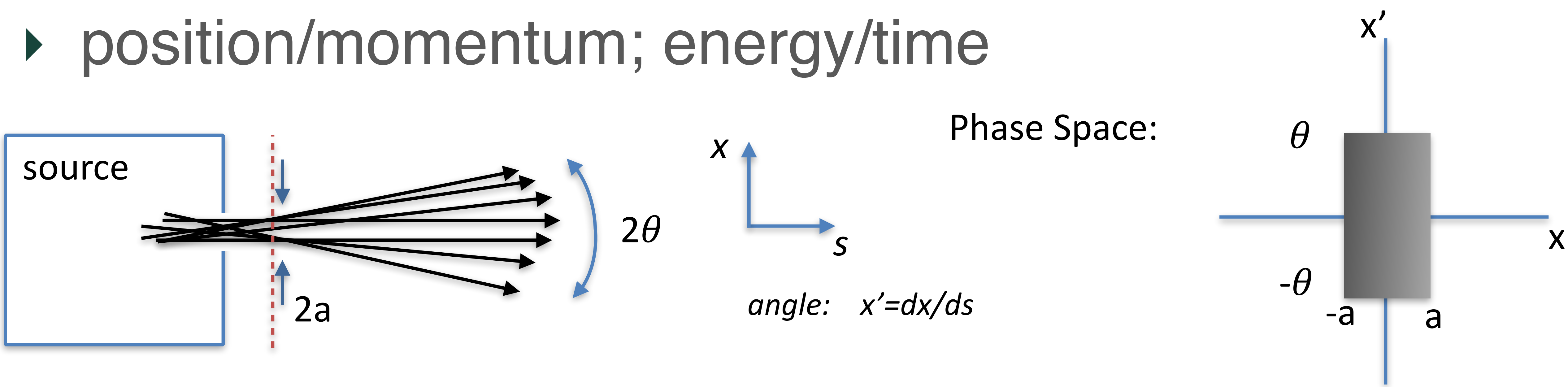
Particle Beams

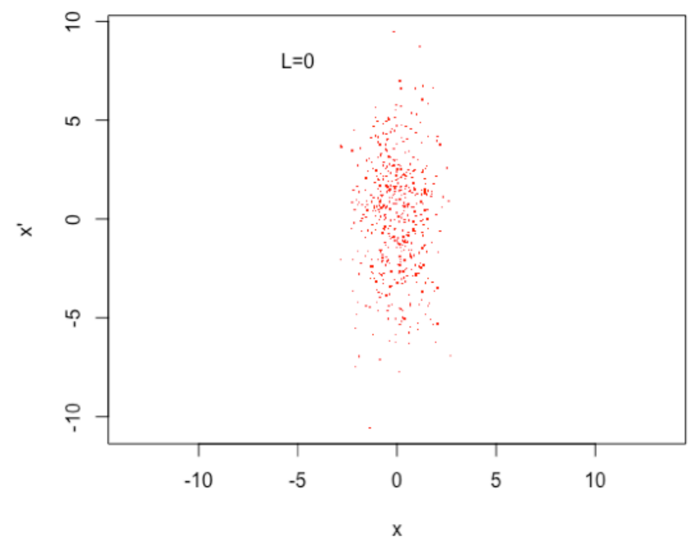
- Not just one particle, but a "bunch" of particles
- Finite spread in particle properties
  - energy / momentum spread
  - position / direction spread
- Characterization in terms of "phase space"
- Adiabatic invariance of phase space variables
  - position/momentum; energy/time

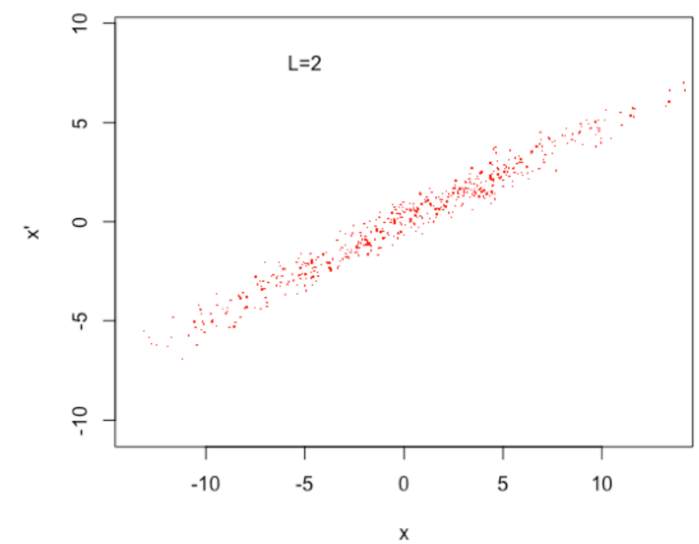


Evolution of the Phase Space

- Suppose transverse positions, *x*, and transverse angles, *x'*, are distributed in a normal (Gaussian) fashion coming from the "source"
- If the distribution is allowed to "drift" a distance L, then $x = x_0 + Lx'_0$ for each particle; the particles with largest $x' = x'_0$ will quickly drift to larger *x* values, and the distribution will "shear"

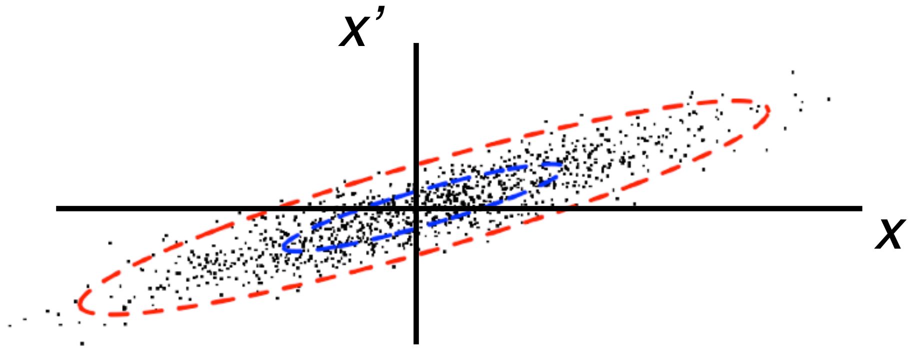
Shape, orientation of distribution in "phase space" will change, but effective "area" of distribution will remain constant





Emittance in Terms of Moments

- Considering the general equation of an ellipse, the area enclosed by the ellipse is related to the coefficients by:



$$ax^2 + bxx' + cx'^2 = 1$$

$$\mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

- from the web site:

http://nicadd.niu.edu/~syphers/uspas/2018w/some-notes-on-ellipses.html

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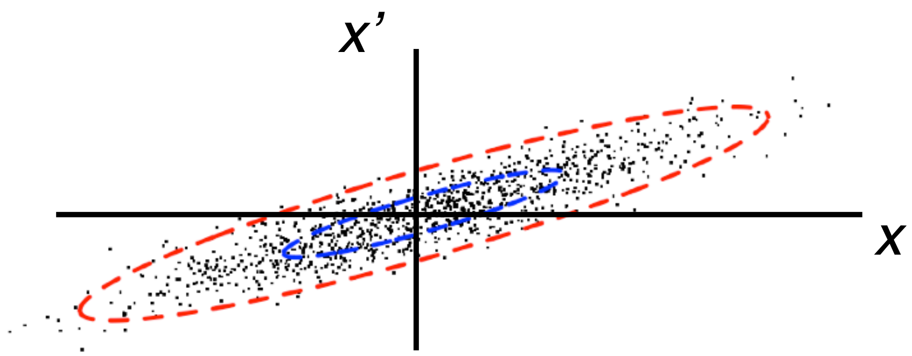


$$ax^2 + bxx' + cx'^2 = 1 \qquad \mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

- Can define scaled quantities from our distribution:

$$\alpha \equiv -\frac{\langle xx' \rangle}{\epsilon/\pi} \qquad \beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \qquad \gamma \equiv \frac{\langle x'^2 \rangle}{\epsilon/\pi}$$

$$\epsilon = \pi \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

α, β, γ collectively are called the *Courant-Snyder* parameters, or *Twiss* parameters

the "rms emittance"

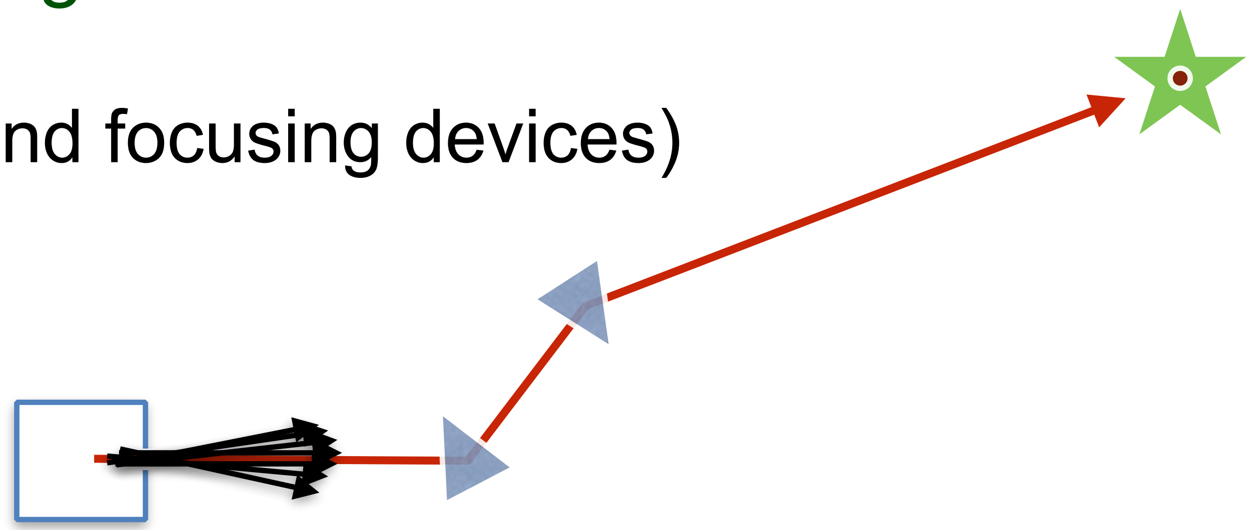
So, equation of the **blue** curve above:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon/\pi$$

The ellipse (red curve above) that contains ~95% has area ~6ϵ

Essential Beam Transport and Focusing

- Can imagine using a section of finite length containing pure uniform magnetic field to bend a charged particle's trajectory through a portion of a circular arc, thus steering it in a new direction. An arrangement of such magnets can thus be used to guide an "ideal" particle from one point to another
- However, most (all?) particles are NOT ideal! Hence, as particles drift away from the ideal trajectory, we wish to guide them (using quadrupole magnets or solenoids) back toward the ideal.
- Will use discrete electromagnets of finite length and assume a linear relationship between a particle's exit trajectory to its entrance trajectory, depending upon the strength of the magnetic field
  - (similar rules for electrostatic bending and focusing devices)



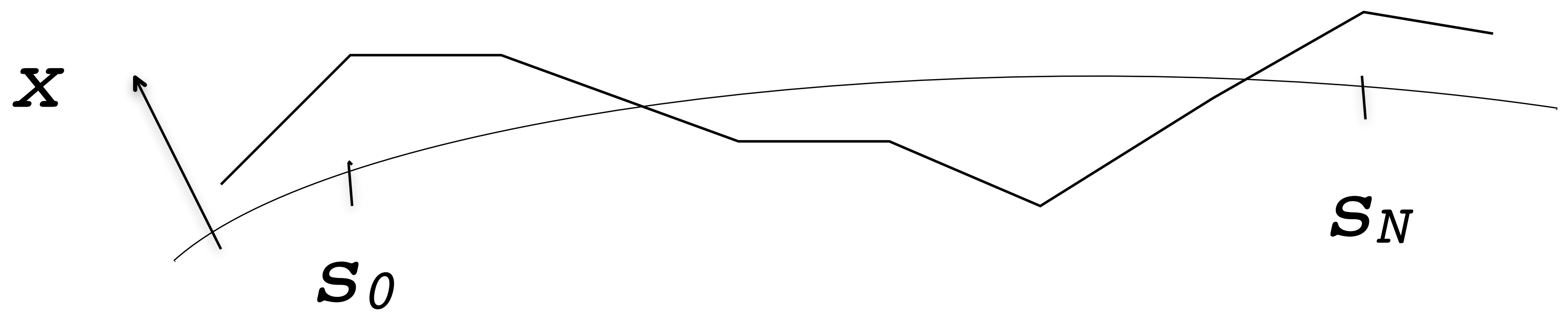
Linear Optics

- Let *x* be the transverse (horizontal, say) displacement of a particle from the ideal beam trajectory. Let the angle it makes to the ideal trajectory be *x'* = *dx*/*ds*, where *s* is the distance along the ideal trajectory. Transport through a magnetic element is then described by a matrix *M*, such that

$$\vec{X} = M\vec{X}_0 \qquad \vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

- An arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication for elements all along the beam line…

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Piecewise Method -- Matrix Formalism

- Write solution to each "piece" of the beam transport system in matrix form
  - for each piece, assume K = const. from s = 0 to s = L

- $K = 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

free space/drift, or bending magnet

focusing (quadrupole) field

- $K > 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K < 0$:

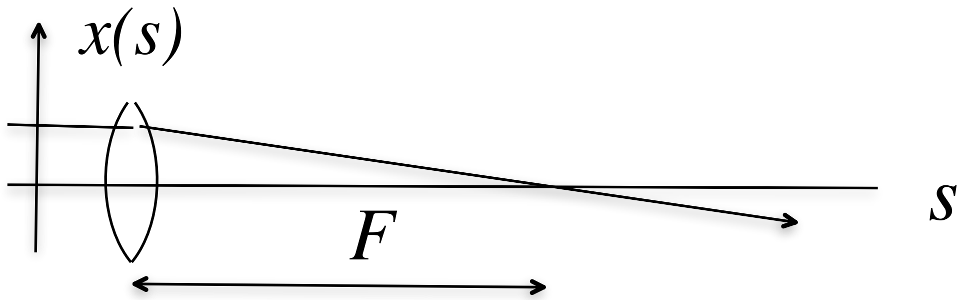
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

defocusing (quadrupole) field

note: det$M$ = 1

"Thin Lens" Quadrupole

- If quadrupole magnet is short enough, particle's offset through the quad does not change by much, but the slope of the trajectory does -- acts like a "thin lens" in geometrical optics



- Take limit as *L* --> 0, while *KL* remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

- (similarly, for defocusing quadrupole)
- valid approximation, if *F* >> L

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

TRANSPORT of Beam Moments

- Transport of particle state vector downstream from position 0

$$\vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix} \qquad \vec{X} = M\vec{X}_0$$

- Create a "covariance matrix" of the resulting vector…

$$\vec{X}\vec{X}^T = \begin{pmatrix} x^2 & xx' \\ x'x & x'^2 \end{pmatrix} = M\vec{X}_0(M\vec{X}_0)^T = M\vec{X}_0\vec{X}_0^T M^T$$

- … then, by averaging over all the particles in the distribution,

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

we get: $$\Sigma = M\Sigma_0 M^T$$

TRANSPORT of Beam Moments

- So, since

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} \sim \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = \epsilon \cdot K$$

- where

$$K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\boxed{K = M\ K_0\ M^T}$$

- If know matrices *M*, then can "transport" beam parameters from one point to any point downstream, which determines beam distribution along the way.

$$\beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \quad \Rightarrow \quad x_{rms}(s) = \sqrt{\epsilon\beta(s)/\pi}$$

Conservation of Emittance

- Note that from

$$\Sigma = M\Sigma_0 M^T$$

$$\Sigma = \epsilon \cdot K \qquad K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\det \Sigma = \det M \ \det \Sigma_0 \ \det M^T = \det \Sigma_0$$

- and

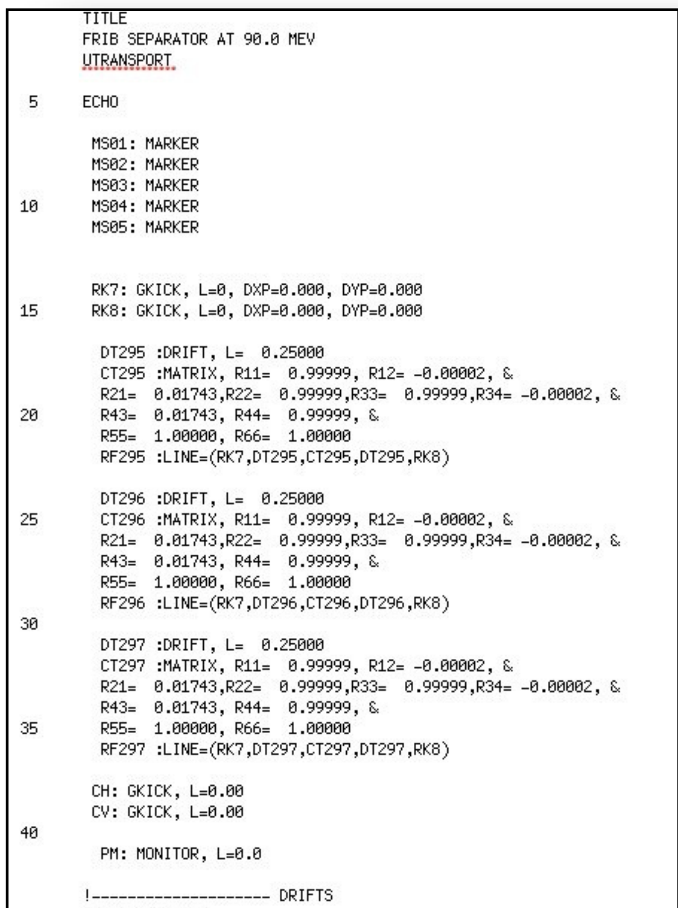
note: det$M$ = 1

$$\det \Sigma = \epsilon^2 \det K = \epsilon^2(\beta\gamma - \alpha^2) = \epsilon^2$$

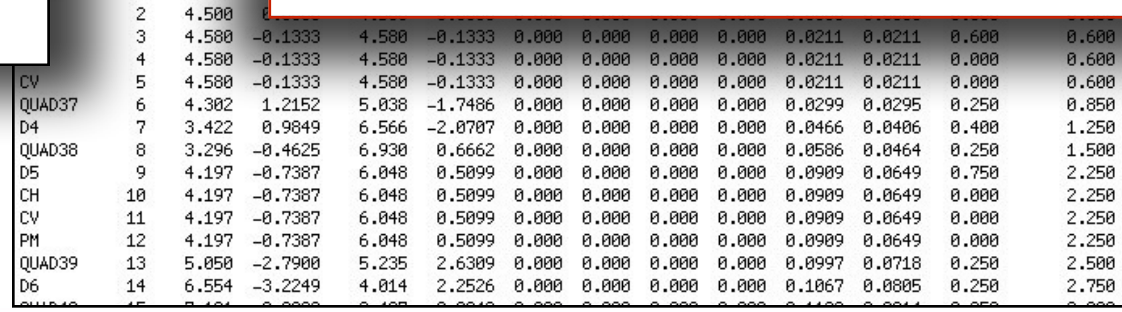
- Thus, the emittance is conserved upon transport through the system

Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT, MAD, DIMAD, TRACE, TRACE3D, COSY, SYNCH, CHEF, many more ...

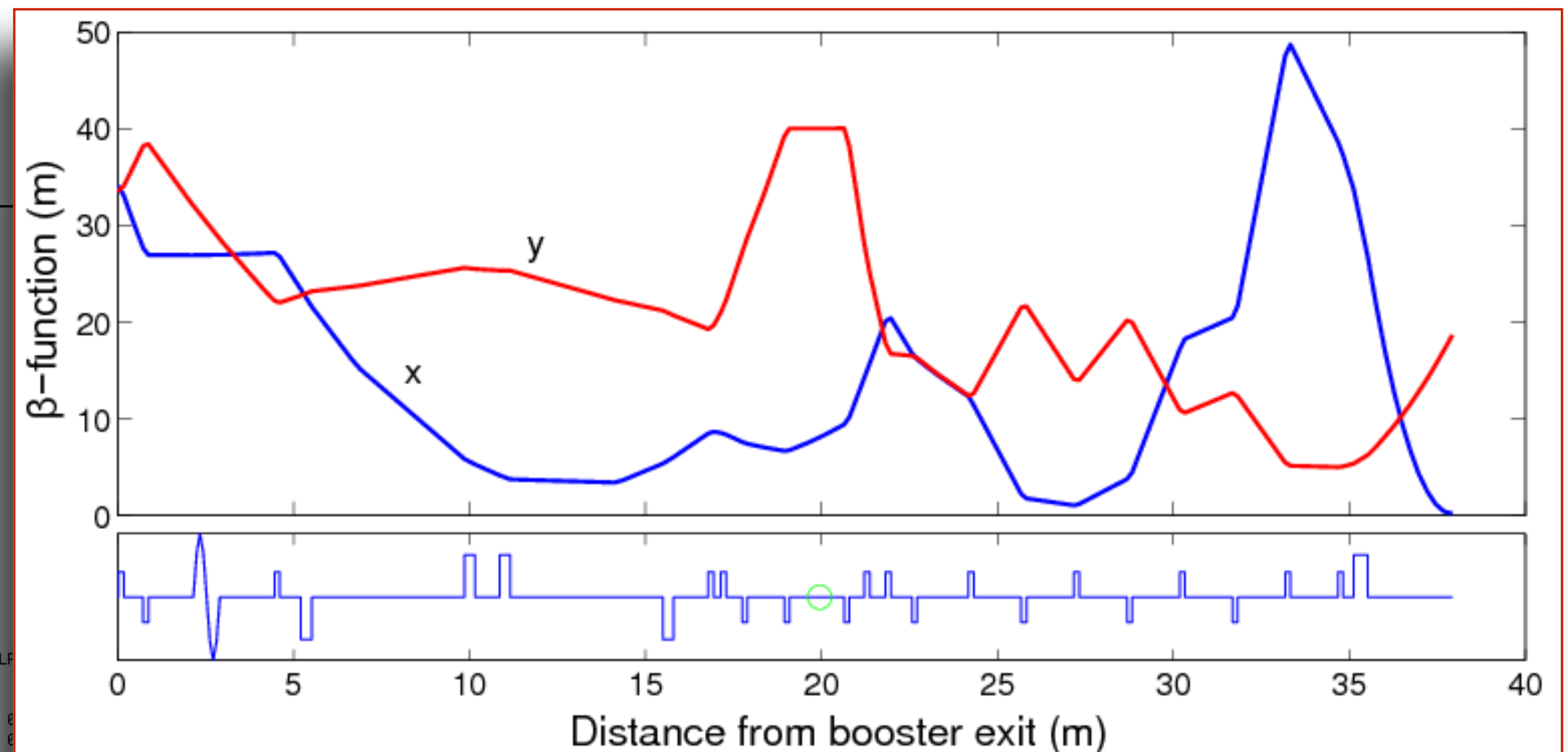


```
TITLE
FRIB SEPARATOR AT 90.0 MEV
UTRANSPORT

ECHO

 MS01: MARKER
 MS02: MARKER
 MS03: MARKER
 MS04: MARKER
 MS05: MARKER


 RK7: GKICK, L=0, DXP=0.000, DYP=0.000
 RK8: GKICK, L=0, DXP=0.000, DYP=0.000

  DT295 :DRIFT, L=  0.25000
  CT295 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF295 :LINE=(RK7,DT295,CT295,DT295,RK8)

  DT296 :DRIFT, L=  0.25000
  CT296 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF296 :LINE=(RK7,DT296,CT296,DT296,RK8)

  DT297 :DRIFT, L=  0.25000
  CT297 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF297 :LINE=(RK7,DT297,CT297,DT297,RK8)

 CH: GKICK, L=0.00
 CV: GKICK, L=0.00

  PM: MONITOR, L=0.0

!-------------------- DRIFTS
D0:DRIFT L=0.0
```



```
D42 K1   -2.987138
T03 K2    0.000000
D43 K1    2.990031

D47 K1    2.370599
D48 K1   -1.605765
D49 K1   -1.641612
D50 K1    1.272052

D51 K1    2.346764
D52 K1   -2.557171
D53 K1    2.031046
D54 K1   -2.003034
RATION LIST ,

ACHINE

 0.1 0 0.01 1 1
 -0.0 0 0
 -0.0 0 0
```

MENT	#	BETAX	ALF										
		4.500											
1	1	4.500											
	2	4.500											
	3	4.580	-0.1333	4.580	-0.1333	0.000	0.000	0.000	0.000	0.0211	0.0211	0.600	0.600
	4	4.580	-0.1333	4.580	-0.1333	0.000	0.000	0.000	0.000	0.0211	0.0211	0.000	0.600
CV	5	4.580	-0.1333	4.580	-0.1333	0.000	0.000	0.000	0.000	0.0211	0.0211	0.000	0.600
QUAD37	6	4.302	1.2152	5.038	-1.7486	0.000	0.000	0.000	0.000	0.0299	0.0295	0.250	0.850
D4	7	3.422	0.9849	6.566	-2.0707	0.000	0.000	0.000	0.000	0.0466	0.0406	0.400	1.250
QUAD38	8	3.296	-0.4625	6.930	0.6662	0.000	0.000	0.000	0.000	0.0586	0.0464	0.250	1.500
D5	9	4.197	-0.7387	6.048	0.5099	0.000	0.000	0.000	0.000	0.0909	0.0649	0.750	2.250
CH	10	4.197	-0.7387	6.048	0.5099	0.000	0.000	0.000	0.000	0.0909	0.0649	0.000	2.250
CV	11	4.197	-0.7387	6.048	0.5099	0.000	0.000	0.000	0.000	0.0909	0.0649	0.000	2.250
PM	12	4.197	-0.7387	6.048	0.5099	0.000	0.000	0.000	0.000	0.0909	0.0649	0.000	2.250
QUAD39	13	5.050	-2.7900	5.235	2.6309	0.000	0.000	0.000	0.000	0.0997	0.0718	0.250	2.500
D6	14	6.554	-3.2249	4.014	2.2526	0.000	0.000	0.000	0.000	0.1067	0.0805	0.250	2.750



Let's Think About the Numbers & Units…

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

- If $<x^2>$ ~ mm², and $<x'^2>$ ~ mrad², then the emittance can have units of mm-mrad (*also* = μm)
- Courant-Snyder parameters

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon}$$

mm²/(mm-mrad) ~ mm/mrad = m

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

(mm-mrad)/(mm-mrad) = dimensionless

$$\gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

mrad²/(mm-mrad) ~ 1/m

The "π" comes from our definition of emittance as an area in phase space; emittance is often expressed in units of "π mm-mrad"

Summary

- Given an initial particle distribution in phase space at the input to a beam transport system, can describe that distribution (sometimes not all that well, but we try…) using Courant-Snyder parameters:

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon} \qquad \gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

- The C-S parameters can then be computed downstream, using

$$\Sigma = M\Sigma_0 M^T$$