Phase Space Propagation

M J Syphers

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Creating a Distribution

Consider a collection of N_{par} particles, each with a transverse coordinate x lying on a corresponding trajectory with slope x' relative to the ideal trajectory. Suppose the particles are distributed "normally" about the ideal trajectory, with rms spreads in position and slope of σ_x and $\sigma_{x'}$, respectively. To model this situation, we call up **R** on the computer and enter the following commands into a console window:

```
Npar = 10000
x  = rnorm(Npar,0, 5)
xp = rnorm(Npar,0,2)
```

The "random normal" function in **R** creates coordinates x from a normal, or *Gaussian*, distribution with a desired mean of zero and an rms spread of $\sigma_x = 5$. (Let's presume that the units are in mm.) The distribution of slopes x' should have an rms spread of $\sigma_{x'} = 10$. (Here, let's presume that the units are in mr $= 10^{-3}$ radians.)

We can see how well we have done by finding the rms of the above variable distributions, using the `var()` function (*variance*) in **R**, which finds $\langle (x - \langle x \rangle)^2 \rangle$:

```
sqrt(var( x))
sqrt(var(xp))
```

We can also plot simple histograms of the distributions in x and x', as well as a "phase space plot" as follows:

```
hist(x)
hist(xp)
plot(x,xp,pch=".")
```

As will be presented in class, and as can be found in the on-line notes (see Appendices on the web site), the phase space distribution can be characterized by a set of parameters, namely an "emittance", ϵ – which is related to the area in phase space occupied by the particles – and the Courant-Snyder parameters, α, β, and γ.

In addition to `var(x)` for finding the variance $\langle x^2 \rangle$ of a variable, the covariance $\langle xy \rangle$ of two variables is found using `cov(x,y)`. The function `CScalc()` defined in the code below computes the Courant-Snyder parameters and the emittance using the variance and covariance functions applied to the x and x' distributions created earlier.

```
CScalc = function(x,xp){
   epsx=sqrt(var(x)*var(xp)-cov(x,xp)^2)
   betx= var(x )/epsx
   alfx=-cov(x,xp)/epsx
   gamx= var(xp)/epsx
   c(alfx,betx,gamx,epsx)
}
```

A simple call to this function prints out four numbers: α, β, γ and ϵ:

```
CScalc(x,xp)
```

- How does the emittance, ϵ, found from `CScalc()`relate to the initial rms values $\sqrt{\langle x^2 \rangle}$ and $\sqrt{\langle x'^2 \rangle}$ generated earlier?

- The value of the covariance `cov(x,xp)` should be nearly zero for our example; check that this is so. What would a non-zero value tell you about the phase space distribution?

```
cov(x,xp)
```

Propagating the Distribution

We know that as the particles move down the beam line unencumbered through a distance L, the angles of their trajectories will not change, but their transverse positions will change according to

$$x \leftarrow x + x' \cdot L.$$

Input the following code into your **R** program to see how the distribution changes…

```
# drift a distance L
L = 3
x = x + xp*L

hist(x)
hist(xp)
plot(x,xp,pch=".")
CScalc(x,xp)
```

You should notice that the Courant-Snyder parameters have changed; the value of α changes significantly, indicating a correlation developing between x and x'.

- Why would this happen?
- How has the value of ϵ changed (or has it)?

Next, since we see the extent of the x distribution getting very large, let's go through a thin lens quadrupole of focal length F. The magnet will alter the slopes of the trajectories according to

$$x' \leftarrow x' - x/F$$

while leaving the positions x unchanged (in the "thin lens" approximation). The code below performs this transformation and generates results.

```
# focus through a quadrupole, focal length F
F = 2
xp = xp-x/F

hist(x)
hist(xp)
plot(x,xp,pch=".")
CScalc(x,xp)
```

Notice that the correlation has changed – in fact, the covariance has changed sign for our chosen parameters.

Create a system of lenses

We now have the tools to model the propagation of our distribution through a series of lenses. Let's create two lists, one a list of lens focal lengths F and the second a list of the spaces between these lenses, L. Let's assume these distances to be in meters (and remember, x is in mm and x' is in mrad).

```
L = c( 5, 2, 7, 4, 3, 3, 6, 8)
F = c(-8, 4,-9, 4,-7, 9,-5, 7)
```

We now write a code that models a particle being focused through a thin lens of focal length F_i, then propagating a distance L_i. We then loop this operation through the entire list of N lenses, where $i \in [1, N]$. At the end of each drift we compute the Courant-Snyder variables and keep track of the results. In the code below, the vector V is $V = (\alpha, \beta, \gamma, \epsilon)$ and at the end of each loop we *bind* the new results with the previous results using the `rbind()` function (*bind by row*) in **R**. The final result will be a matrix of values where each row will be the values of V at the end of the i-th drift.

```
i = 0
V = CScalc(x,xp)  # values at s=0
while(i < length(L)){
   i = i+1
   xp = xp-x/F[i]
   x  = x+xp*L[i]
   V  = rbind(V,CScalc(x,xp))
}
```

After exectuing the above code we can see the final result simply by typing

```
V
```

in the console window.

To ease the analysis we'll define new varibles in order to capture the values found in each of the 4 rows of V:

```
alpha = V[,1]
beta  = V[,2]
gamma = V[,3]
eps   = V[,4]
```

Finally, we can make a plot of the beam envelope through our beam line. First, to get the path length coordinate, let $s_{i+1} = s_i + L_i$. The `cumsum()` function in **R** makes this fairly easy, though we must include the initial value of $s = 0$ to get aligned with the proper number of values found in our loop above:

```
s = c(0,cumsum(L))

plot(s,     beta ,typ="l",ylim=c(0,1.2*max(     beta )))
plot(s,sqrt(beta),typ="l",ylim=c(0,1.2*max(sqrt(beta))))
```

- Why might we plot $\sqrt{\beta}$, as we did in the second plot, above rather than β ?
- Use **R** to plot the value of $\beta\gamma - \alpha^2$ at each of the points in our beam line calculation.
- With the result of this plot, write the general relationship between γ and the other two Courant-Snyder parameters.
