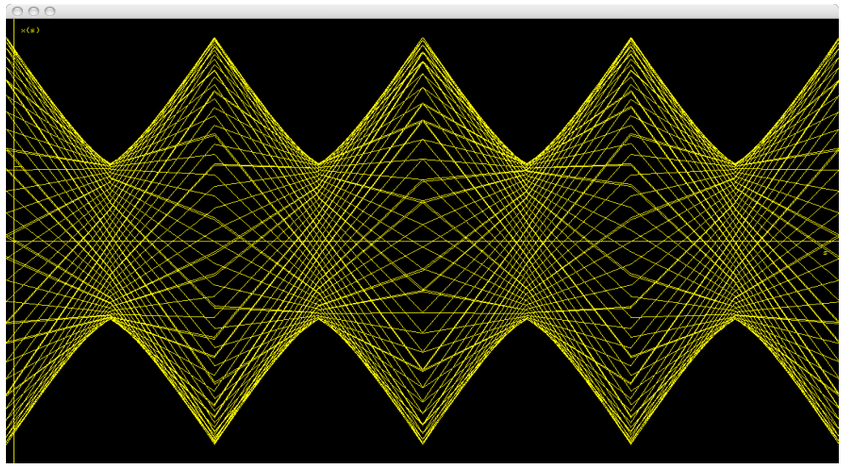


  - Therefore, parabola is ***always*** concave up

- $K > 0$, $K < 0$: sinusoidal + *constant*

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$$

Summary



$$x'' + K(s)x = 0$$ Hill's Equation

trial solution: $x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$

requires:

$$\psi' = 1/\beta \quad \longrightarrow \quad \psi(s) = \int \frac{ds}{\beta(s)}$$

and

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \quad \longrightarrow \quad \beta'' + 4K\beta = const.$$

(for *K'* = 0)

for *K* = 0 : $\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$

for *K* > 0 : $\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$

Courant-Snyder Parameters, & Connection to Matrix Approach

- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Previously have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,

$$\boxed{\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}}$$

- Collectively, β, α, γ are called the *Courant-Snyder Parameters* (sometimes called "Twiss parameters" or "lattice parameters")

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \qquad \textit{becomes} \qquad K\beta = \gamma + \alpha'$$

Courant-Snyder Parameters, & Connection to Matrix Approach

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YES! They ARE the same
α, β, γ seen earlier!

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- Collectively, β, α, γ are called the *Courant-Snyder Parameters* (sometimes called "Twiss parameters" or "lattice parameters")

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \qquad \textit{becomes} \qquad K\beta = \gamma + \alpha'$$

Solutions using Courant-Snyder Parameters

- Our previous results become
  - drift space:

$$\beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$

$$\text{–>} \quad \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

  - gradient field:

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$$

$$\text{–>} \quad \beta(s) = \frac{\beta_0}{2}[1 + \cos(2\sqrt{K}s)] - \frac{\alpha_0}{\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\gamma_0}{2K}[1 - \cos(2\sqrt{K}s)]$$

The Transport Matrix

- We can always write:

$$x(s) = a\sqrt{\beta}\sin\Delta\psi + b\sqrt{\beta}\cos\Delta\psi$$

- Solve for *a* and *b* in terms of initial conditions and write in matrix form
  - we get:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

So, can write any of our transport matrices in terms of values of C-S parameters at the two end points, and the phase advance between them.

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

Tracking β, α, γ ...

- Saw earlier that if given values of the Courant-Snyder parameters at one location in the beam line, and if know the matrix of the linear motion between that location and another location downstream, then can compute the values at the second location *via*:

$$\boxed{K = M\ K_0\ M^T}$$

where $K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$

- Have not explicitly proven that the ellipse coefficients found earlier are the SAME as the parameters above, but they are — and, we will.

Evolution of the Phase Advance

- Also, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can “transport” phase *and* the Courant-Snyder parameters along a beam line from one point to another

Simple Examples

- Propagation through a Drift:

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = \tan^{-1}\left(\frac{L}{\beta_1 - L\alpha_1}\right)$$

$$\beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha = \alpha_0 - \gamma_0 L$$

$$\gamma = \gamma_0$$

- Propagation through a Thin Lens:

$$M = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = 0$$

$$\beta = \beta_0$$

$$\alpha = \alpha_0 + \beta_0/F$$

$$\gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2$$

$$\boxed{K = M\ K_0\ M^T}$$

- Given *α, β* at one point, can calculate *α, β* at all downstream points

Betatron Oscillation Amplitude

- Transverse oscillations in a synchrotron (or beam line) are called Betatron Oscillations (first observed/analyzed in a "betatron" accelerator)
- Write *x,x'* in terms of initial conditions x_0, x'_0 :

$$x(s) = a\sqrt{\beta}\cos\Delta\psi + b\sqrt{\beta}\sin\Delta\psi$$

$$x' = \frac{1}{\sqrt{\beta}}([b - a\alpha]\cos\Delta\psi - [a + b\alpha]\sin\Delta\psi)$$

$$\downarrow$$

$$a = \frac{x_0}{\sqrt{\beta_0}}, \qquad b = \frac{\alpha_0 x_0 + \beta_0 x'_0}{\sqrt{\beta_0}}$$

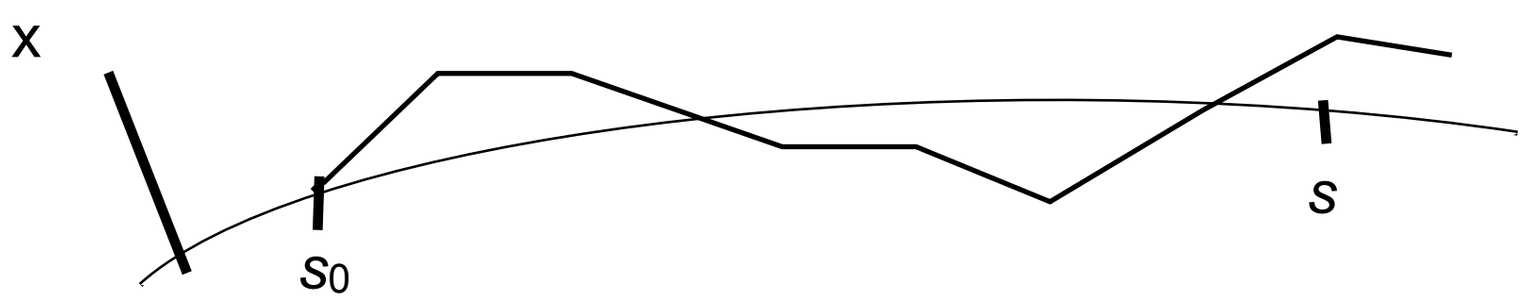
$$\boxed{\Longrightarrow x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\,[x_0\cos\Delta\psi + (\alpha_0 x_0 + \beta_0 x'_0)\sin\Delta\psi]}$$

$$\text{amplitude: } A = \sqrt{\frac{x_0^2 + (\alpha_0 x_0 + \beta_0 x'_0)^2}{\beta_0}}$$

Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle $\Delta x' = x'_0 = \Delta\theta$
- Then, downstream, we have

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin[\psi(s) - \psi_0]$$



Example:
Suppose $\Delta\theta = 0.4$ mrad, $\beta_0 = 4.0$ m, $\beta(s) = 6.4$ m, and $\Delta\psi = n \times 2\pi + 30°$. Then $x(s) = 1$ mm.

Courant-Snyder Invariant

- In general,

$$
\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
&= A\sqrt{\beta}\cos\psi - \alpha x \\
\boxed{\beta x' + \alpha x} &\boxed{= A\sqrt{\beta}\cos\psi}
\end{aligned}
$$

$$
\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}
$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

While C-S parameters evolve along the beam line, the combination above remains constant.

Properties of the Phase Space Ellipse

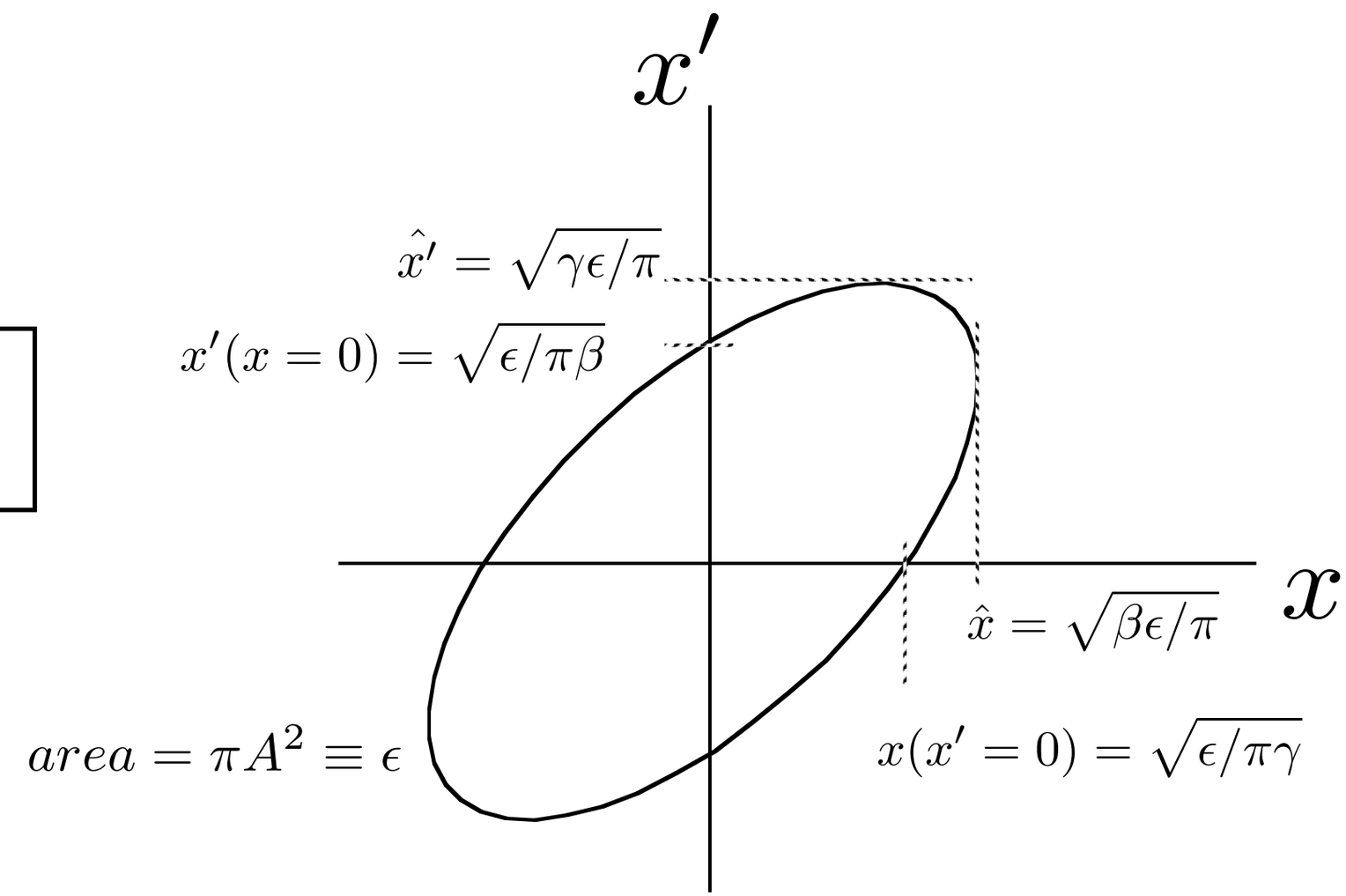
- The eqn. for the C-S invariant is that of an ellipse.
- If compute the area of the ellipse, find that …

$$\boxed{area = \pi A^2}$$

i.e., while the ellipse changes shape along the beam line, its area remains constant

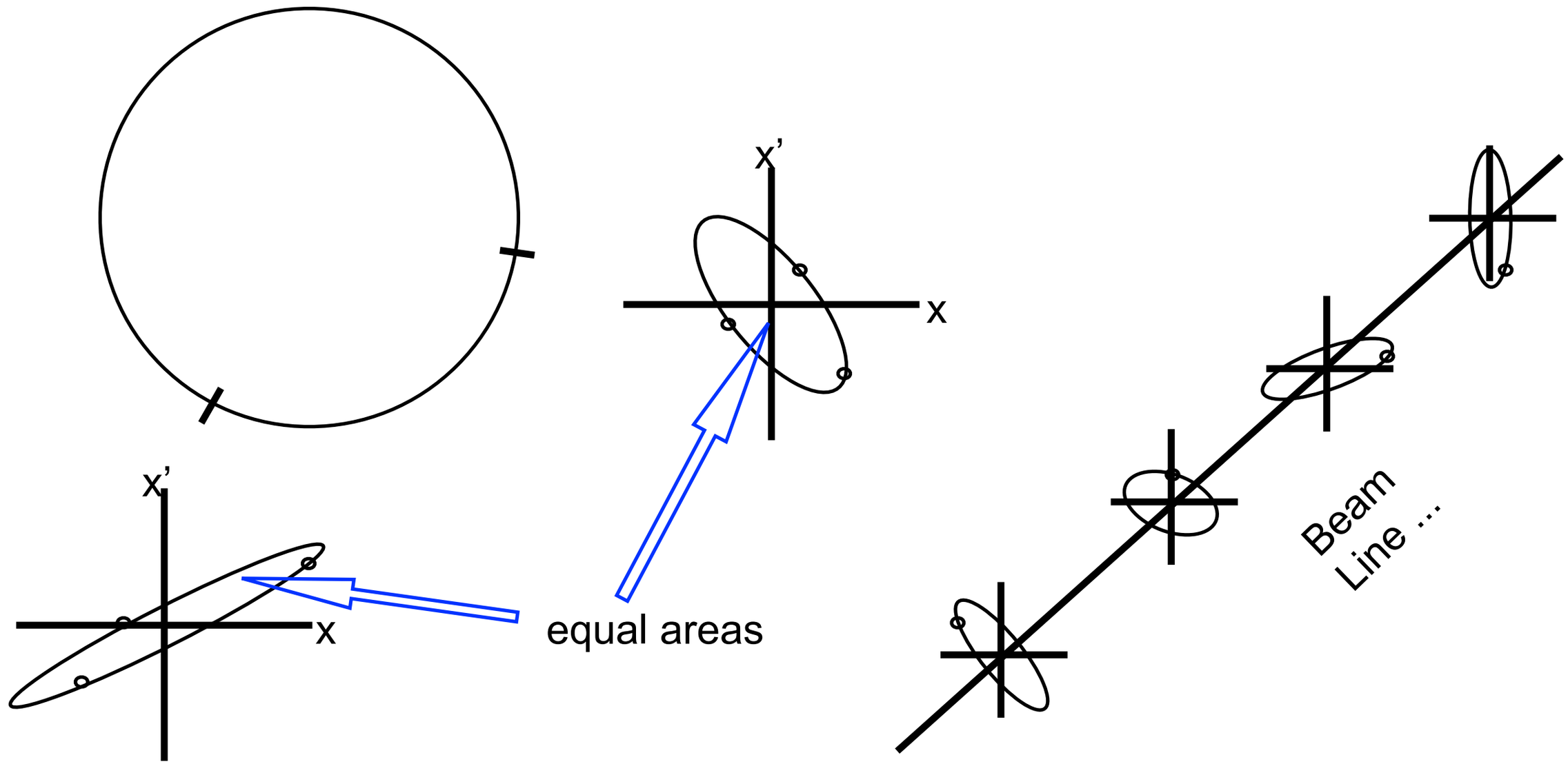
Emittance = area within a phase space trajectory

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2$$



Motion in Phase Space

- Follow phase space trajectory...



Courant-Snyder Parameters

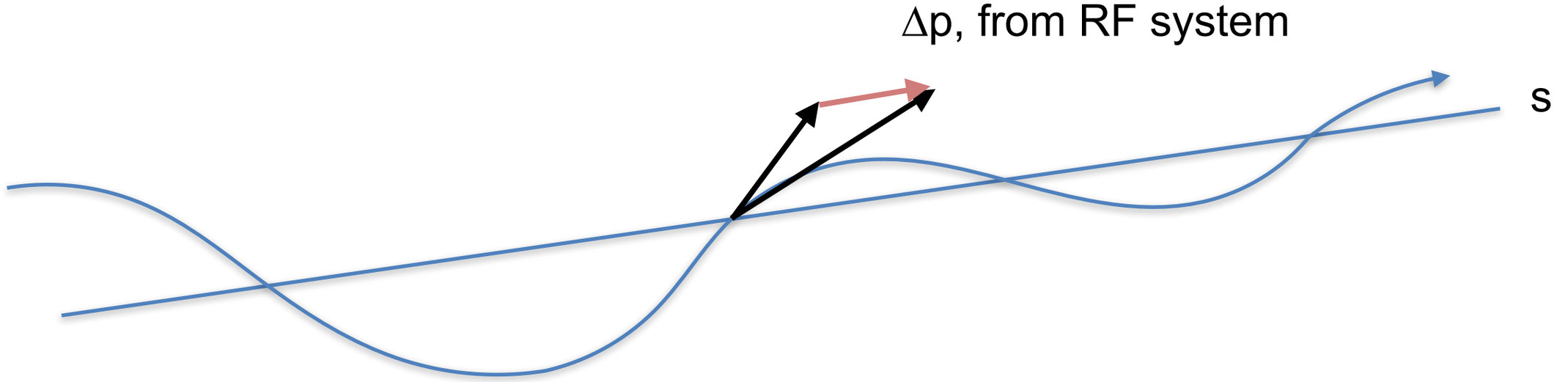
- So, we see that the solution to Hill's Equation leads to an amplitude function β(s) and a related phase advance ψ(s) such that dψ(s)/ds = 1/β(s). The general solution is
  - $x(s) = A\sqrt{\beta(s)}\ \sin[\psi(s) + \delta]$
- We see that if we define functions $\alpha(s)$ = -1/2 dβ(s)/ds and γ(s) = (1+α(s)^2)/β(s), then the phase space trajectory of a single particle lies on an ellipse given by
  - $\gamma(s)\, x^2 + 2\, \alpha(s)\, xx' + \beta(s)\, x'^2 = A^2$
- AND, the area of the ellipse is given by πA^2
- THUS, α, β, γ are the same parameters found to describe the phase space distributions discussed earlier

Choice of Initial Conditions

- Have seen how β, α can be propagated from one point to another. Still, have the choice of initial values...
- If periodic system, like a "ring," then natural to choose the *periodic* solutions for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- In a system like a linac, wish to "match" to desired initial conditions at the input to the system (somewhere after the source, say) using an arrangement of focusing elements

Adiabatic Damping from Acceleration

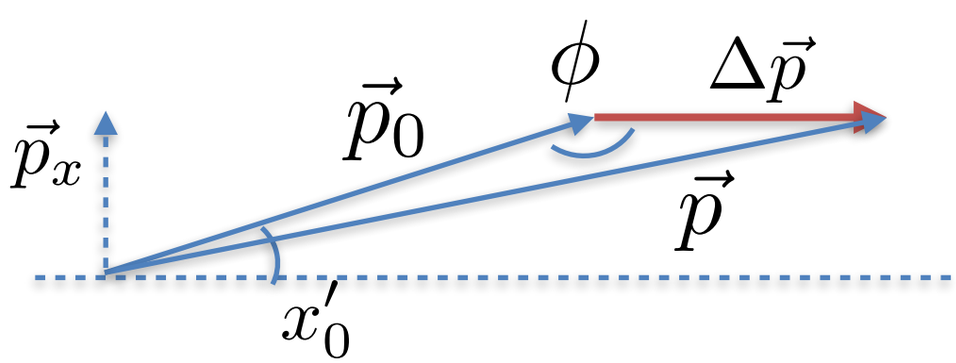
- Transverse oscillations imply transverse momentum. As accelerate, momentum is "delivered" in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.



- The coordinates *x*-*x'* are not canonical conjugates, but x-p_x are; thus, from classical mechanics, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Adiabatic Damping from Acceleration

details…



Note: assuming that ALL particles receive the same *Δp* from the cavity

$$x' = \frac{p_x}{p} = \frac{p_x}{\sqrt{p_0^2 + \Delta p^2 - 2\Delta p\, p_0 \cos\phi}} = \frac{p_x}{p_0}\left(1 - \frac{\Delta p}{p_0} + \ldots\right) \approx x'_0\left(1 - \frac{\Delta p}{p_0}\right)$$

Note: particles at peak of their betatron oscillation will have little/no change in *x'*, while particles with large transverse angles will have their *x'* affected most

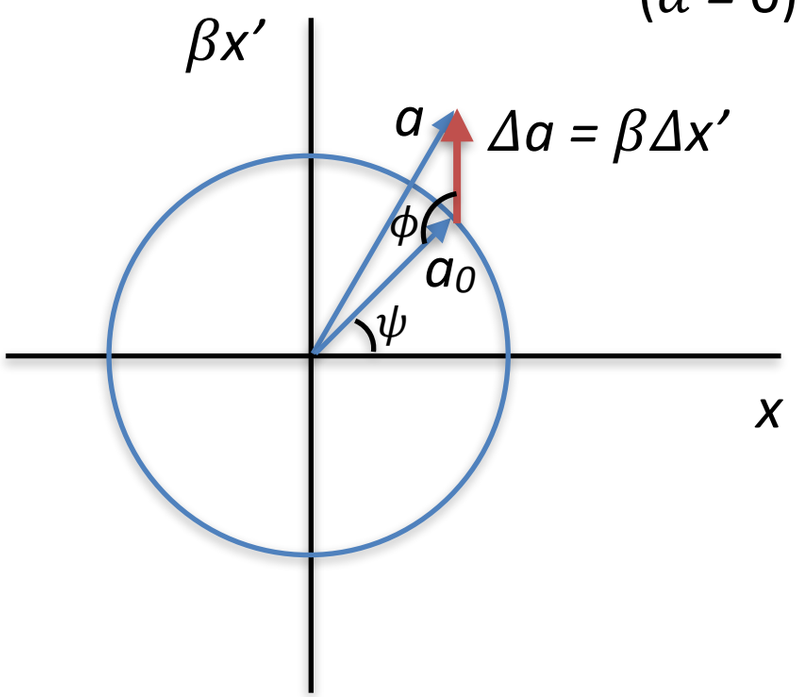
$$\Longrightarrow \quad \Delta x' = -x'_0\,\frac{\Delta p}{p_0}$$

Adiabatic Damping from Acceleration

details…

Now assuming that the incremental change in momentum is small compared to the overall momentum, and that the changes occur gradually on time scales large compared to those of the betatron motion…

($\alpha = 0$)



$$\boxed{\Delta x' = -x'_0 \frac{\Delta p}{p_0}}$$

$$\begin{aligned}
a^2 &= a_0^2 + \Delta a^2 - 2\Delta a\; a_0 \cos\phi \\
&= a_0^2 + \Delta a^2 + 2\Delta a\; a_0 \sin\psi \\
&= a_0^2 + \Delta a^2 + 2\Delta a\; \beta\; x'_0 \\
&= a_0^2 + (-\beta x'_0 \frac{\Delta p}{p_0})^2 + 2(-\beta x'_0 \frac{\Delta p}{p_0})\beta x'_0 \\
&= a_0^2 + (\beta x'_0)^2 \cancel{(\frac{\Delta p}{p_0})^2} - 2(\beta x'_0)^2 \frac{\Delta p}{p_0}
\end{aligned}$$

$$\Longrightarrow \quad \langle a^2 \rangle = \langle a_0^2 \rangle - 2\langle (\beta x'_0)^2 \rangle \frac{\Delta p}{p_0}$$

Note: $2\langle (\beta x')^2 \rangle = 2\langle x^2 \rangle = \langle a^2 \rangle$

So, $\Delta\langle a^2 \rangle = -\langle a^2 \rangle \frac{\Delta p}{p}$ $\qquad \frac{\Delta\epsilon}{\epsilon} = -\frac{\Delta p}{p}$

$$\boxed{\epsilon \propto \frac{1}{p} \qquad x_{rms} \propto \frac{1}{\sqrt{p}}}$$

Normalized Beam Emittance

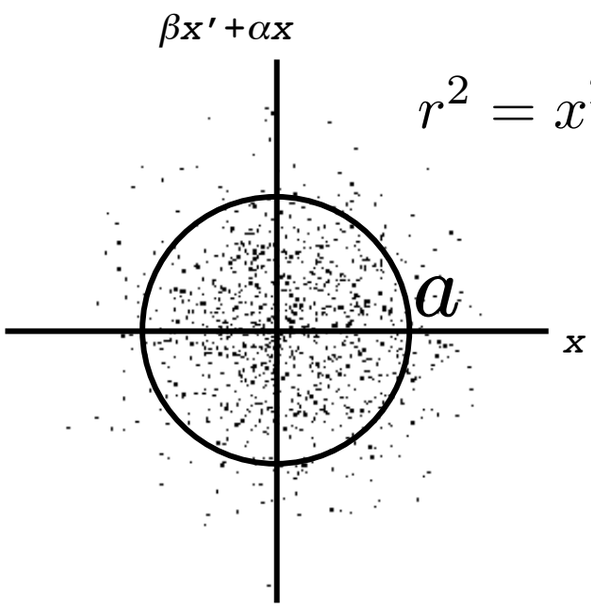
- Hence, as particles are accelerated, the area in *x-x'* phase space is not preserved, while area in x-p_x **is** preserved. Thus, we define a "normalized" beam emittance, as

$$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)_{Lorentz}$$

- In principle, the *normalized* beam emittance should be preserved during acceleration, and hence along the chain of accelerators from source to target. Thus it is a measure of beam quality, and its preservation a measure of accelerator performance.
- In practice, it is not preserved -- non-adiabatic acceleration, especially at the low energy regime; non-linear field perturbations; residual gas scattering; charge stripping; field errors and setting errors; *etc.* -- all contribute at some level to increase the beam emittance. Best attempts are made to keep the emittance as small as possible.

Emittance of a Stationary Gaussian Distribution

- Imagine a transverse distribution of particles with a Gaussian profile in transverse coordinate *x* with zero mean and standard deviation σ.
  - The distribution can be described as follows:



$$r^2 = x^2 + (\beta x' + \alpha x)^2$$

$$\rho(r,\theta) r dr d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r \, dr d\theta$$

$$\int_0^{2\pi} \int_0^a \rho \, r dr d\theta = f$$

Radius, *a*, containing fraction, *f*, of particles, corresponding to phase space area with emittance, ϵ:

$$a^2 = -2\sigma^2 \ln(1-f) = \epsilon\beta/\pi$$

Emittance of a Stationary Gaussian Distribution

- So, the normalized emittance that contains a fraction *f* of a Gaussian beam is:

$$\epsilon_N = \frac{-2\pi \ln(1-f)\sigma^2(s)}{\beta(s)}(\beta\gamma)$$

Lorentz!

- Common values of *f* :

f	$\epsilon_N/(\beta\gamma)$
95%	$6\pi\sigma^2/\beta$
86.5%	$4\pi\sigma^2/\beta$
39%	$\pi\sigma^2/\beta$
15%	σ^2/β

Perhaps most commonly used, typically called the "rms" emittance; but, always ask if not clear in context!

more typical for light sources, e- colliders