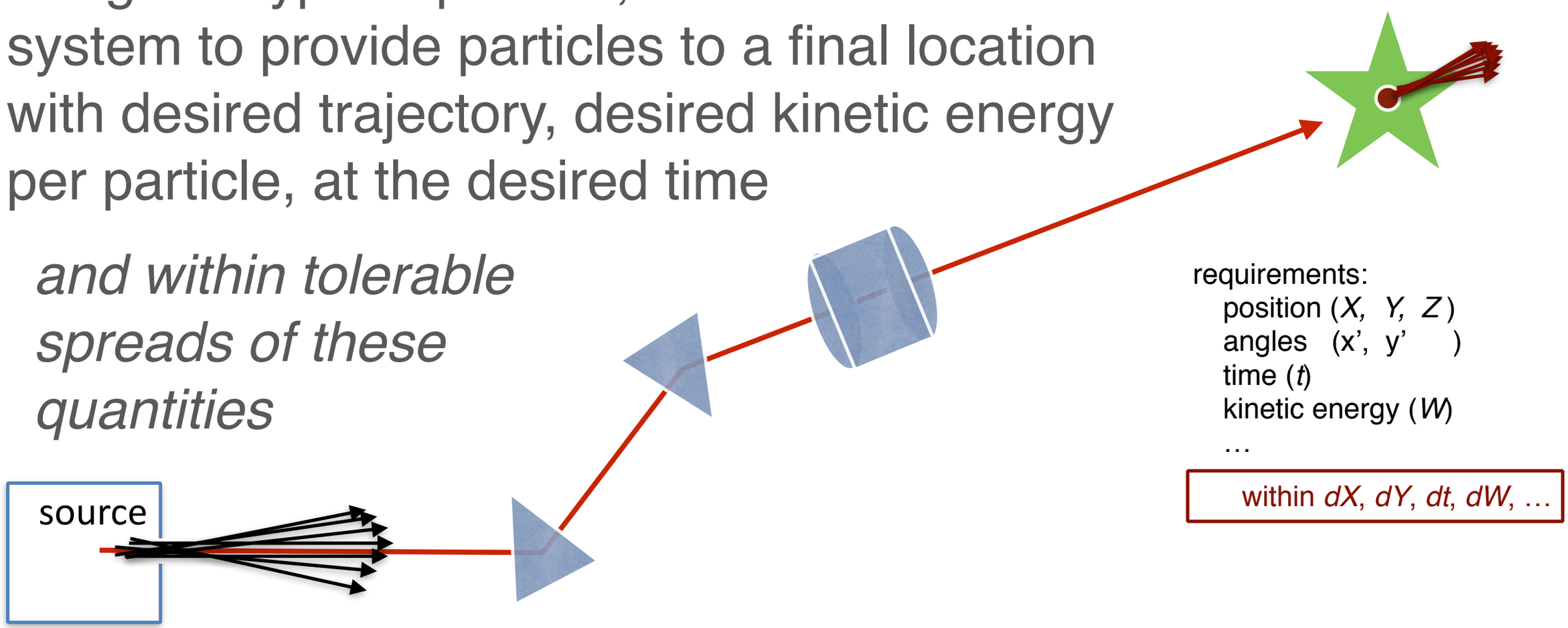
The Problem

- 1927: Lord Rutherford requested a “copious supply” of projectiles “more energetic than natural alpha and beta particles”

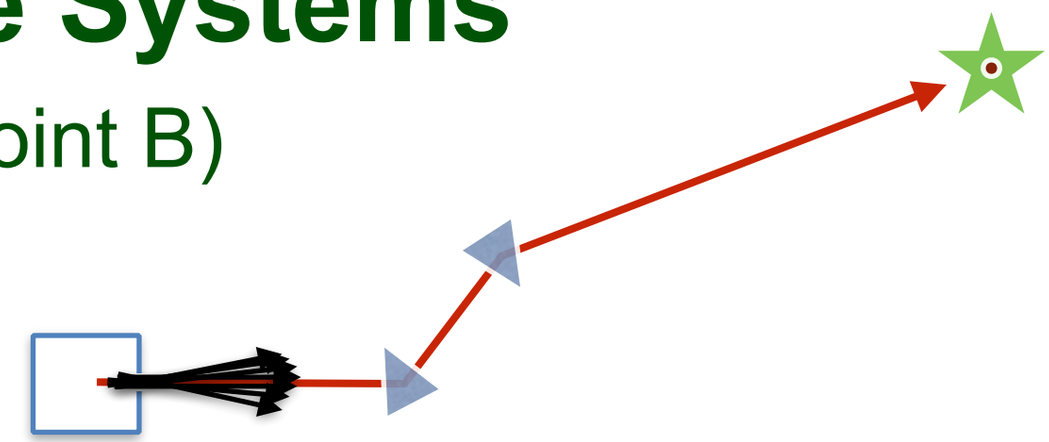
- For given type of particle, create an ideal system to provide particles to a final location with desired trajectory, desired kinetic energy per particle, at the desired time

  and within tolerable spreads of these quantities

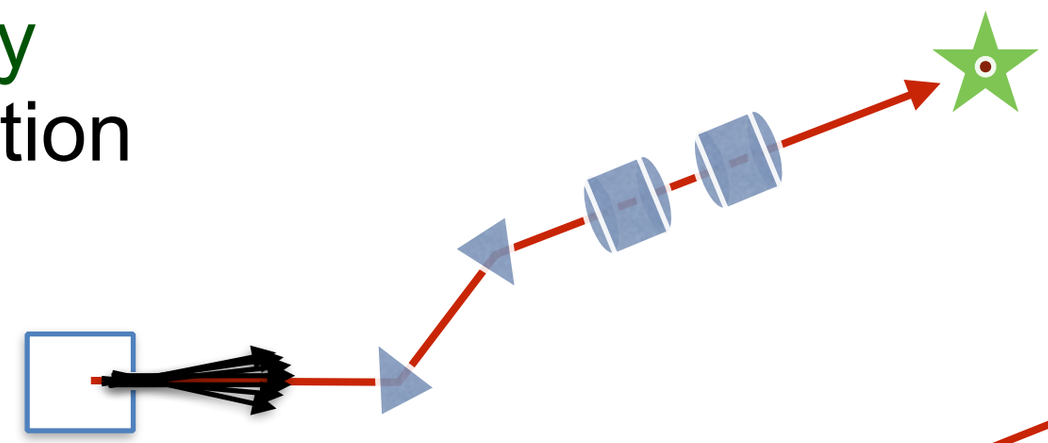


Single-Pass vs. Repetitive Systems

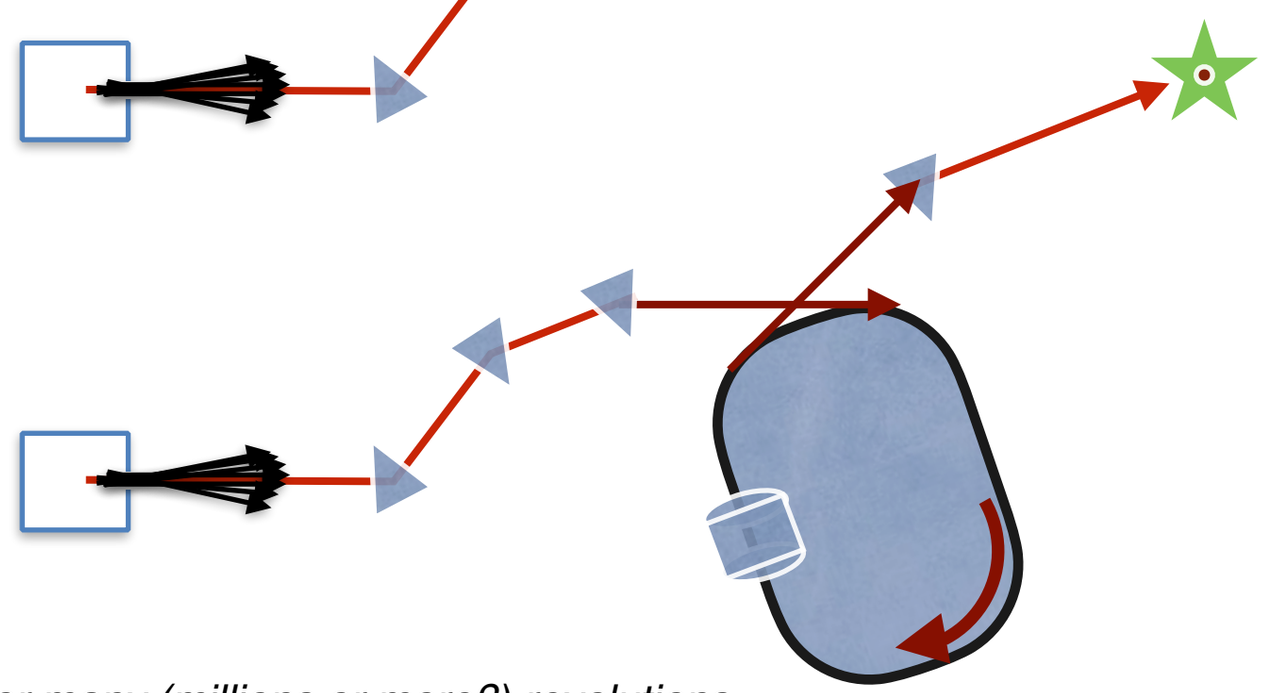
- Beam Transport (from point A to point B)



- Acceleration along the way
  - single-pass with acceleration



  - multi-pass acceleration



may need motion in such a system to be stable for many (millions or more?) revolutions

Stability of Motion Near the Ideal

- Not all particles (any??) begin “on” the design trajectory with *exactly* the ideal energy/momentum
- We wish to have a system that will keep particles near the ideal conditions as they are transported (and possibly accelerated) through the system
- Particles emerge from their “source” with a slight divergence and will need to be guided back toward the ideal trajectory
- Also, particles with different energies/momenta will travel at different speeds, and hence may not arrive at cavities, experiments, etc., at the ideal time

Reduction of the Problem

- Will treat transverse motion of particles through the accelerator as independent of the longitudinal motion, and study these two cases separately. Must show along the way that this is viable approach.
- Certainly not always be the case...
  - electric fields used for focusing at low energies can also accelerate the particles as well;
  - fields in the gaps of cavities will have focusing effects; etc.
- However, much of the “cross talk” can be minimized, and for much of the particle’s journey, especially at higher energies, the major transverse focusing can be performed by magnetic fields -- particle’s energy not changed
- Look at “linear” fields, *i.e.* linear restoring forces

Equations of Motion

- Lorentz Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

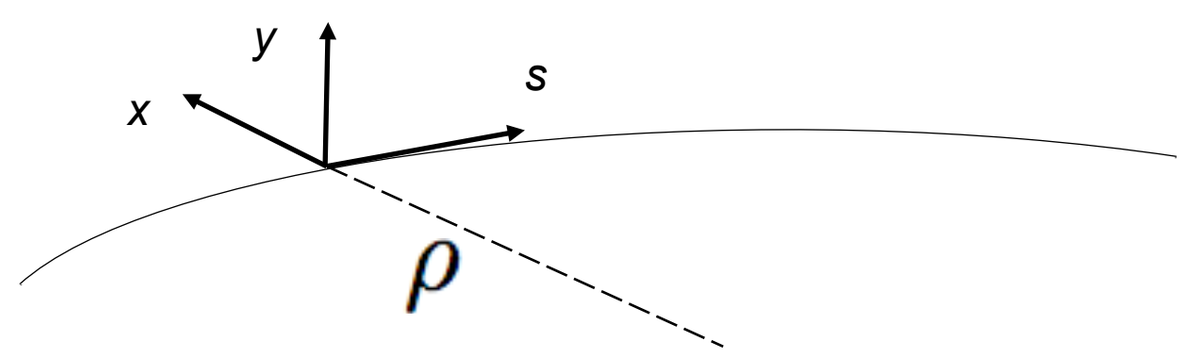
- Magnetic Rigidity
  - particle of unit charge, q = e:

$$B\rho \equiv \frac{p}{q} = \frac{p}{e}$$

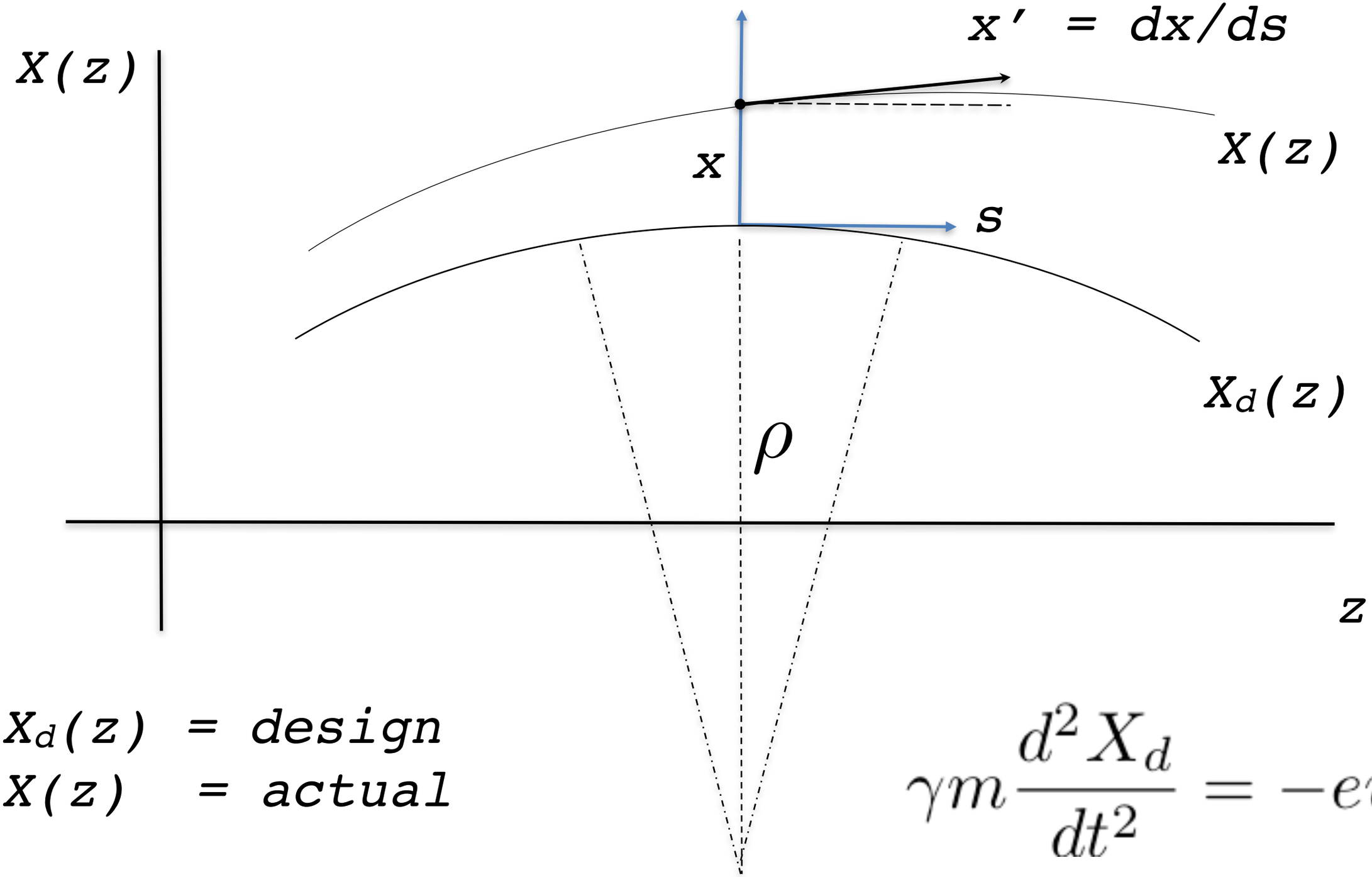
  - ion w/ mass A (atomic units, u), charge Q:

$$B\rho = \frac{A}{Q}\left(\frac{1}{300}\frac{\mathrm{T}\cdot\mathrm{m}}{\mathrm{MeV/c/u}}\right)\ p_u$$

- Reference Trajectory
  - Local Coordinate System



Linear Magnetic Fields for Guiding & Focusing



$X_d(z)$ = design
$X(z)$ = actual

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Transverse Fields ($B_z = 0$)

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

- Drift Space: $B_x = 0$, $B_y = 0$
- Bending Region: $B_x = 0$, $B_y = B_0$
  - *(dipole magnet)*
- Focusing Region: $B_x = B'y$, $B_y = B'x$
  - *(quadrupole magnet)*
- Combined Function Region: $B_x = B'y$, $B_y = B_0 + B'x$
  - *(gradient magnet)*
- Accelerating Device: $B_x = 0$, $B_y = 0$; $E_z = V/g$
  - *(cavity)*

Linear Restoring Forces

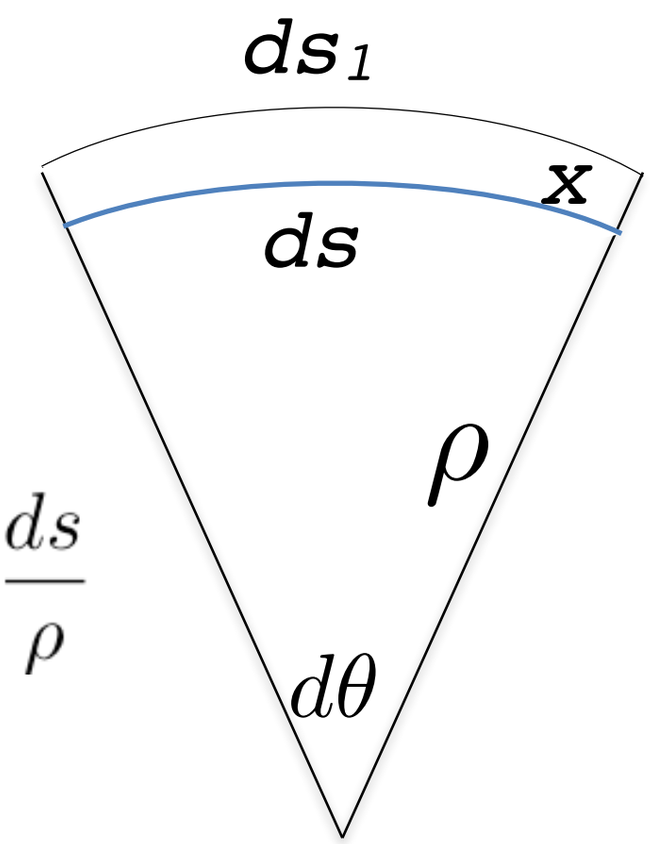
- Assume linear guide fields: --

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

- Look at radial motion:

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x' v_s$$

$$\frac{ds_1}{\rho + x} = \frac{ds}{\rho}$$



$$\begin{aligned}
\gamma m \frac{d^2(X_d)}{dt^2} &= -e v_s B_0 \\
\gamma m \frac{d^2(X_d + x)}{dt^2} &= -e v_{s1} B_y(X) \\
\gamma m (X_d'' + x'') v_s^2 &= -e v_{s1} B_y(X) \\
\gamma m v_s x'' &= -e \frac{v_{s1}}{v_s} B_y + e B_0 \\
\gamma m v_s x'' &= -e \left[B_y \left(1 + \frac{x}{\rho} \right) - B_0 \right] \\
x'' &= -\frac{e}{p} \left[(B_y - B_0) + B_y \frac{x}{\rho} \right] \\
&\approx -\frac{1}{B\rho} \left[B' x + B_0 \frac{x}{\rho} \right]
\end{aligned}$$

Hill's Equation

- Now, for vertical motion:

$$\begin{aligned} \gamma m \frac{d^2(Y_d)}{dt^2} &= 0 \\ \gamma m \frac{d^2(Y_d + y)}{dt^2} &= e v_{s1} B_x(Y) \\ \gamma m v_s^2 y'' &= e v_{s1} B_x(Y) \\ \gamma m v_s y'' &= e \frac{v_{s1}}{v_s} B_x \\ \gamma m v_s y'' &= e B_x \left(1 + \frac{x}{\rho}\right) \end{aligned}$$

$$\begin{aligned} B_y &= B_0 + B'x \\ B_x &= B'y \end{aligned}$$

- So we have,
- to lowest order,

$$\begin{aligned} y'' &= \frac{e}{p}\left[B_x(1 + \frac{x}{\rho})\right] \\ &\approx \left(\frac{B'}{B\rho}\right) y \end{aligned}$$

$$\begin{aligned} x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right) x &= 0 \\ y'' - \left(\frac{B'}{B\rho}\right) y &= 0 \end{aligned}$$

Hill's Equation

General Form: $\longrightarrow$ $x'' + K(s)x = 0$

- As accelerate, scale K with momentum; becomes purely geometrical

Piecewise Method of Solution

- Hill's Equation:

$$x'' + K(s)x = 0$$

  - Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: drift $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$

- *K* > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$

- *K* < 0: Quad, Gradient Magnet, edge, ... $$x(s) = x_0 \cosh(\sqrt{|K|} s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|} s)$$

Here, x refers to horizontal or vertical motion, with relevant value of K

Piecewise Method -- Matrix Formalism

- Write solution to each piece in matrix form
  - for each, assume K = const. from s=0 to s=L

- *K = 0:*

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- *K > 0:*

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- *K < 0:*

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

Determining *K*

$$x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right) x = 0$$

$$y'' - \left(\frac{B'}{B\rho}\right) y = 0$$

- Quadrupole Magnets

$$K_x = \frac{B'}{B\rho}$$

$$K_y = -\frac{B'}{B\rho}$$

- Sector Bend Dipole Magnets

$$K_x = \frac{1}{\rho^2}$$

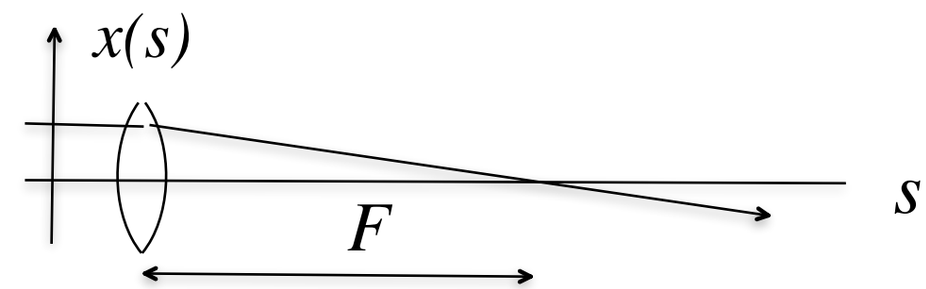
$$K_y = 0$$

- Other Considerations
  - Rectangular Bend
  - Bend with other Edge Angles

Note:
"combined-function"
sector bend

“Thin Lens” Quadrupole

- If quadrupole magnet is short enough, particle’s offset through the quad does not change by much, but the slope of the trajectory does -- acts like a “thin lens” in geometrical optics



- Take limit as *L* --> 0, while *KL* remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

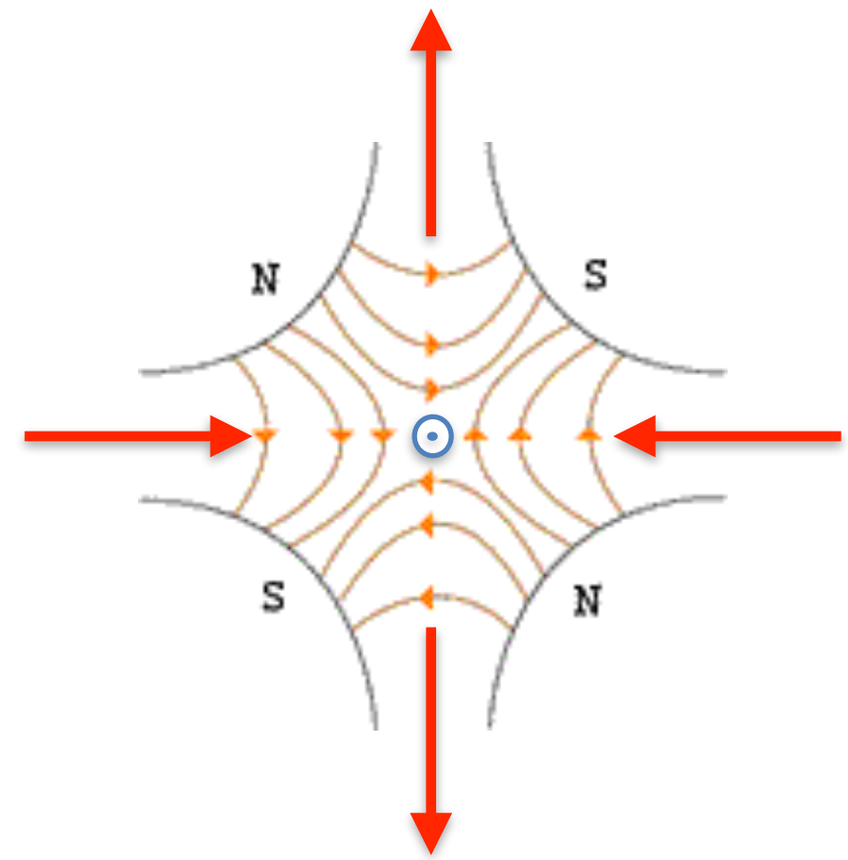
  - (similarly, for defocusing quadrupole)

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

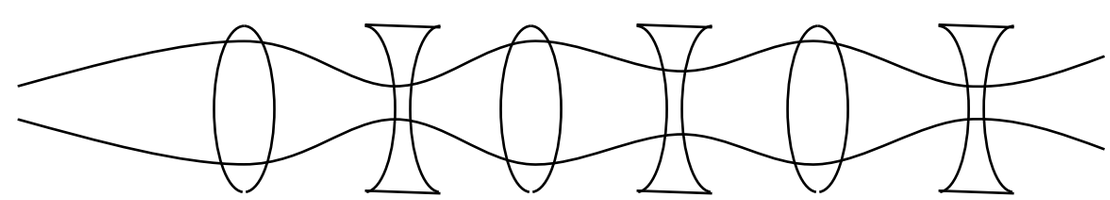
- Valid approx., if *F* >> L

Strong (Alternating Gradient) Focusing

- Think of standard focusing scheme as alternating system of focusing and defocusing lenses (today, use quadrupole magnets)

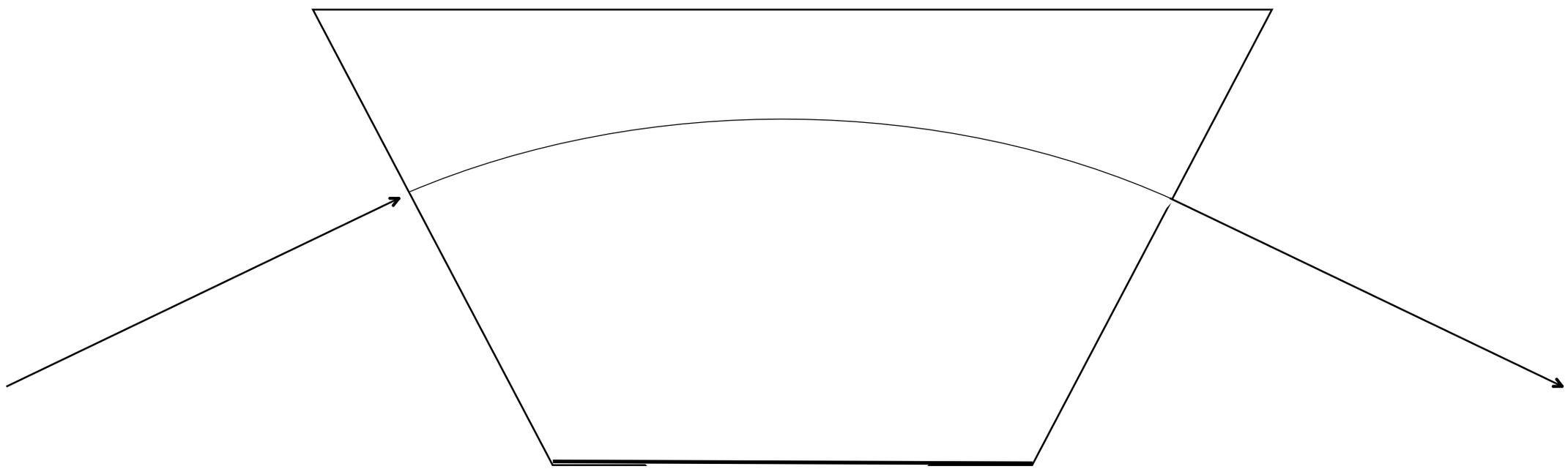


- Quadrupole will focus in one transverse plane, but defocus in other; if alternate, can have net focusing in both
  - alternating gradients:



Sector Magnets

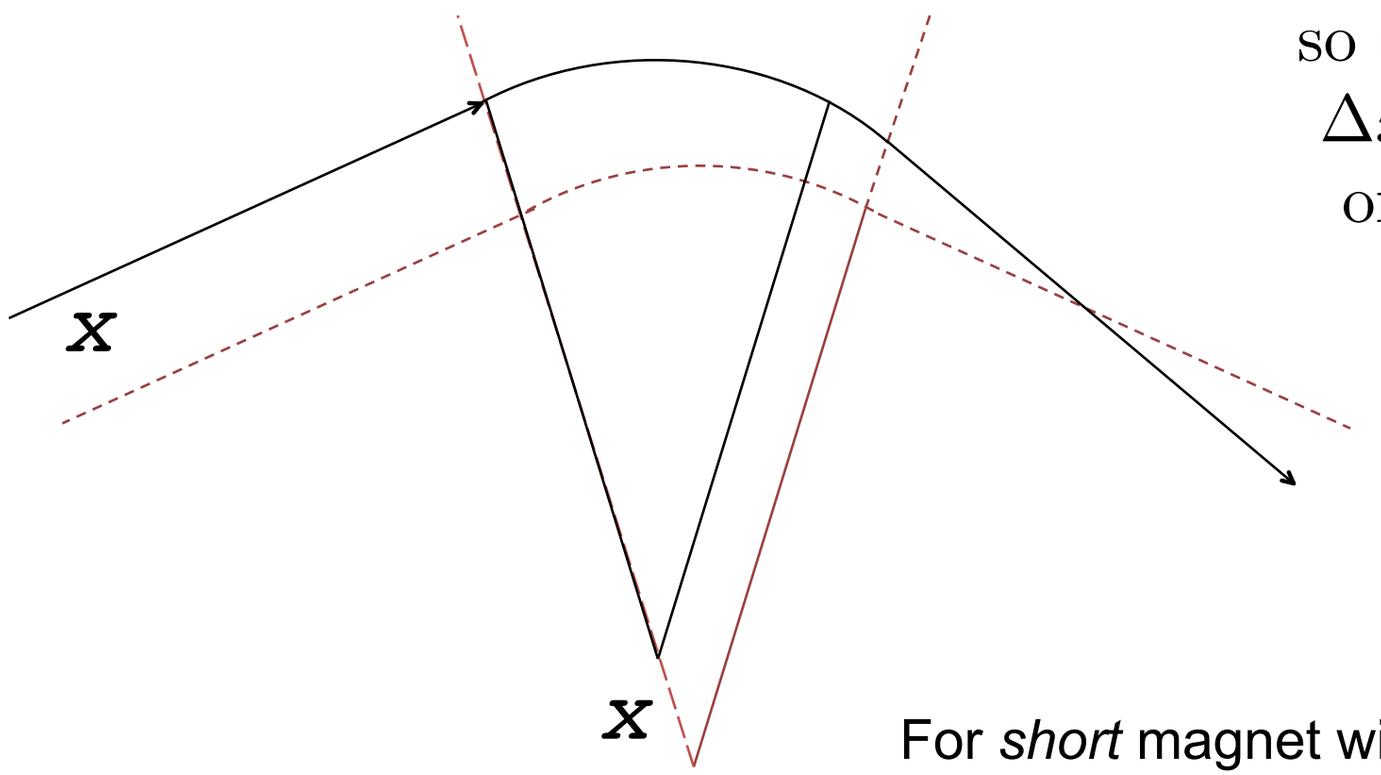
- Sector Dipole Magnet: “edge” of magnetic field is perpendicular to incoming/outgoing design trajectory:



Field points “*out of the page*”

Sector Magnets & Sector Focusing

- Incoming ray displaced from ideal trajectory will experience more/less bending field, thus is "focused" toward axis in the bend plane:



$$\text{Extra path length} = \Delta s = x\ \theta$$
$$\text{so extra bend angle} = \Delta x' = -\Delta s/\rho$$
$$\Delta x' = -(\theta/\rho)x = -(\ell/\rho^2)x$$
$$\text{or, } x'' = dx'/ds = -(1/\rho^2)x$$

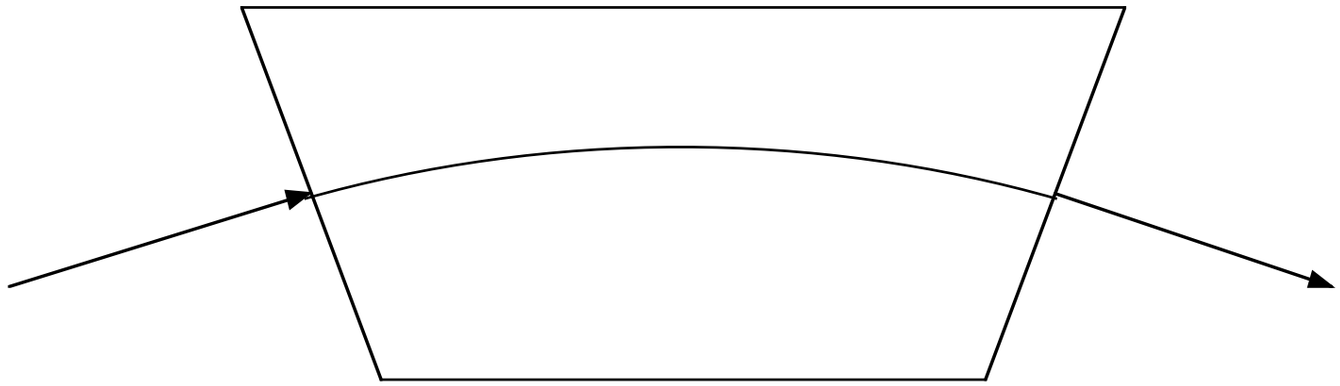
$$\text{Thus, } K_x = 1/\rho^2,\ K_y = 0.$$

For *short* magnet with small bend angle, will also acts like lens in the bend plane

$$\Longrightarrow \quad \frac{1}{f_x} = \frac{\theta}{\rho}$$

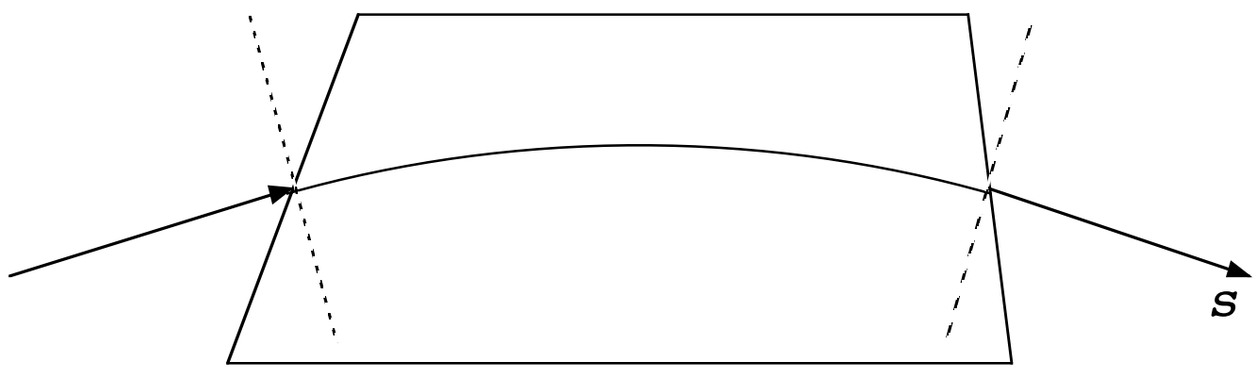
Edge Focusing

- In an ideal *sector magnet*, the magnetic field begins/ends exactly at $s = 0$, L independent of transverse coordinates x, y relative to the design trajectory.
  - *i.e.*, the face of the magnet is perpendicular to the design trajectory at entrance/exit



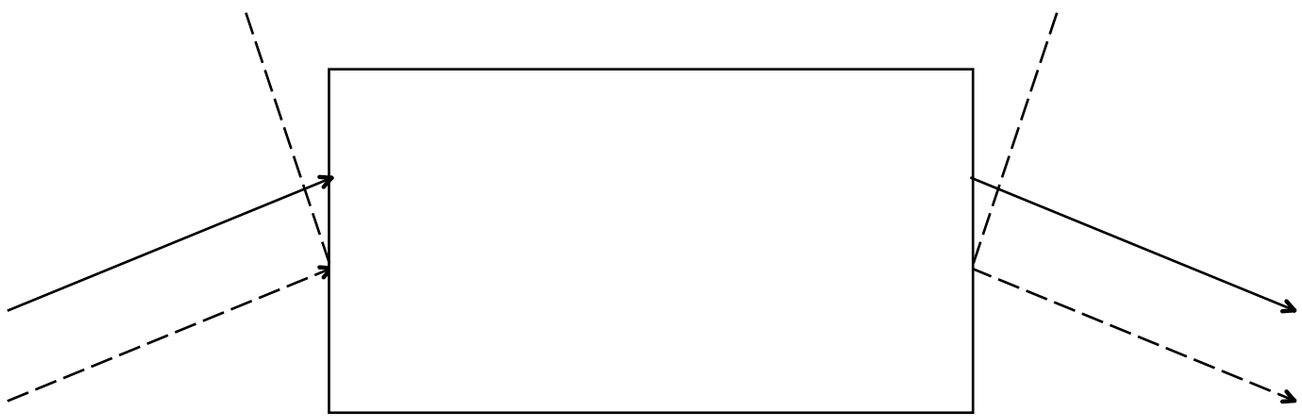
Edge Focusing

- However, could (and often do) have the faces at angles w.r.t. the design trajectory -- provides “edge focusing”



- Since our transverse coordinate *x* is everywhere perpendicular to *s*, then a particle entering with an offset will see more/less bending at the interface...

Rectangular Bending Magnet



In the bending plane, each edge acts as a defocusing lens with focal length:

$$\frac{1}{f} = -\frac{\theta/2}{\rho} = -\frac{\theta}{2\rho}$$

For Pure Sector Magnet,

hor: $\frac{1}{f_x} \approx \frac{\theta}{\rho}$

ver: $\frac{1}{f_y} \approx 0$

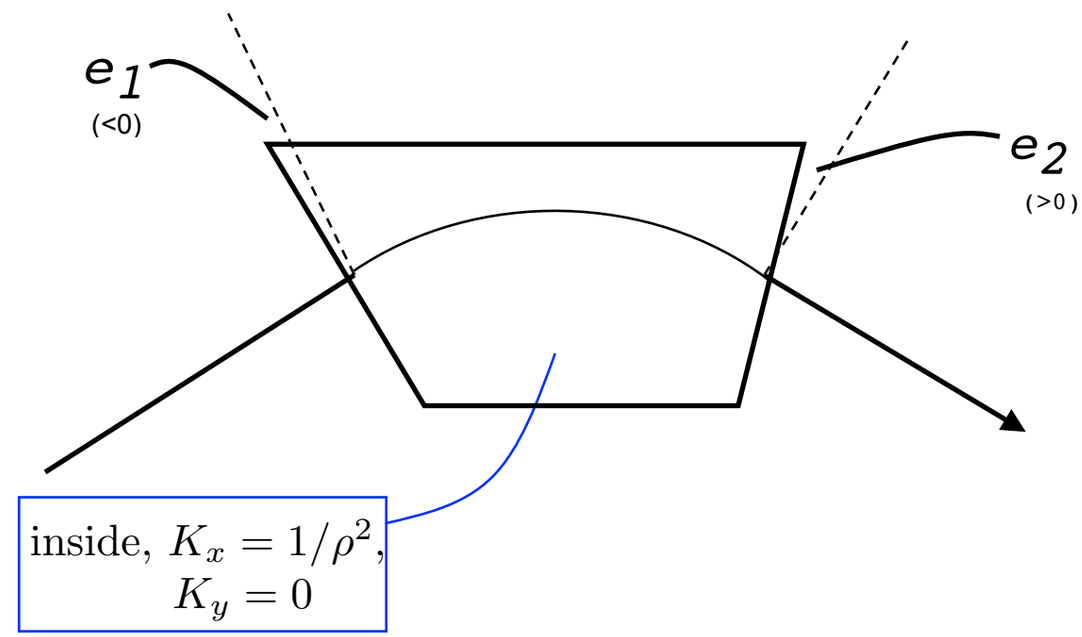
For Rectangular Magnet,

hor: $\frac{1}{f_x} \approx -\frac{\theta}{2\rho} + \frac{\theta}{\rho} - \frac{\theta}{2\rho} = 0$

ver: $\frac{1}{f_y} \approx \frac{\theta}{2\rho} + \frac{\theta}{2\rho} = \frac{\theta}{\rho}$

Transport through a General Bending Magnet

- Put all the pieces together…



- $M_{total} = M_{e2}\, M_{body}\, M_{e1}$

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan e_2}{\rho} & 1 \end{pmatrix} \begin{pmatrix} \cos(\ell/\rho) & \frac{1}{\rho}\sin(\ell/\rho) \\ -\frac{1}{\rho}\sin(\ell/\rho) & \cos(\ell/\rho) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\tan e_1}{\rho} & 1 \end{pmatrix}$$

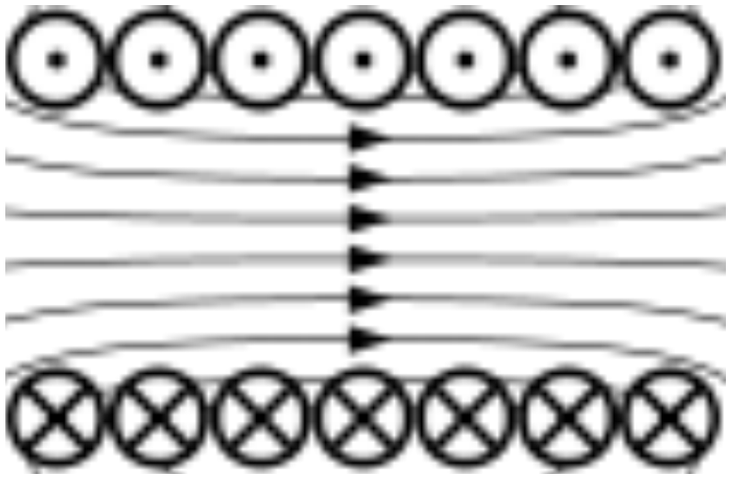
$$M_y = \begin{pmatrix} 1 & 0 \\ -\frac{\tan e_2}{\rho} & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\tan e_1}{\rho} & 1 \end{pmatrix}$$

Comments

- Very often, especially for highly-relativistic particles, the bend radii with bending magnets can be large, and hence the sector focusing can be a small effect. However, in accelerators with dozens, hundreds, or thousands of elements, it can certainly add up.
- Same can be said for edge effects in many circumstances.
- One must always seek to understand the particular situation and determine what assumptions can be made for the level of detail one is studying

Solenoid Focusing

- Solenoid Field



$$B = \mu_0 N' I$$

FRIB linac solenoids ~ 8-9 T

- Particle trajectory in a uniform field

Helical, with radius $a = mv_\perp/(qB_0)$

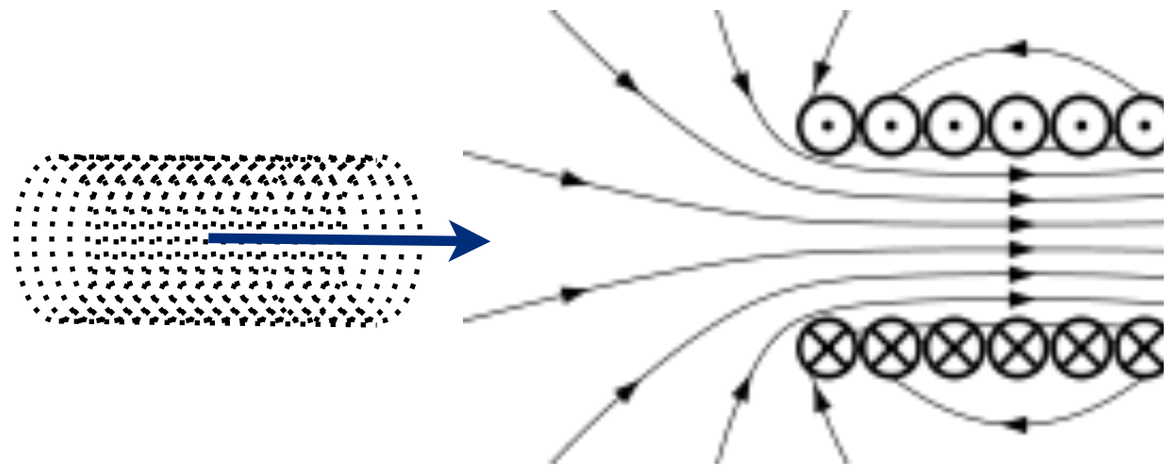
where $v_\perp$ is the velocity perpendicular to $\vec{B}$

$$\omega = \frac{v_\perp}{a} = qB_0/m$$

- So, how does a solenoid "focus"?

Acquisition of Angular Momentum

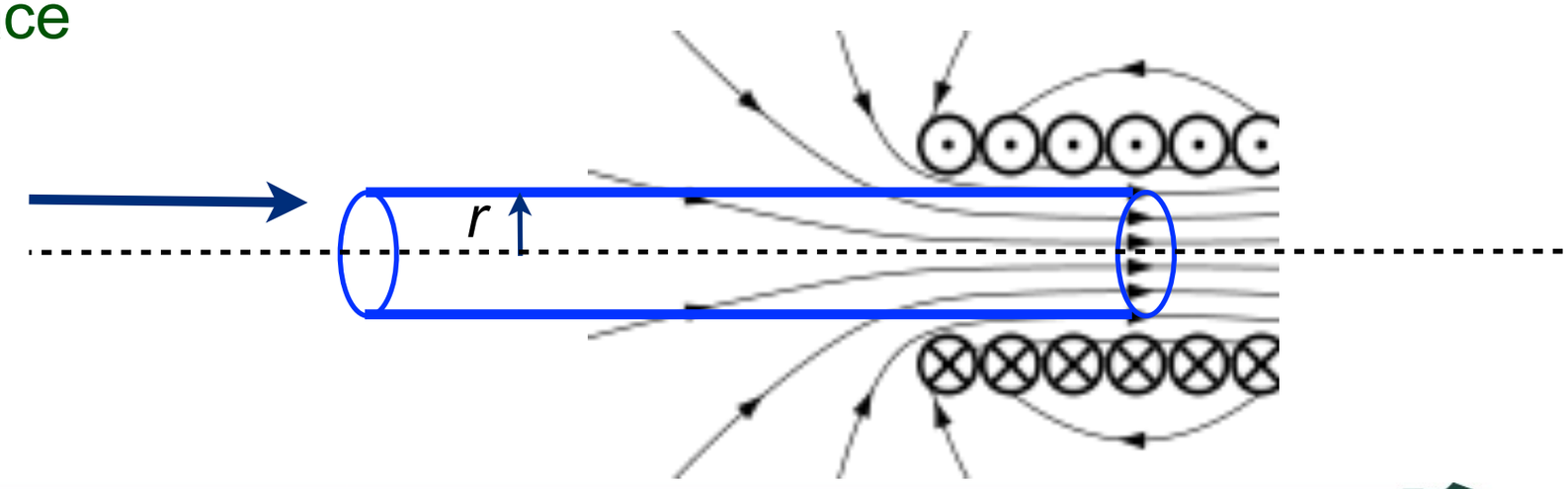
- Imagine particle distribution entering a solenoid magnet, centered on the axis of the magnet, and assume (for now) all trajectories are parallel...



- Treat the "edge" (entrance) of the solenoid as an impulse, but estimate its effect by integrating through the interface

$$\Delta p_\theta \approx q \int_{-\infty}^{0} (\vec{v} \times \vec{B})_\theta dt$$

$$= -\frac{qB_0}{2}\, r$$



Solenoid Focusing

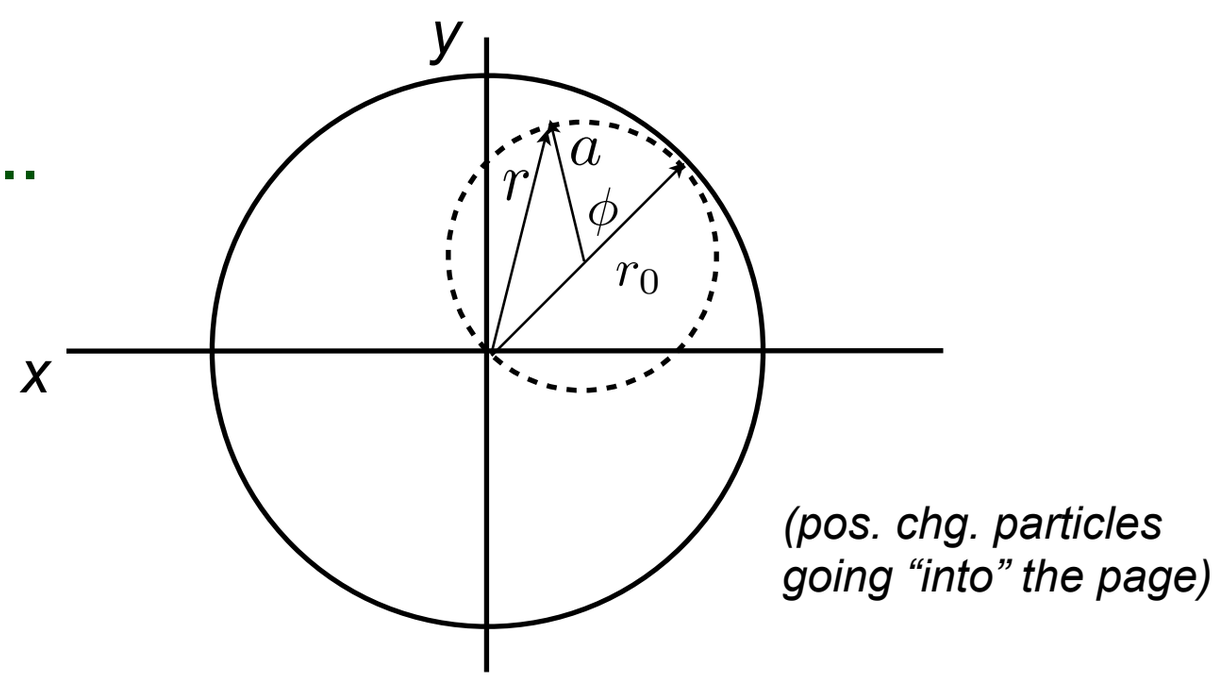
- So, the momentum gained in the theta direction will depend upon its distance from the solenoid axis, and thus its radius of gyration will be given through

$$\omega = qB_0/m = \frac{v_\perp(r_0)}{r_0/2} \qquad \longrightarrow \qquad a = r_0/2$$

- The resulting trajectory will be helical, with radius $a = r_0/2$, and the rotation will advance by an amount

$$\phi(s) = \frac{qB_0}{m}\,\frac{s}{v}$$

- Thus, ...



$$r = r_0 \cos[\phi(s)/2]$$

- Upon exit, the angular momentum of the beam will be removed.

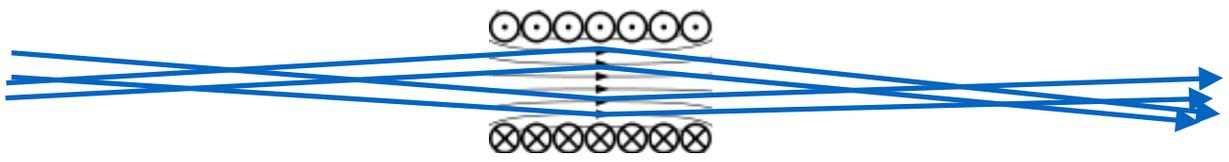
Solenoid Focusing

- We see that the equation of motion of the particle radius is

$$r'' = d^2r/ds^2 = d(r')/ds = -\left(\frac{qB_0}{2mv}\right)^2 r$$

- Thus, a "short" solenoid can be interpreted as a "thin lens" of focal length *f* given by…

$$\frac{1}{f} = \left(\frac{qB_0}{2mv}\right)^2 \ell = \left(\frac{B_0}{2(B\rho)}\right)^2 \ell = \frac{\theta_0^2}{\ell} \qquad \text{where } \theta_0 = B_0\ell/2(B\rho)$$



- Note: "thin" lens $\rightarrow \quad \ell/f = \theta_0^2 \ll 1$
- Use solenoids when Q is high and momentum is low.

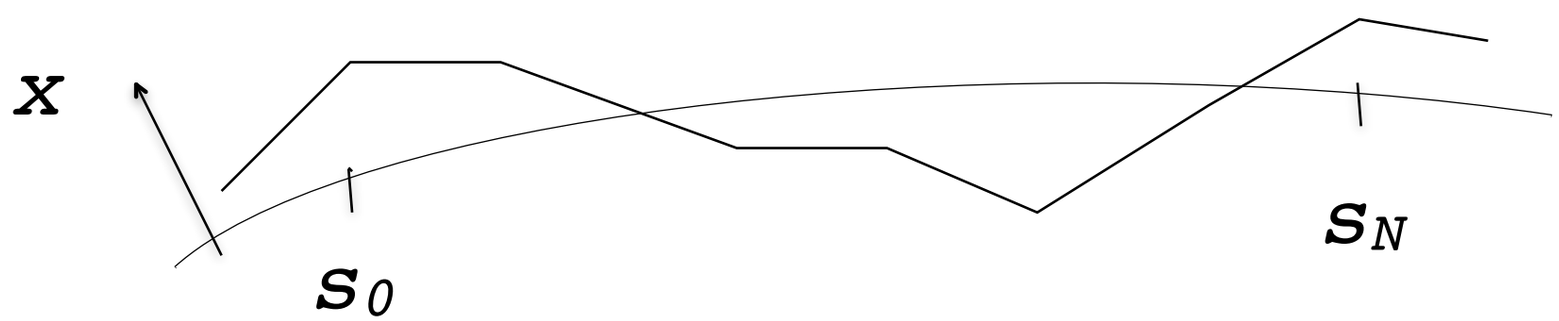
Solenoids vs. Quadrupoles

- Solenoids are most useful when
  - dealing with cylindrically symmetric systems
  - speeds are low and/or charge state is high
    - » focusing strength $1/f \sim 1/(B\rho)^2 \sim q^2 / p^2$
- Thus, often found in low-energy systems such as ion sources and low energy transport systems
- Since $F = q\ vxB$, and since v is mostly parallel to B in a solenoid, the useful force is smaller than when v is perpendicular to B
- Also, solenoids couple the motion — x and y motion not independent from each other
- Hence, at higher energies, typically use quadrupole fields

Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Example: A Beam Line Calculation

- Will consider two particle trajectories, starting with
  - $(x,x') = (0, 0.5 \text{ mrad})$, and $(x,x') = (5 \text{ mm}, 0)$
- A distance 6 m later, the trajectories enter a thin lens quadrupole of focal length $F = 3$ m. This is followed by a second quadrupole of focal length $-F$, a distance 1 m later.

  - Find the trajectories (x,x') for each case at the exit of the second quad

Example: A Beam Line Calculation

- Will consider two particle trajectories, starting with
  - $(x,x') = (0, 0.5 \text{ mrad})$, and $(x,x') = (5 \text{ mm}, 0)$
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$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 1\ m \\ \frac{1}{3\ m} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ -\frac{1}{3\ m} & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

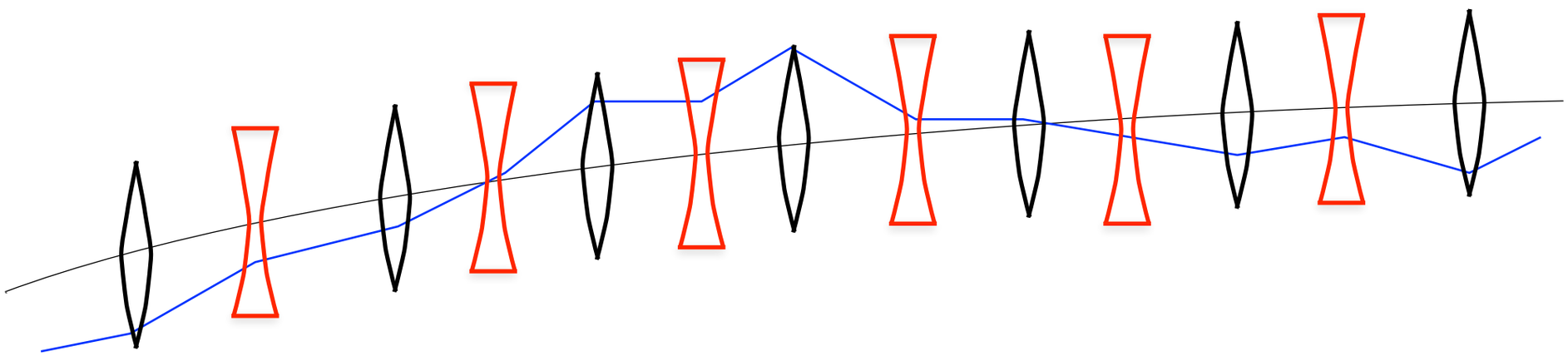
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 5\ m \\ -\frac{1}{9\ m} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

x_0 = 0 mm, x'_0 = 0.5 mr —> x = 2.5 mm, x' = 0.33 mr

x_0 = 5 mm, x'_0 = 0.0 mr —> x = 3.3 mm, x' = -0.6 mr

Can now make LARGE accelerators!

- Since the lens spacing can be made arbitrarily short, with corresponding focusing field strengths, then in principal can make beam transport systems (and linacs and synchrotrons, for instance) of arbitrary size

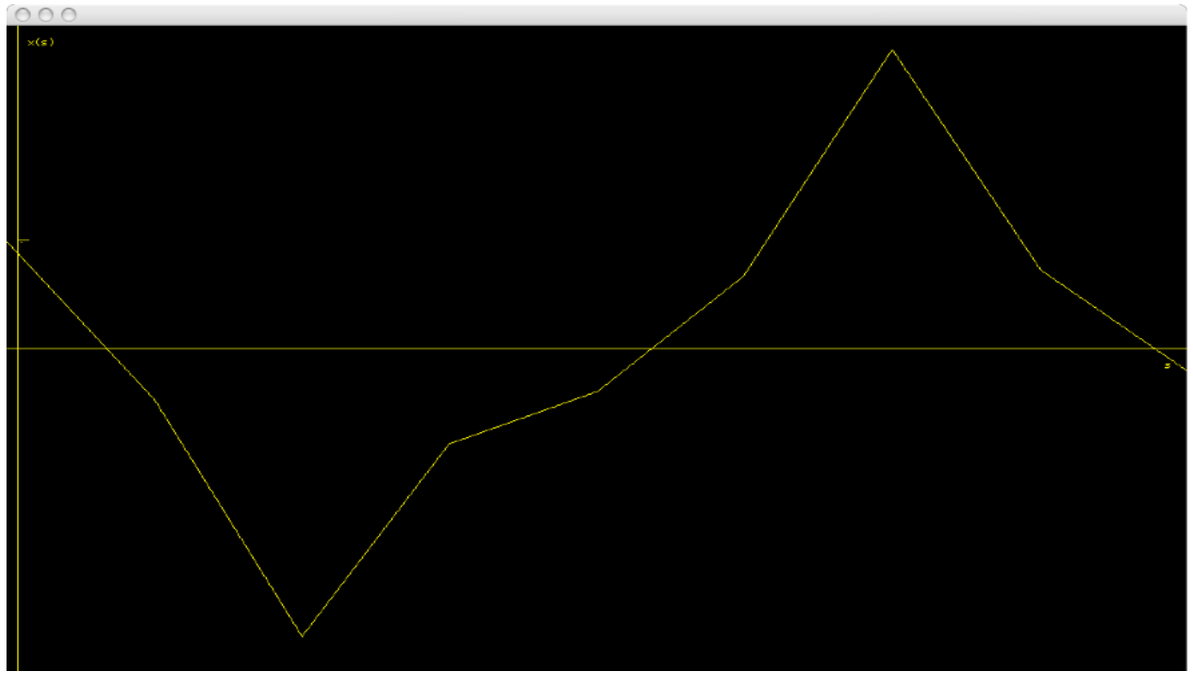


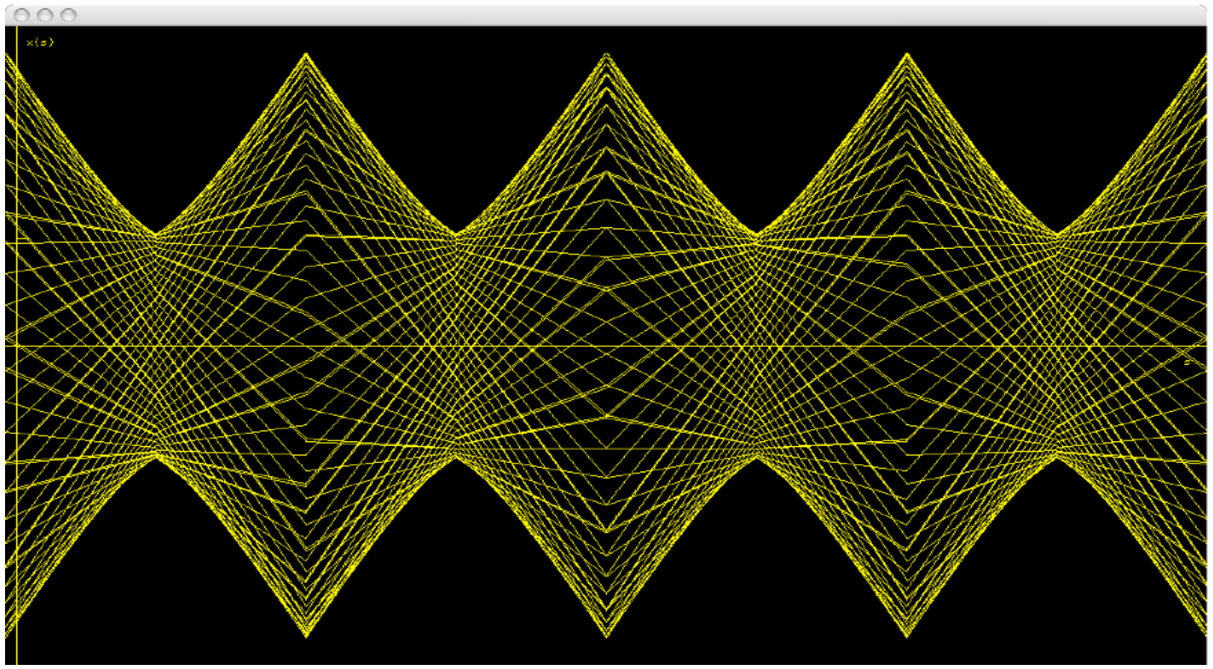
- Instrumental in paving the way for very large accelerators, both linacs and especially synchrotrons, where the bending and focusing functions can be separated into distinct magnet types

The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line





Can we describe the maximum amplitude of particle excursions in analytical form?

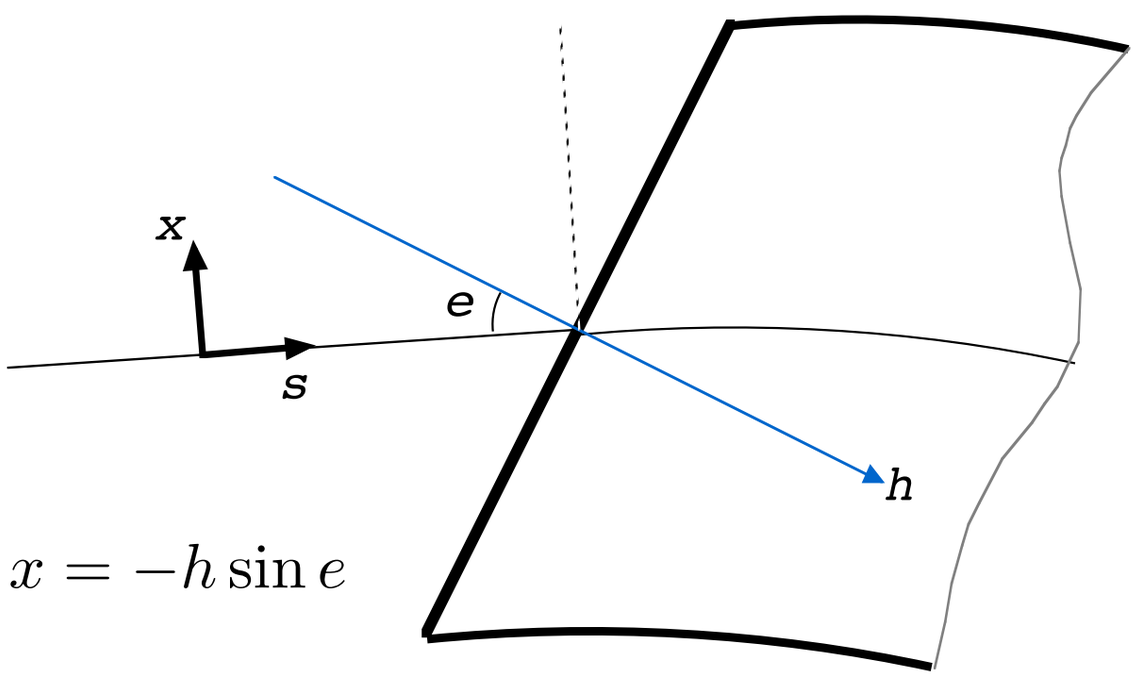
of course! *coming up soon* …

extra slides…

So, How to Model Edges?

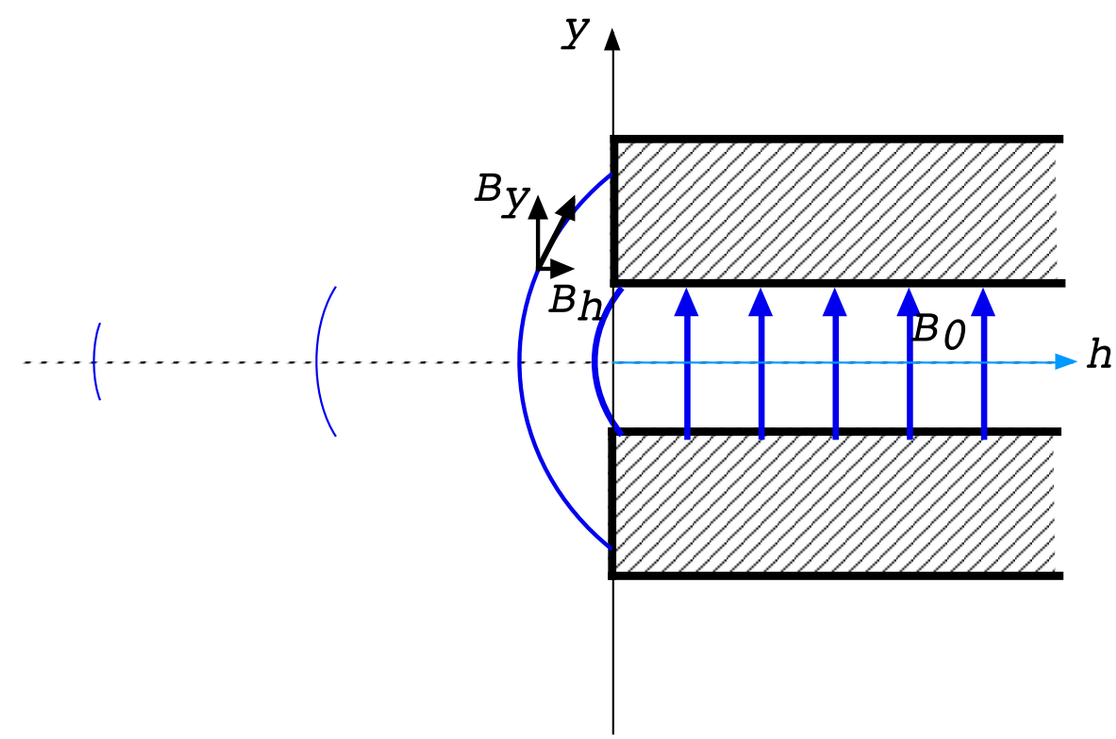
- In many cases, can consider edge effects to be perturbations to main motion, and treat as “impulse” kicks -- a “hard edge model” (can do better modeling, if required...)

From Above:



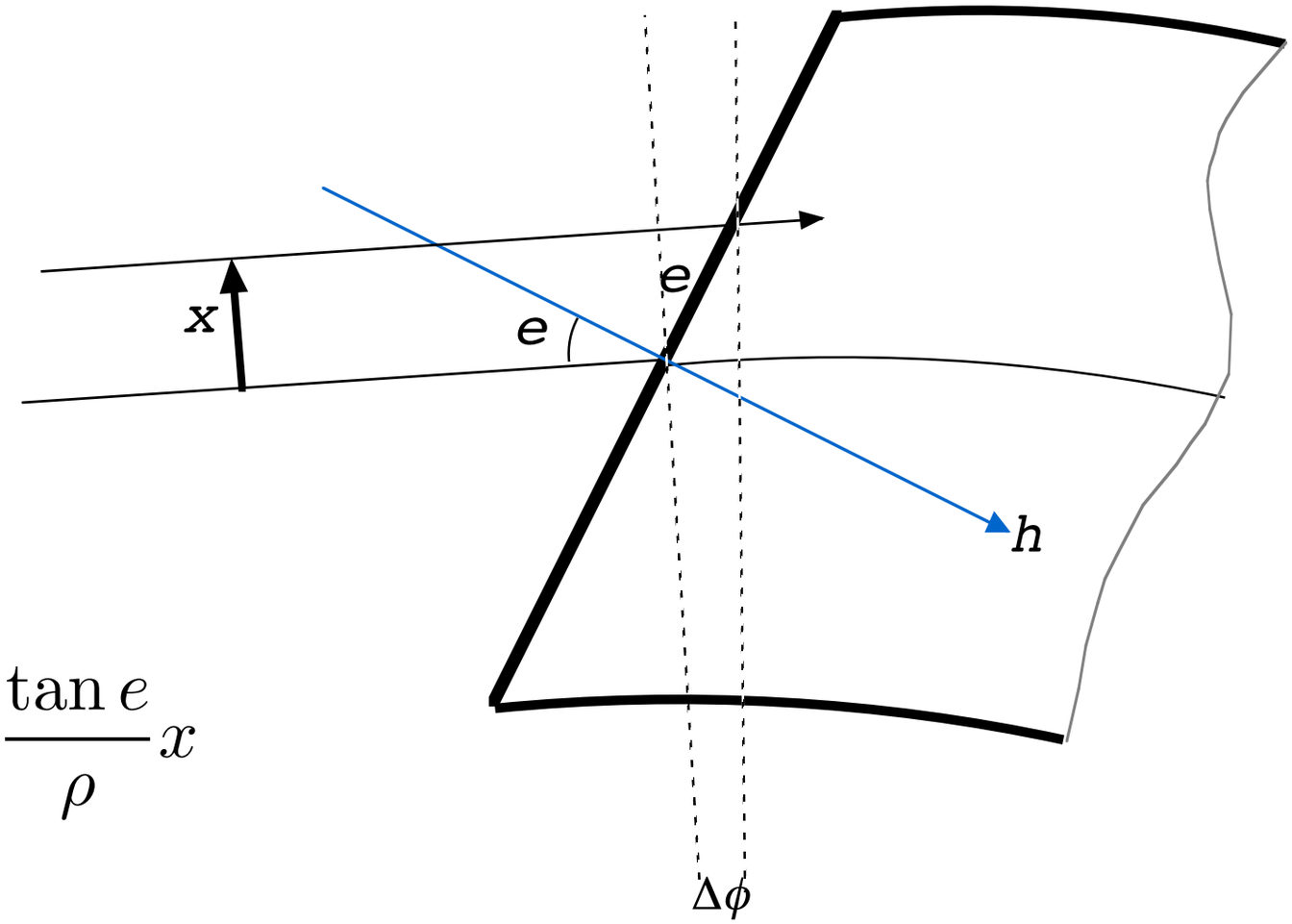
$$x = -h \sin e$$

From Side, “on edge”:



Edge Focusing -- radial

- Radial Defocusing:



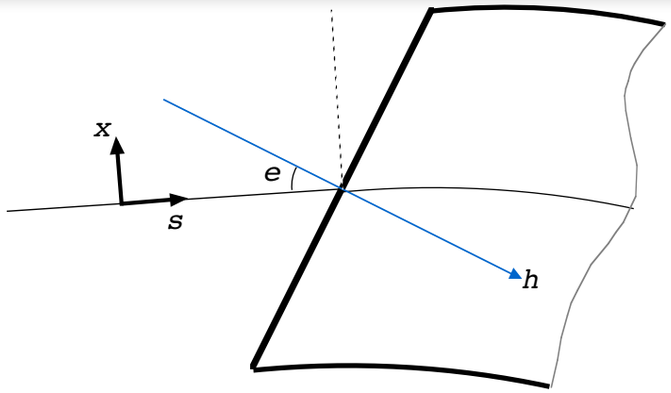
$$\Delta x' = \Delta\phi = \frac{\Delta s}{\rho} = \frac{x \tan e}{\rho} = \frac{\tan e}{\rho} x$$

- So, for positive *x*, design trajectory "curves away" before particle reaches edge of magnetic field; thus, "defocusing" effect

- Similarly, upon exit

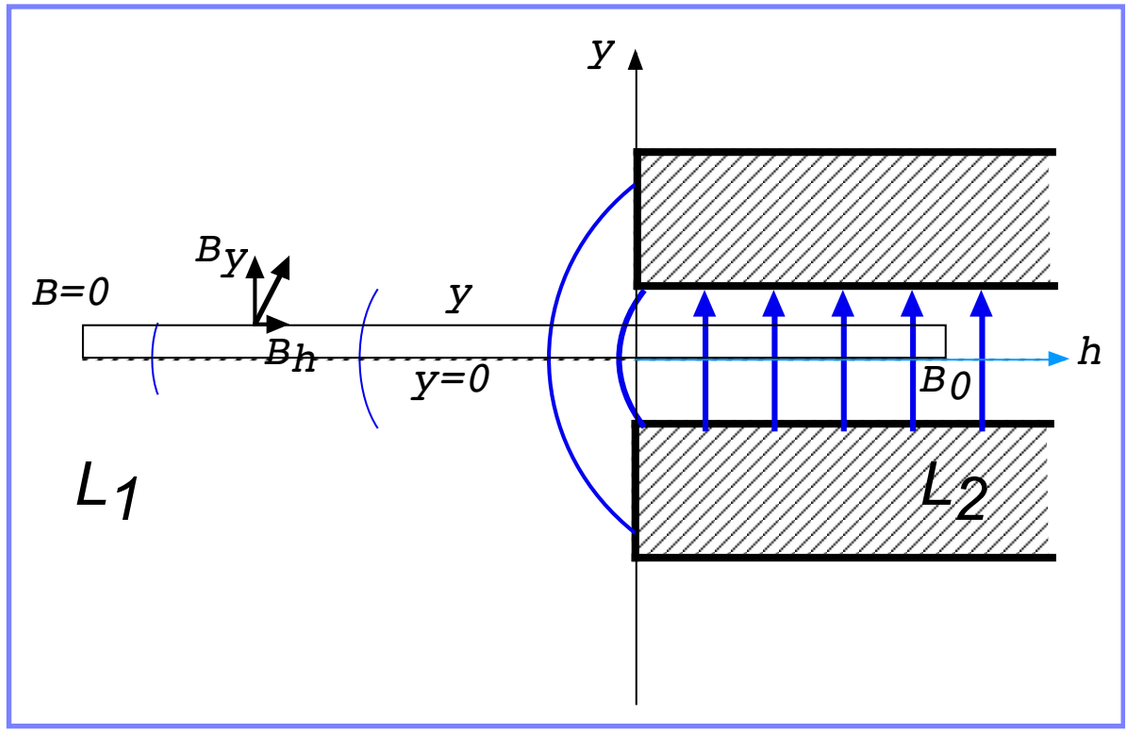
Edge Focusing -- axial

- Vertical (axial) Focusing:



$$\Delta y' = \frac{\Delta p_y}{p} = \frac{ev\int B_x(y)ds}{pv} = \frac{1}{B\rho}\int B_x ds$$
$$= -\frac{\sin e}{B\rho}\int B_h ds = -\frac{\tan e}{B\rho}\int (B_h \cos e)ds$$
$$= -\frac{\tan e}{B\rho}\int_{L_1}^{L_2} \vec{B}\cdot\vec{ds}$$

$$B_x(y=0) = 0$$
$$B_x = -B_h \sin e$$

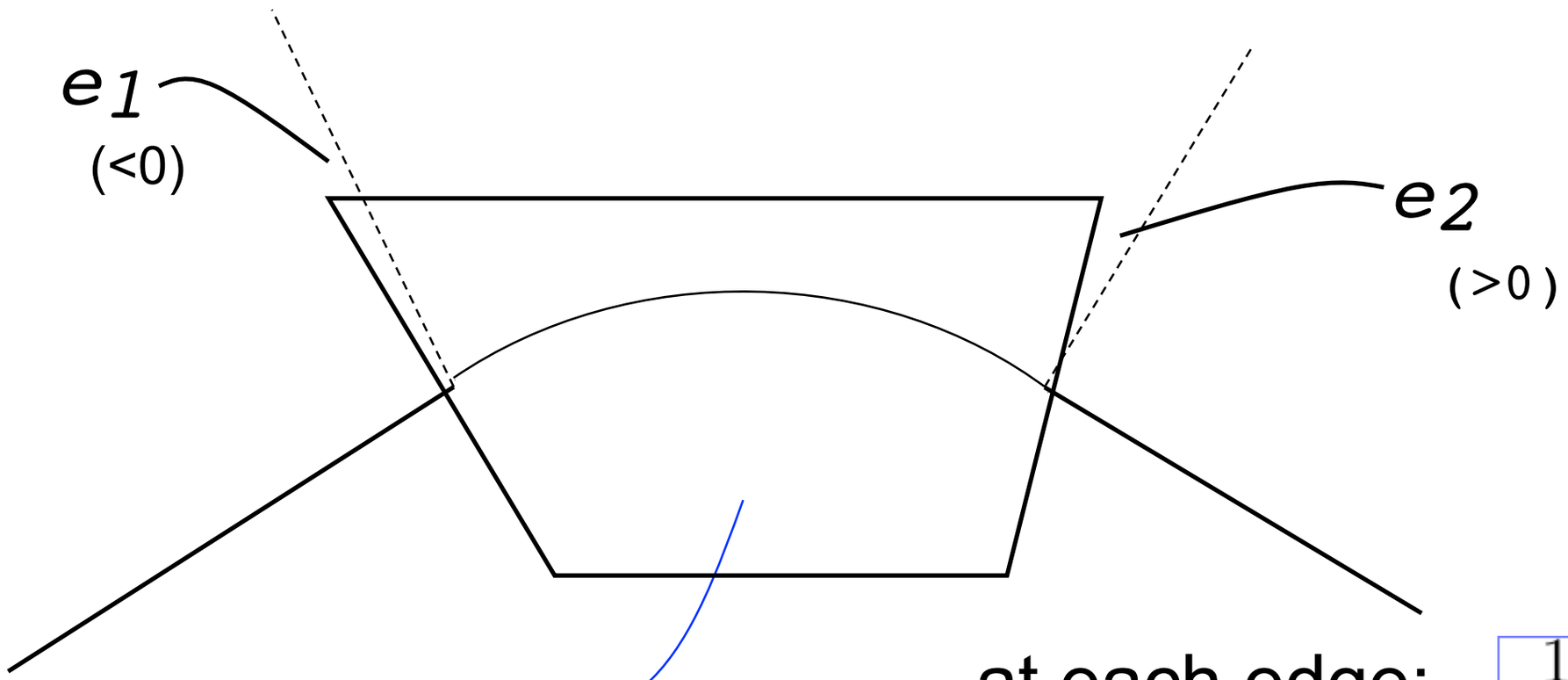


$$\oint \vec{B}\cdot\vec{ds} = 0 + \int_{L_1}^{L_2} \vec{B}\cdot\vec{ds} - B_0\cdot y + 0 = 0$$
$$\Downarrow$$
$$\int_{L_1}^{L_2} \vec{B}\cdot\vec{ds} = yB_0$$

$$\Longrightarrow \Delta y' = -\left(\frac{\tan e}{B\rho}\right)B_0 y = -\frac{\tan e}{\rho}y$$

Total Bend Magnet: Sector + Edges

- Treat arbitrary edge angles as separate “lenses” at each end of a sector magnet…

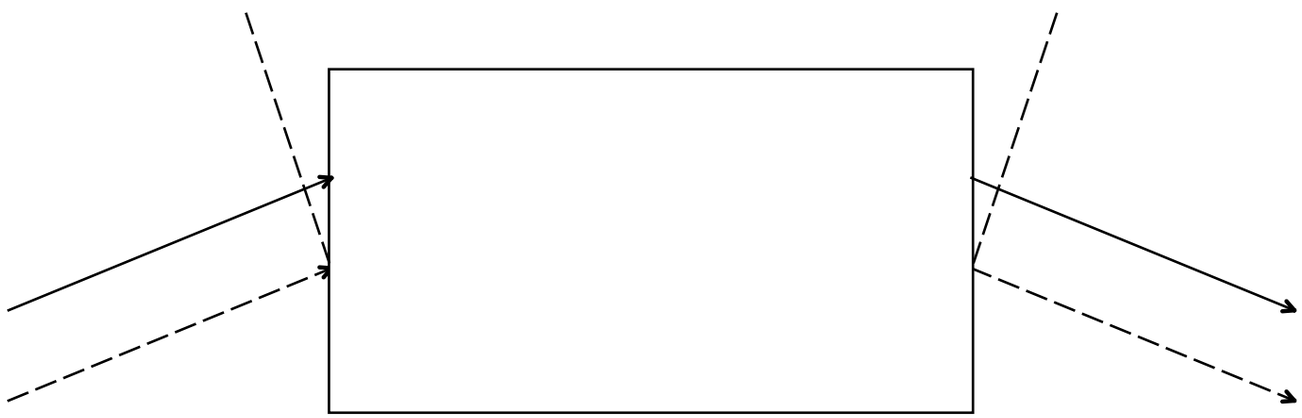


at each edge: $\frac{1}{f_{x,y}} \approx \pm \frac{\tan e}{\rho}$

inside, $K_x = 1/\rho^2$, $K_y = 0$

Thin lenses -- *KL = 1/f*

Rectangular Bending Magnet



In the bending plane, each edge acts as a defocusing lens with focal length:

$$\frac{1}{f} = -\frac{\theta/2}{\rho} = -\frac{\theta}{2\rho}$$

For Pure Sector Magnet,

hor: $\frac{1}{f_x} \approx \frac{\theta}{\rho}$

ver: $\frac{1}{f_y} \approx 0$

For Rectangular Magnet,

hor: $\frac{1}{f_x} \approx -\frac{\theta}{2\rho} + \frac{\theta}{\rho} - \frac{\theta}{2\rho} = 0$

ver: $\frac{1}{f_y} \approx \frac{\theta}{2\rho} + \frac{\theta}{2\rho} = \frac{\theta}{\rho}$