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Part III

The Stability Criterion

Discovery of Strong Focusing

Periodic Optics and Tune Calculations

Momentum Dispersion

Arbitrary Focusing System

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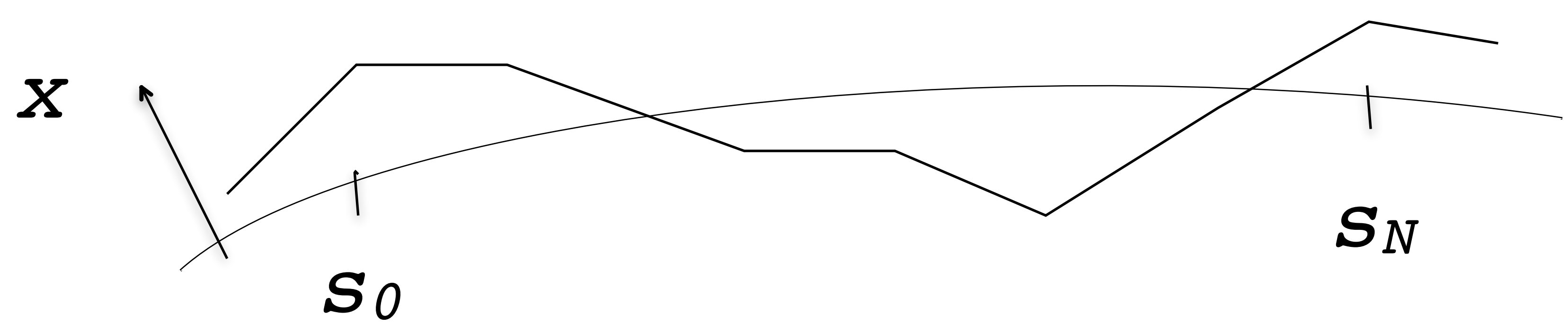
- What if the focusing is not *continuous* but rather varies with location *s*?
- Generate a single-turn matrix of the linear motion, made from matrices of individual elements (Note: each with unit determinant)
- Look at matrix describing motion for one passage through a repetitive period:

$$M = M_N M_{N-1} \cdots M_2 M_1$$



- Now suppose repeat this operation k times. We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \to \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

The Stability Criterion

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- From the discipline of "linear algebra", we know that any vector within a vector space (i.e., that is operated on, say, by a matrix *M*) can be written in terms of the eigenvectors of the matrix *M*

  - Eigenvector: V $MV = \lambda V$
    - »
    - » *where* λ is an eigenvalue of *M* *(real or imaginary)*

- A 2x2 matrix *M* will have two eigenvalues, λ_1 and λ_2 and two corresponding eigenvectors, V_1 and V_2; so any vector that *M* operates on can be written as

$$X = c_1 V_1 + c_2 V_2$$

The Stability Criterion

- So, if the matrix M is applied to vector X_0 a number of times, the resulting vector X_k after the k-th iteration will be

$$X_k = M^k X_0 = M^k (c_1 V_1 + c_2 V_2) = c_1 \lambda_1^k V_1 + c_2 \lambda_2^k V_2$$

V = eigenvector
λ = eigenvalue

- Now, also from linear algebra, the determinant of the matrix M will be the product of the eigenvalues. So,

$$\det M = 1 = \lambda_1 \lambda_2 \rightarrow \lambda_2 = 1/\lambda_1 \rightarrow \lambda = e^{\pm i\mu}$$

The Stability Criterion

- Since for our case the eigenvalues are reciprocals of each other, and since we can write $\lambda = e^{\pm i\mu}$, then

$$X_k = M^k X_0 = c_1 \lambda_1^k V_1 + c_2 \lambda_2^k V_2 = c_1 e^{ik\mu} V_1 + c_2 e^{-ik\mu} V_2$$

If μ is imaginary, then repeated application of *M* gives exponential growth; if μ is real, gives oscillatory solutions…

- To find the eigenvalues, we solve the "characteristic equation": from $MV = \lambda V$,

$$\text{characteristic equation: } \det(M - \lambda I) = 0$$

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } (a-\lambda)(d-\lambda) - bc = 0$$

The Stability Criterion

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- Solving for the eigenvalues,

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{then } \det(M - I\lambda) = \left| \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \right| = (a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$
$$\lambda^2 - trM\lambda + 1 = 0$$
$$\lambda + 1/\lambda = trM$$
$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = trM$$

$$ad - bc = \det M = 1$$

$$a + d = trM = \text{“}trace\text{''} \ of \ M$$

$$\text{So, } \mu \text{ real (stability)}$$
$$\rightarrow |trM| < 2$$

The Stability Criterion

Check: The Weak Focusing Synchrotron

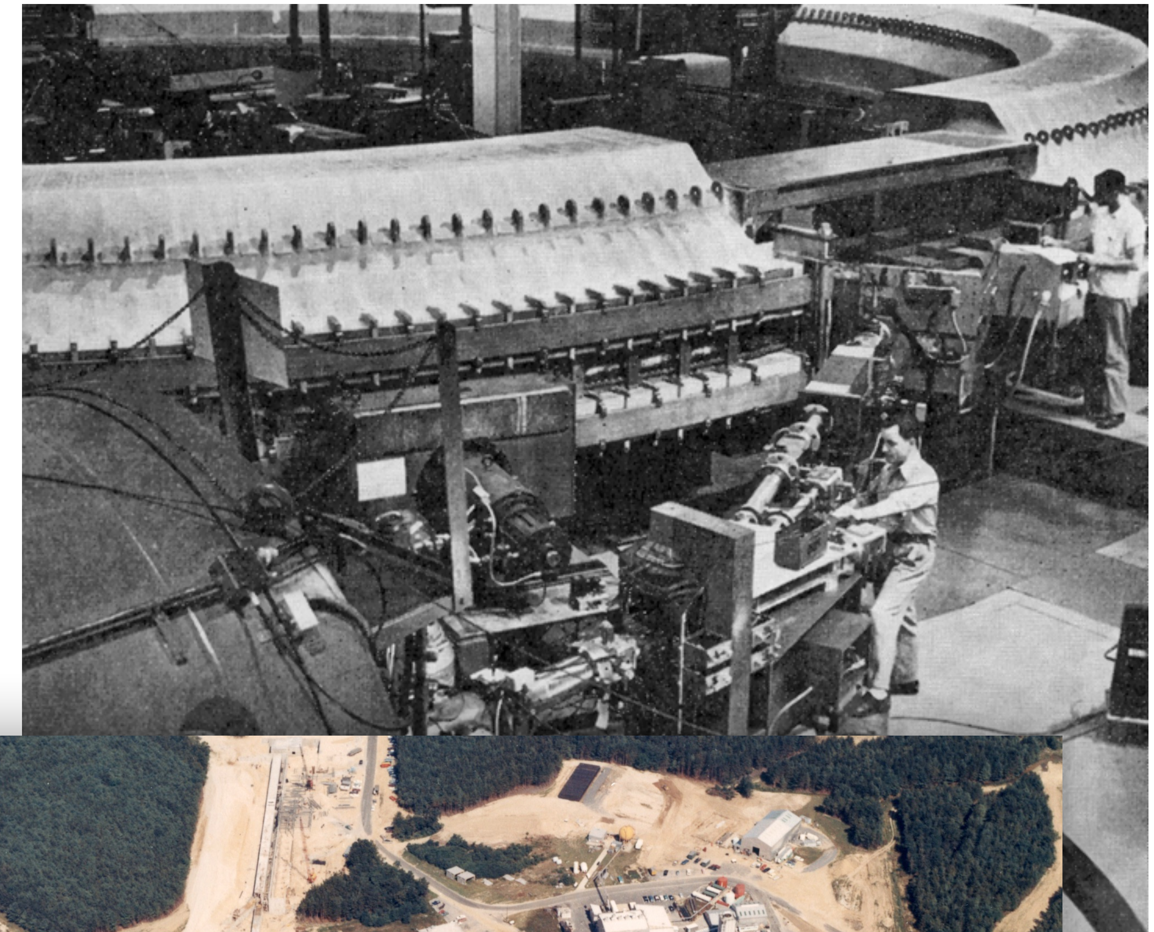
- We had, for example: $$y'' + \frac{n}{R_0^2}\, y = 0$$
  - » from which follows:
  $$y = y_0 \cos(\frac{\sqrt{n}}{R_0}\, s) + y_0' \frac{R_0}{\sqrt{n}} \sin(\frac{\sqrt{n}}{R_0}\, s)$$
  $$y' = -y_0 \frac{\sqrt{n}}{R_0} \sin(\frac{\sqrt{n}}{R_0} s) + y_0' \cos(\frac{\sqrt{n}}{R_0} s)$$
  - » or, in matrix form:
  $$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\frac{\sqrt{n}}{R_0} s) & \frac{R_0}{\sqrt{n}} \sin(\frac{R_0}{\sqrt{n}} s) \\ -\frac{\sqrt{n}}{R_0} \sin(\frac{R_0}{\sqrt{n}} s) & \cos(\frac{\sqrt{n}}{R_0} s) \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}_0$$
- For 1 revolution, $s = 2\pi R_0$ and the trace of *M* is …

$$|trM| = |2\cos(2\pi\sqrt{n})| \leq 2 \qquad (|2\cos(2\pi\sqrt{1-n})| \leq 2, \text{ for horizontal})$$

$$0 \leq n \leq 1$$
for stability

Discovery of Strong Focusing

- The Cosmotron (BNL)
  - (weak focusing)



- Through looking at upgrade options, strong focusing was discovered and the decision was made to go for a new, much larger synchrotron
  - The Alternating Gradient Synchrotron (AGS)

Discovery of Strong Focusing*

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- Consider the "weak-focusing" magnet system just discussed. Suppose the ring is made up of $2N$ identical magnets, each with field index n

- Take every other magnet and have the magnet *open* to the inside, instead of the outside: $n \rightarrow -n$

  - All have the same central field value, B_0, but the field "gradients" will alternate n, $-n$, n, $-n$, …

- Analyze the resulting system by using a matrix approach and applying the stability criterion

*Courant, Livingston, and Snyder, 1952.
Christofolis, c. 1950

Discovery of Strong Focusing [2]

- Consider one degree of freedom, say the vertical
  - for one of the N cells, the matrix would be…

(B'>0) (B'<0) $K = |B'|/B\rho$

$$M_c = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} \cosh(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}L) \\ \sqrt{K}\sinh(\sqrt{K}L) & \cosh(\sqrt{K}L) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\sqrt{K}L)\cosh(\sqrt{K}L) + \sin(\sqrt{K}L)\sinh(\sqrt{K}L) & \cdots \\ \cdots & \cos(\sqrt{K}L)\cosh(\sqrt{K}L) - \sin(\sqrt{K}L)\sinh(\sqrt{K}L) \end{pmatrix}$$

from which

$$tr M = 2\cos(\sqrt{K}L)\cosh(\sqrt{K}L)$$

- So, for stability, we would need:

$$|\cos(\sqrt{K}L)\cdot\cosh(\sqrt{K}L)| < 1$$

*Courant, Livingston, and Snyder, 1952.
Christofolis, c. 1950

Discovery of Strong Focusing [3]

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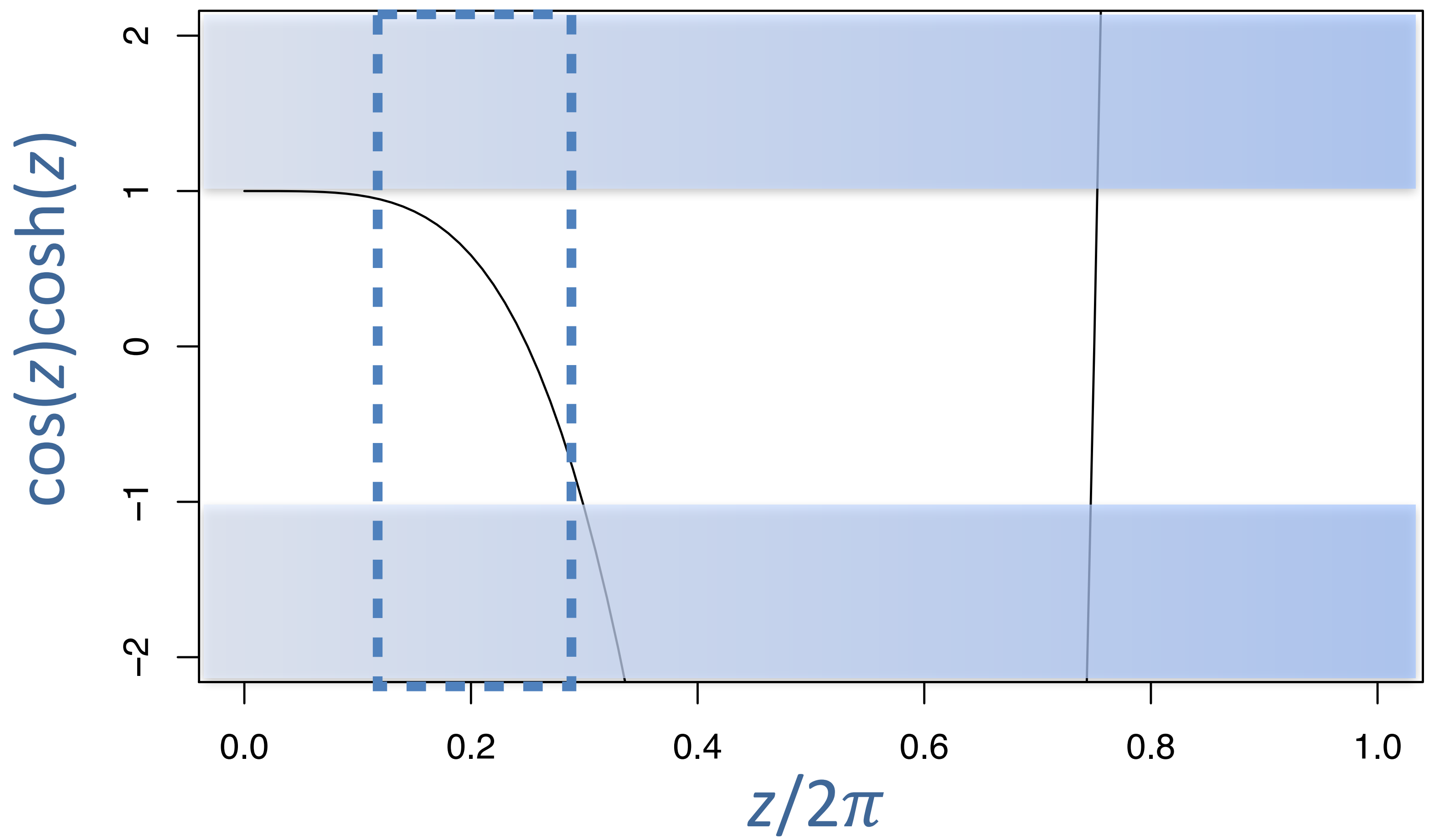
- We see a range in which the system would be stable



- Choose $z = \sqrt{K}\,L$
  $K = (z/L)^2$
- Also, $L = 2\pi R_0 / 2N$ $\quad K = (zN/\pi R_0)^2$
- In the weak focusing case, $K_0 = n/R_0^2$. So,… $$\frac{K}{K_0} = \left(\frac{z}{\pi}\right)^2 \frac{N^2}{n}$$
- Let's pick $z/\pi = 0.4$, $n = 0.5$, and $N = 25$:

$$K/K_0 = 200!$$

Another Example: FODO system

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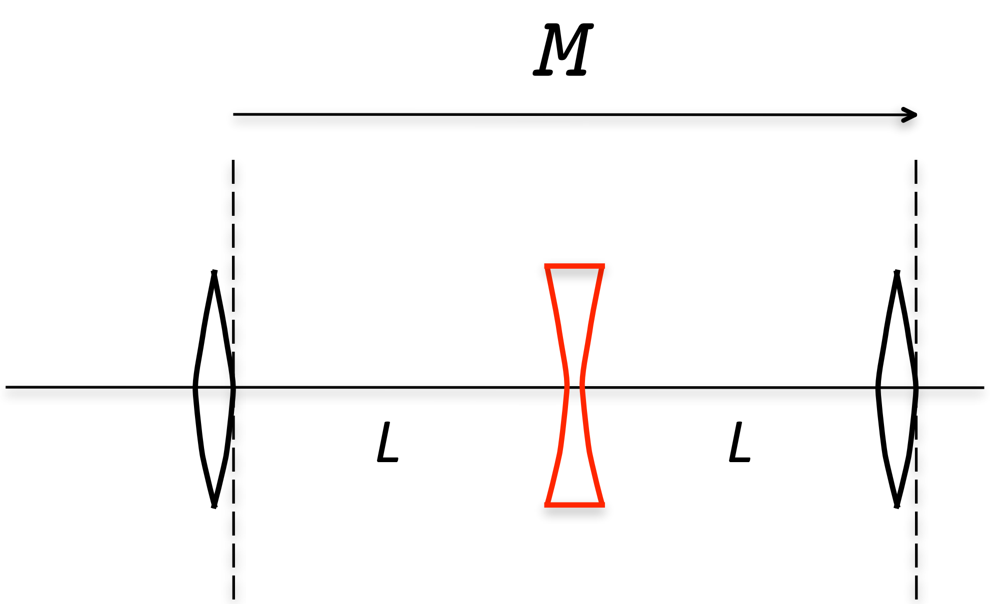
$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1-L/F \end{pmatrix}\begin{pmatrix} 1 & L \\ 1/F & 1+L/F \end{pmatrix} \\
&= \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix}
\end{aligned}
$$



and repeat…

So, $tr\,M = 2 - L^2/F^2$ and thus, for stability,

$$-2 < 2 - L^2/F^2 < 2$$

$$-4 < -L^2/F^2 < 0$$

$$F > L/2$$

Particle Trajectories in a Periodic Lattice

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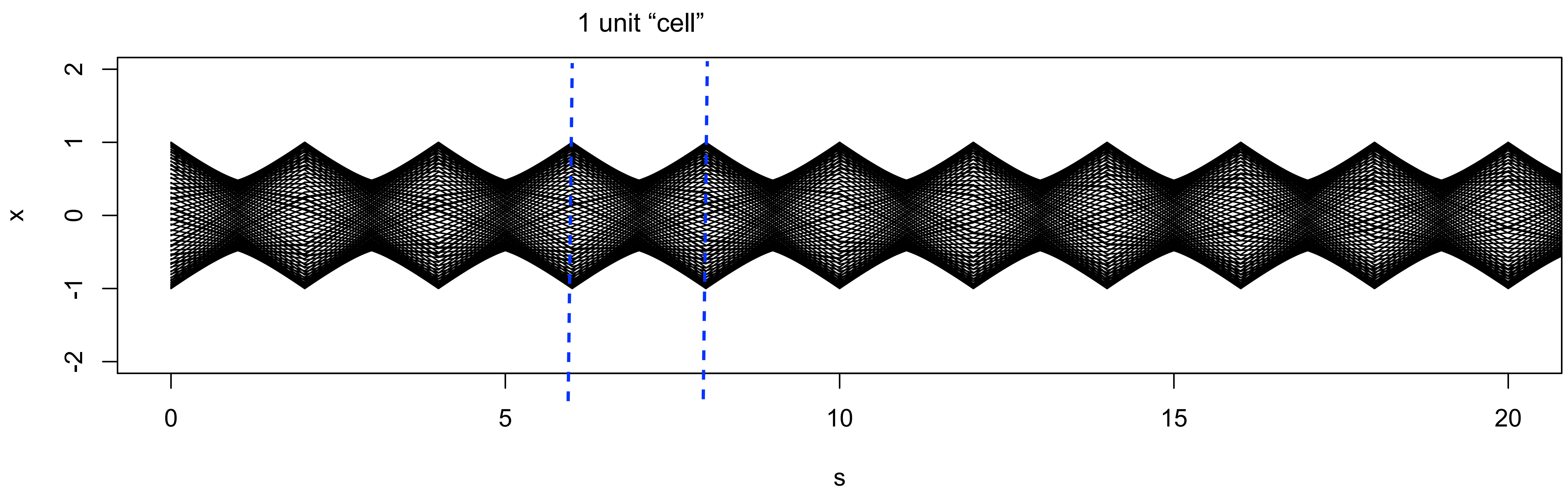

$$\frac{dx'}{ds} = \frac{d^2x}{ds^2} = -\frac{eB'(s)}{p}x$$

$$x'' + K(s)x = 0$$

(Hill's Equation)

$$K(s) = \frac{e}{p}\frac{\partial B_y}{\partial x}(s)$$

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

The Periodic Amplitude Function

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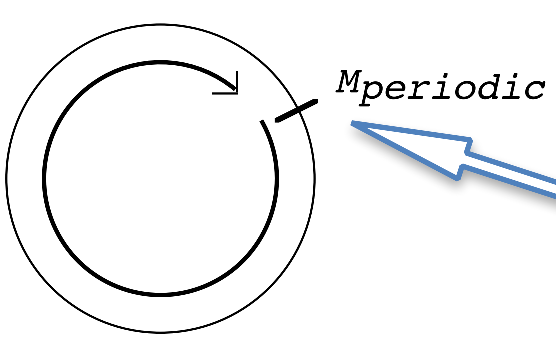
- Previously, …
  - Transport matrix, in terms of amplitude function at end points, and phase advance between:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

$$M_{periodic} = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$

sometimes "μ" is used to denote the **periodic** phase advance



Natural choice in a circular accelerator, when values of β, α above correspond to one particular point in the ring

Choice of Initial Conditions

- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a “ring,” then natural to choose the **periodic** solutions for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- In a system that takes a distribution from a source or off of a target, wish to “match” to desired initial conditions at the input to the downstream beam line system by using an arrangement of tunable focusing elements

Computation of Courant-Snyder Parameters

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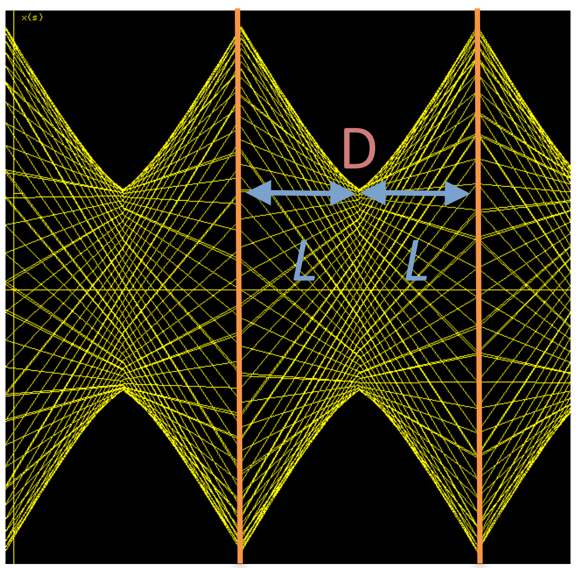
- As an example, consider again the FODO system



F

-F

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$

- Let's, use above matrix of the periodic section to compute functions at exit of the F quad..

FODO Cell Courant-Snyder Parameters

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- From the matrix: $$M_{periodic} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4 numbers

$$traceM = a + d = 2 - L^2/F^2 = 2\cos\mu \quad \Rightarrow \quad \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1 + L/2F}{1 - L/2F}} \qquad \alpha = \frac{a - d}{2\sin\mu} = \sqrt{\frac{1 + L/2F}{1 - L/2F}}$$

- If go from D quad to D quad, simply replace *F* --> *-F* in matrix *M* above
  - So, at exit of the D quad:

$$\beta = 2F\sqrt{\frac{1 - L/2F}{1 + L/2F}} \qquad \alpha = -\sqrt{\frac{1 - L/2F}{1 + L/2F}}$$

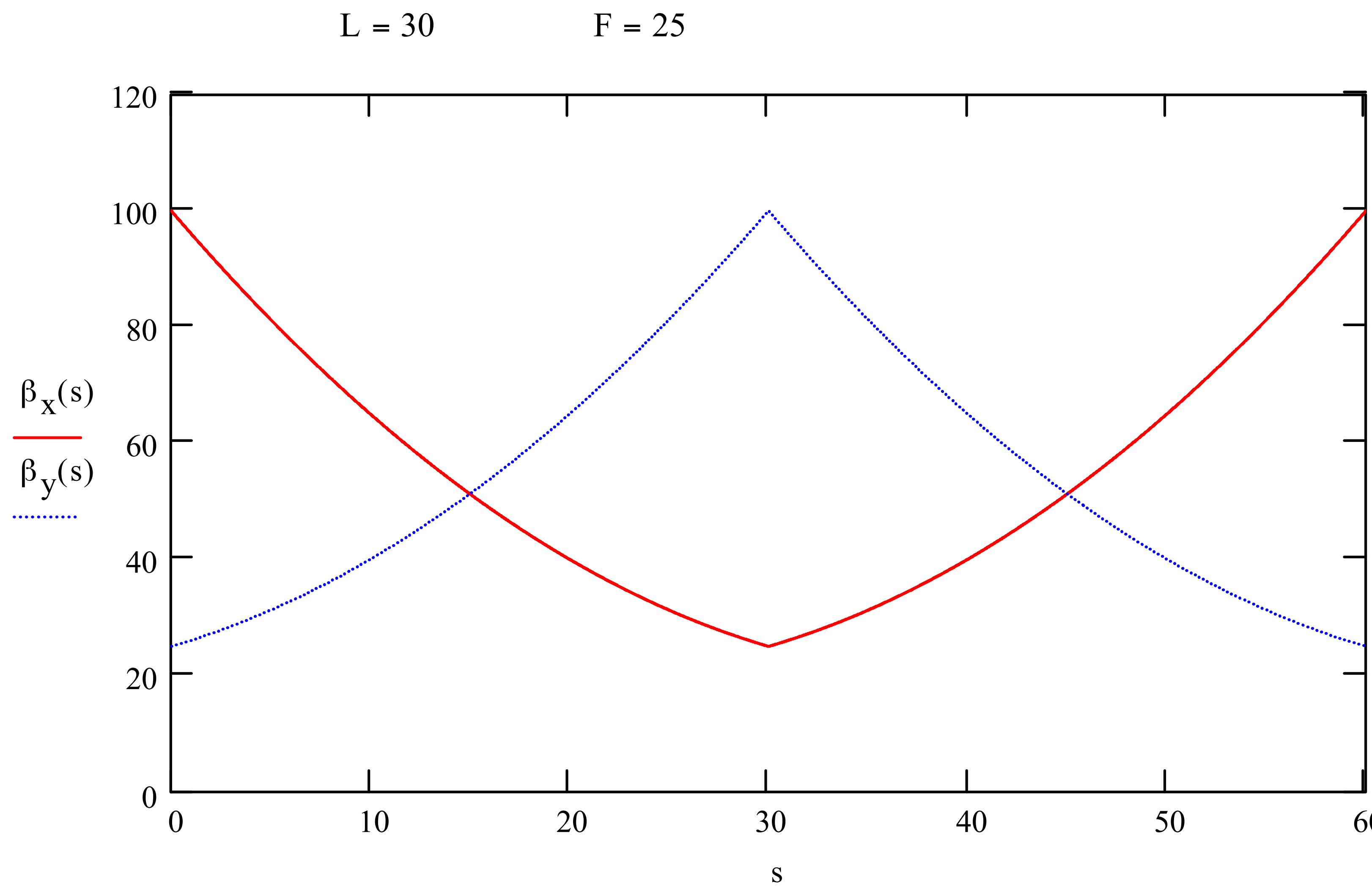
for completeness,

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

Periodic FODO Cell Functions

- Numerical Example: Standard FODO Cell of the old Fermilab Tevatron (~100 of these made up the ring)

$$\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ)$$
$$\beta_{max} = 2(25\text{ m})\sqrt{1.6/0.4} = 100\text{ m}$$
$$\beta_{min} = 2(25\text{ m})\sqrt{0.4/1.6} = 25\text{ m}$$
$$\nu \approx 100 \times 1.2/2\pi \sim 20$$



Note: this thin lens example is actually accurate to a few percent!

$$(F/\ell = 25/2 >> 1)$$

Propagation of Periodic Courant-Snyder Parameters

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- We can write the matrix of a periodic section as:

$$
\begin{aligned}
M_0 &= \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\Delta\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin\Delta\psi \\
&= I\cos\Delta\psi + J\sin\Delta\psi \qquad = e^{J\Delta\psi}
\end{aligned}
$$

- where

$$
J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}
$$

$$
det J = 1, \quad trace(J) = 0; \quad J^2 = -I
$$

α, β are values at the beginning/end of the periodic section described by matrix M

Tracking β, α, γ ...

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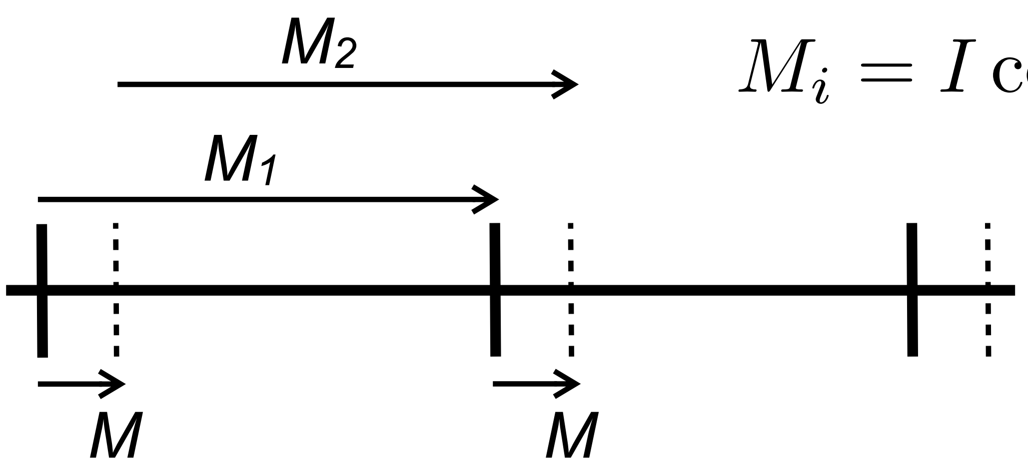
- Let M_1 and M_2 be the "periodic" matrices as calculated at two points, and M propagates the motion between them. Then,



$$M_i = I\cos\mu + J_i \sin\mu \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$M_2 = M \; M_1 \; M^{-1}$$

- Or, equivalently,
  - if know C-S parameters (i.e., J) at one point, can find them at another point downstream if given the matrix for motion in between:

$$J_2 = M \; J_1 \; M^{-1}$$

- Doesn't have to be part of a periodic section; valid between any two points of an arbitrary arrangement of elements

For comparison, remember $K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, and $K = M \; K_0 \; M^T$; these are equivalent

Evolution of the Phase Advance

- Again, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase and the Courant-Snyder parameters along a beam line from one point to another

The Betatron Tune

- In a cyclic accelerator (synchrotron), the particles will oscillate (betatron oscillations) with a certain oscillation frequency — the betatron frequency.
- The betatron frequency is determined by the total phase advance once around the ring:

$$\Delta\psi_{total} = \oint \frac{ds}{\beta(s)}$$

$$\nu \equiv \Delta\psi_{total}/2\pi$$

Betatron Tune: # of oscillations per revolution

$$\mathrm{tr}M = 2\cos(2\pi\nu)$$

$$f_{betatron} = \nu f_{rev}$$

Ex: Tune of a FODO synchrotron

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- Suppose a ring is made up of *N* FODO cells
- Each cell has phase advance given by the lens spacing and lens focal length:

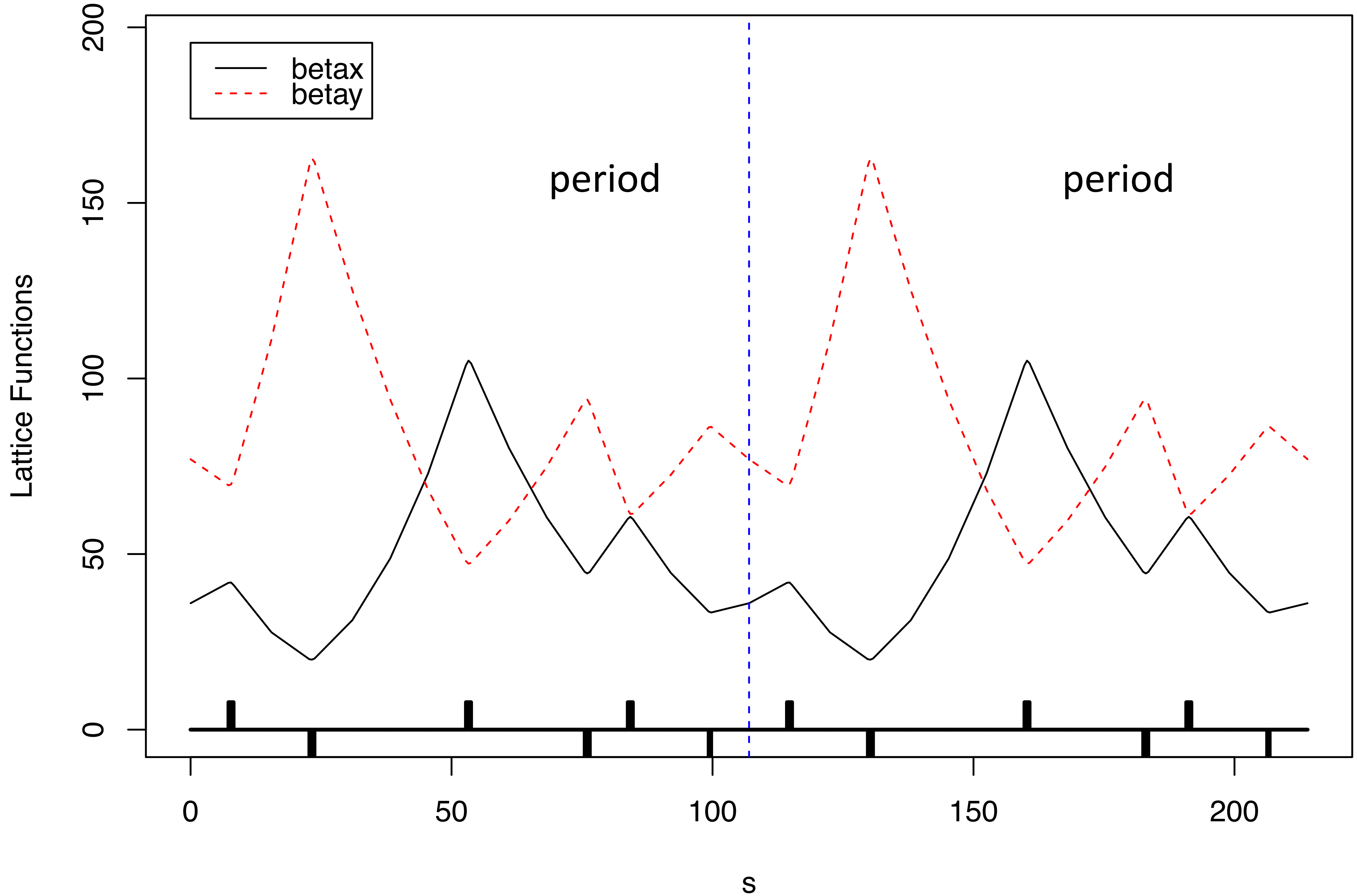
$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

- So, the tune of this simple synchrotron would be:

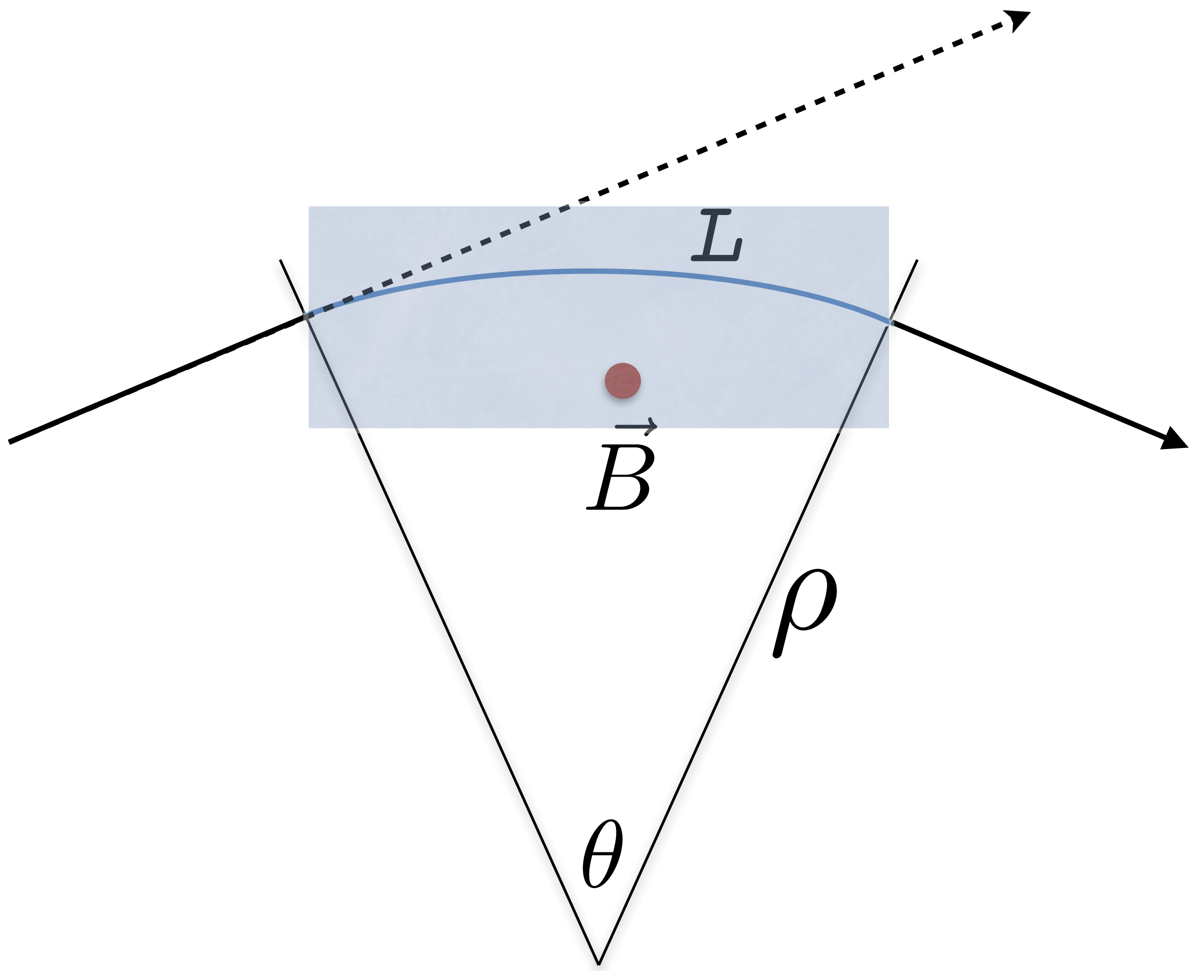
$$\nu = N\mu/2\pi \approx N\frac{L}{2\pi F} = \frac{2LN}{4\pi F} = \frac{C}{2\pi}\frac{1}{2F} = \frac{R}{2F}$$

- Ex: Main Injector at Fermilab: *R* ~ 500 m; *F* ~ 13 m
  - so, ν ~ 20
  - thus, if initiate a betatron oscillation in this synchrotron it will oscillate ~20 times per revolution around the ring

Arbitrary Distribution of Quadrupoles



Bending through Dipole Field



$$\theta = \frac{L}{\rho} = \frac{B \cdot L}{(B\rho)}$$

$$= \frac{q \cdot B \cdot L}{p}$$

Dispersion

$$B\rho = \frac{p}{q} \qquad \theta = \frac{qB \cdot \ell}{p}$$

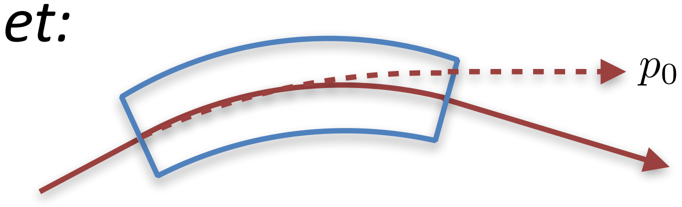
The bend angle (and/or focusing strength) depends upon momentum

Similar to index of refraction depending upon frequency

dipole steering "error" due to a different momentum —> "dispersion"

focusing "error" due to a different momentum —> "chromatic aberration"

dipole magnet:



$$\frac{\Delta\theta}{\theta_0} = -\frac{\Delta p}{p}$$

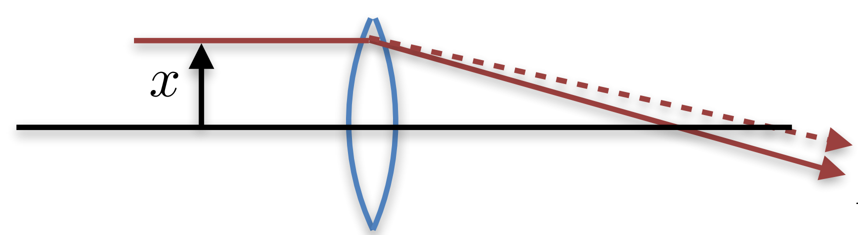
[i.e., in "opposite" direction of bend]

at exit, to lowest order,

$$\Delta x' = \theta_0 \frac{\Delta p}{p}$$

$$\Delta x \approx \frac{1}{2}\ell\theta_0 \frac{\Delta p}{p}$$

likewise, for quadrupole:



$$f = f_0(1 + \frac{\Delta p}{p})$$

Trajectory differences due to momentum differences referred to as "dispersion"

and,

$$D(s, \Delta p/p) \approx D(s) \equiv \frac{\Delta x(s)}{\Delta p/p}$$

"dispersion function"

Dispersion [2]

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(see E&S text for details…)

Equation of Motion:

$$x'' + (\frac{B'}{B\rho} + \frac{1}{\rho^2})x = 0$$ becomes $$x'' + \left\{ \left(\frac{1}{1+\Delta p/p} \right) \frac{B'}{B\rho} + \frac{p_0 - \Delta p}{p_0 + \Delta p} \cdot \frac{1}{\rho_0(s)^2} \right\} x = \frac{1}{\rho_0} \frac{\Delta p}{p_0 + \Delta p}$$

let $x = D\ \Delta p/p,$ particular solution

(must add the homogeneous solution, which we have found previously)

betatron oscillation

then,

$$D'' \frac{\Delta p}{p_0} + \left\{ \left(\frac{1}{1+\Delta p/p} \right) \frac{B'}{B\rho} + \frac{p_0 - \Delta p}{p_0 + \Delta p} \cdot \frac{1}{\rho_0(s)^2} \right\} D \frac{\Delta p}{p_0} = \frac{1}{\rho_0} \frac{\Delta p}{p_0 + \Delta p}$$

keep only terms linear in the relative momentum deviation,

$$D'' \frac{\Delta p}{p_0} + \left(\frac{B'}{B\rho} + \frac{1}{\rho_0^2} \right) D \frac{\Delta p}{p_0} = \frac{1}{\rho_0} \frac{\Delta p}{p_0}$$

a driven betatron oscillation, with a *constant* driving term. The "driver" is the dipole field within a bending magnet

$$D'' + K\ D = \frac{1}{\rho_0}$$

so, solutions are $\sin\sqrt{K}\ell$ & $\cos\sqrt{K}\ell$ plus *const.*

Dispersion [3]

$$D'' + K\ D = \frac{1}{\rho_0}$$

In terms of matrices…

$K = 0$:

$$D'' = \frac{1}{\rho}, \quad D' = \frac{s}{\rho} + D'_0$$

$$D = D_0 + D'_0 s + \frac{1}{2}\frac{s^2}{\rho}$$

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & s & \frac{1}{2}s^2/\rho \\ 0 & 1 & s/\rho \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

in the limit of short, or "thin" elements, a bending magnet primarily changes the slope of the dispersion function by an amount equal to the bend angle of the magnet

$1/\rho = 0$:

same 2x2 as before

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} M & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_0 \\ D'_0 \\ 1 \end{pmatrix}$$

otherwise, the *D* transports roughly like a betatron oscillation

So, can use matrix methods (3x3 now; and 2x2 in "vertical" plane) to solve for:

$$\beta_x, \quad \alpha_x, \quad \psi_x$$

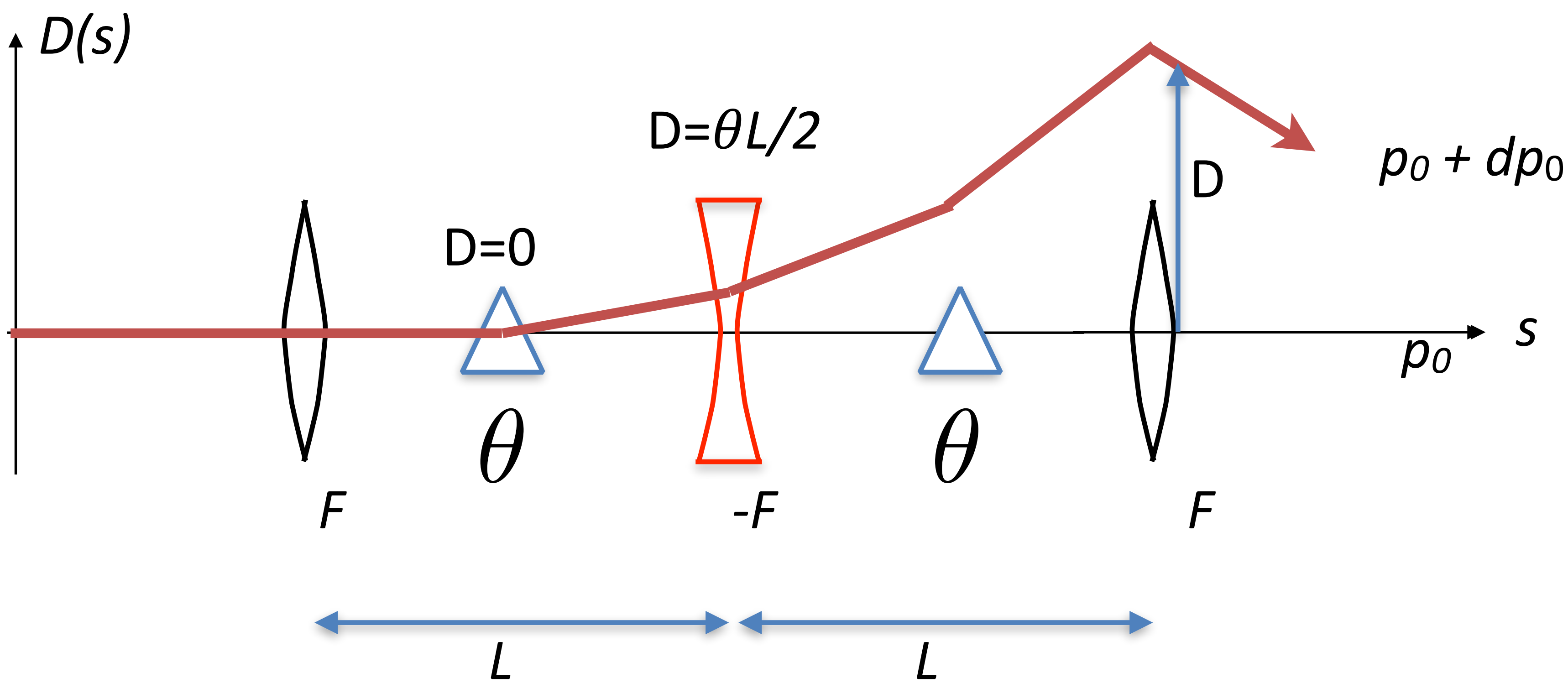
$$\beta_y, \quad \alpha_y, \quad \psi_y$$

$$D_x, \quad D'_x$$

(& D_y, D'_y , if also have vertical bending)

Generating Dispersion

- System of quadrupoles with alternating-sign gradients (F, D, F, …) separated by distance *L*, and with bending magnets present…



D=θL/2(1+L/2F)

Ex: D = 3 m, dp/p = 0.3%, then Δx = 9 mm

Beam Size Including Dispersion

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- Total excursion due to “off momentum” plus betatron oscillation:

$$x = x_\beta + D\ \delta \qquad\qquad \delta \equiv \Delta p/p$$

$$x^2 = x_\beta^2 + 2x_\beta D\delta + D^2\delta^2$$

- Assuming no correlation between x_β and particle’s momentum:

$$\langle x^2\rangle = \langle x_\beta^2\rangle + D^2\langle\delta^2\rangle$$

$$\langle x^2\rangle = \epsilon\beta/\pi + D^2\langle\delta^2\rangle$$

Momentum Compaction Factor

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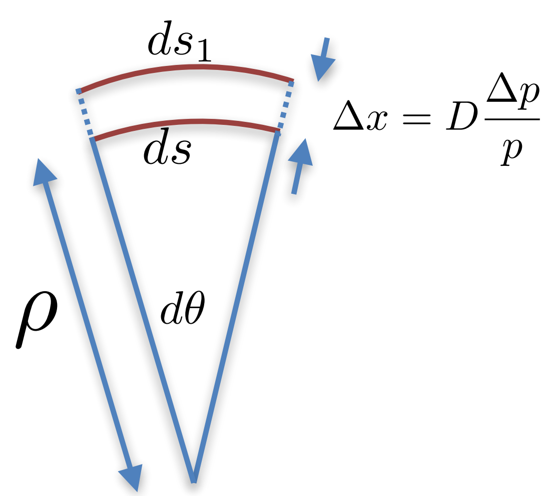
- How does path length along the beam line depend upon momentum?
  - in straight sections, no difference; in bending regions, *can* be different

Look closely at an infinitesimal section along the ideal trajectory…



$$d\theta = \frac{ds}{\rho} = \frac{ds_1}{\rho + \Delta x}$$

$$ds_1 - ds = \left(\frac{\rho + \Delta x}{\rho} - 1\right) ds$$

$$= \frac{\Delta x}{\rho} ds = \frac{D}{\rho}\frac{\Delta p}{p} ds$$

if L = path length along ideal trajectory between 2 points, then

$$\frac{\Delta L}{L} = \boxed{\frac{\int \frac{D(s)}{\rho(s)} ds}{\int ds}} \cdot \frac{\Delta p}{p}$$

The relative change in path length, per relative change in momentum, is called the *momentum compaction factor,*

$\alpha_p = \langle D/\rho \rangle$ along the ideal path

Periodic Dispersion Function

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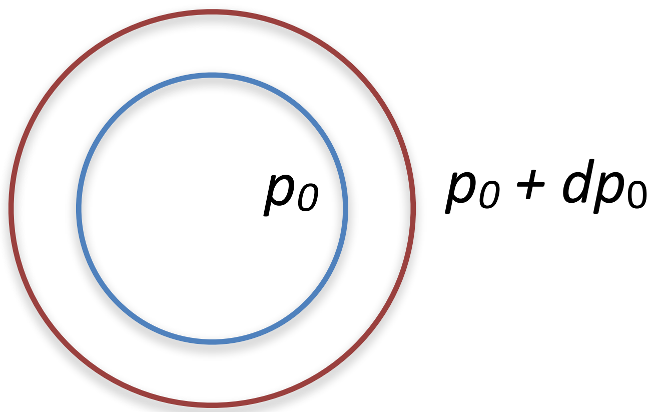
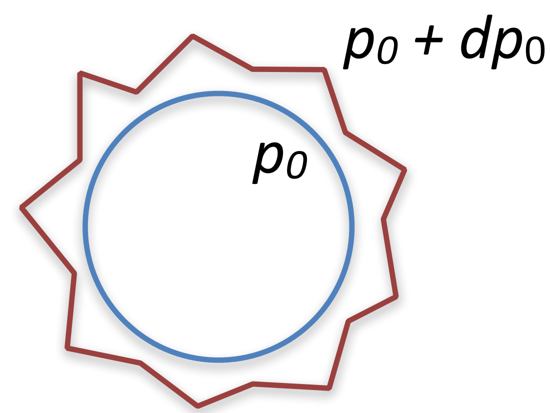
uniform bend field:



add gradients…



the trajectory "closed" orbit for momentum p+Δp

$$D(s,p) = \frac{\Delta x(s,p)}{\Delta p/p}$$

the orbit of an off-momentum particle which closes on itself is described by the *periodic* dispersion function

Ex: FODO Cells with Bending Magnets

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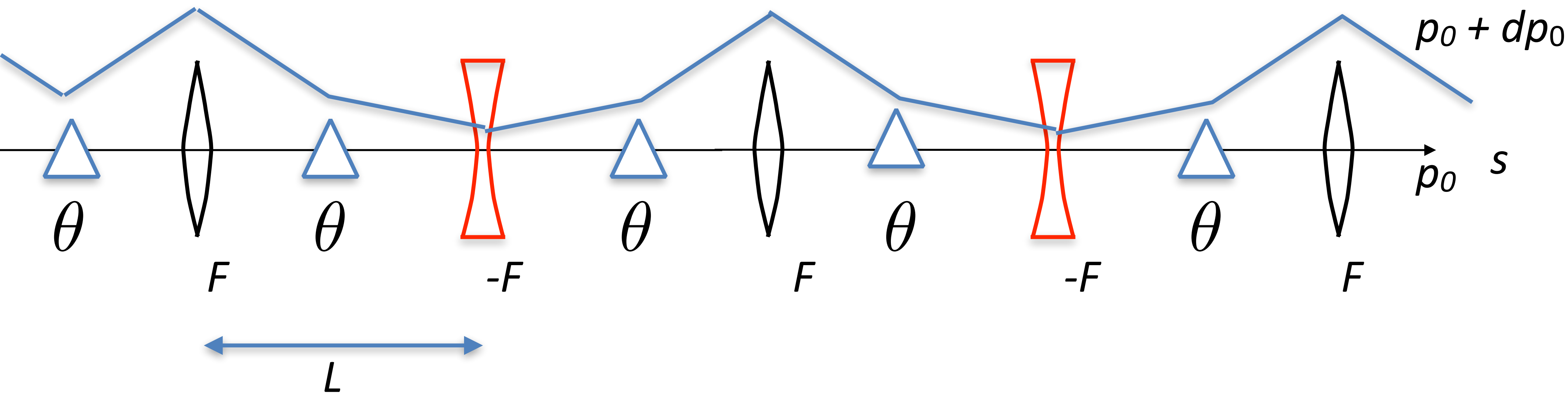


$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

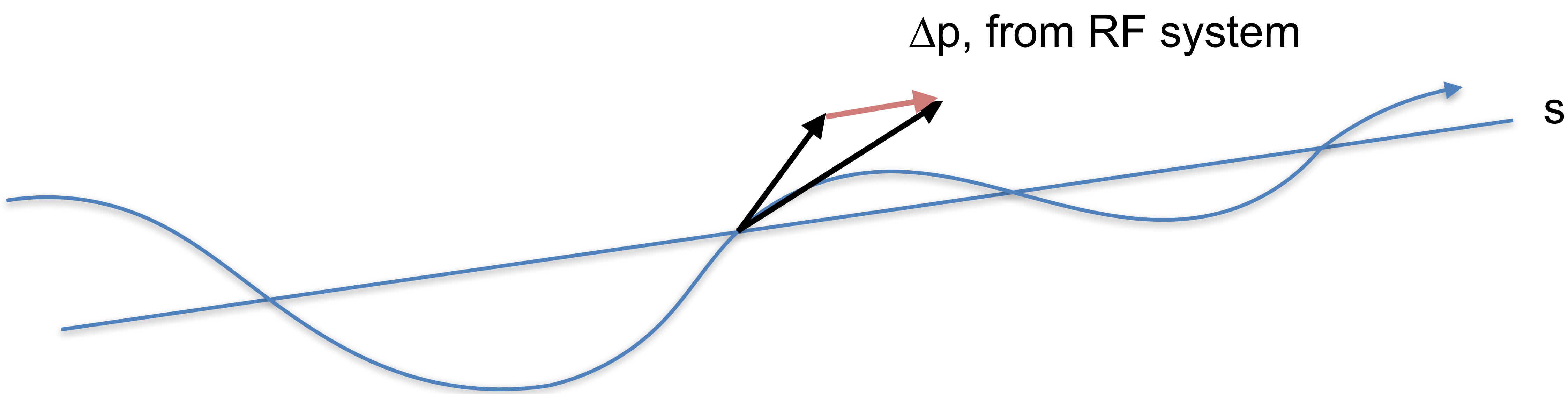
$$D_{max,min} = \frac{L\theta}{2\sin^2(\mu/2)}\left(1 \pm \frac{1}{2}\sin(\mu/2)\right)$$

Values of dispersion function are typically ~ few meters

Note: in a weak-focusing synchrotron, would have $D = R_0$!

Adiabatic Damping from Acceleration

- Transverse oscillations imply transverse momentum. As accelerate, momentum is “delivered” in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.



- The coordinates *x-x’* are not canonical conjugates, but x-p_x are; thus, from classical mechanics, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Adiabatic Damping from Acceleration

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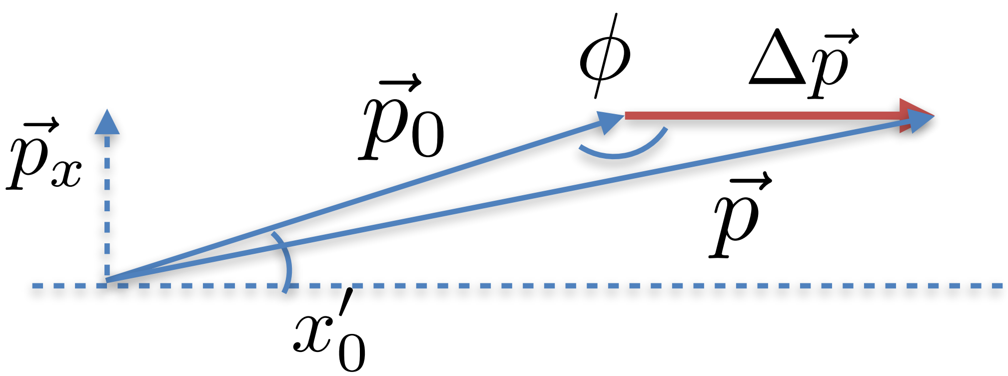
details…



Note: assuming that ALL particles receive the same *Δp* from the cavity

$$x' = \frac{p_x}{p} = \frac{p_x}{\sqrt{p_0^2 + \Delta p^2 - 2\Delta p\, p_0 \cos\phi}} = \frac{p_x}{p_0}\left(1 - \frac{\Delta p}{p_0} + \ldots\right) \approx x_0'\left(1 - \frac{\Delta p}{p_0}\right)$$

Note: particles at peak of their betatron oscillation will have little/no change in *x'*, while particles with large transverse angles will have their *x'* affected most

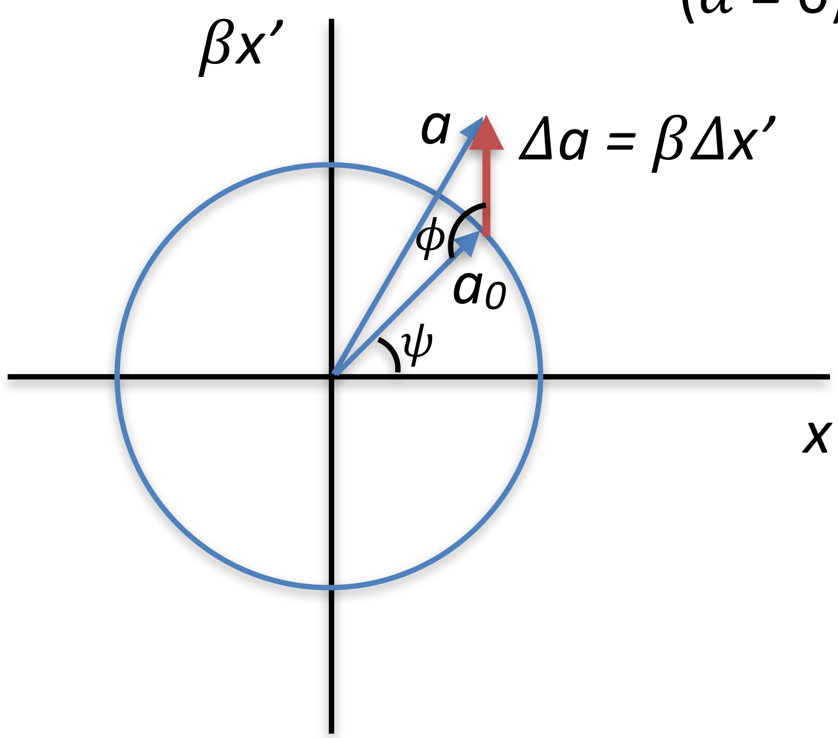
$$\Longrightarrow \quad \Delta x' = -x_0' \frac{\Delta p}{p_0}$$

Adiabatic Damping from Acceleration

details…

Now assuming that the incremental change in momentum is small compared to the overall momentum, and that the changes occur gradually on time scales large compared to those of the betatron motion…

($\alpha = 0$)



$$\Delta x' = -x_0' \frac{\Delta p}{p_0}$$

$$\begin{aligned} a^2 &= a_0^2 + \Delta a^2 - 2\Delta a\, a_0 \cos\phi \\ &= a_0^2 + \Delta a^2 + 2\Delta a\, a_0 \sin\psi \\ &= a_0^2 + \Delta a^2 + 2\Delta a\, \beta\, x_0' \\ &= a_0^2 + (-\beta x_0' \frac{\Delta p}{p_0})^2 + 2(-\beta x_0' \frac{\Delta p}{p_0})\beta x_0' \\ &= a_0^2 + (\beta x_0')^2 (\frac{\Delta p}{p_0})^2 - 2(\beta x_0')^2 \frac{\Delta p}{p_0} \end{aligned}$$

$$\Longrightarrow \quad \langle a^2 \rangle = \langle a_0^2 \rangle - 2\langle (\beta x_0')^2 \rangle \frac{\Delta p}{p_0}$$

Note: $2\langle (\beta x_0')^2 \rangle = 2\langle x_0^2 \rangle = \langle a_0^2 \rangle$

So, $\Delta\langle a^2 \rangle = -\langle a_0^2 \rangle \frac{\Delta p}{p_0}$ $\qquad \frac{\Delta \epsilon}{\epsilon} = -\frac{\Delta p}{p}$

$$\epsilon \propto \frac{1}{p} \qquad x_{rms} \propto \frac{1}{\sqrt{p}}$$

Normalized Beam Emittance

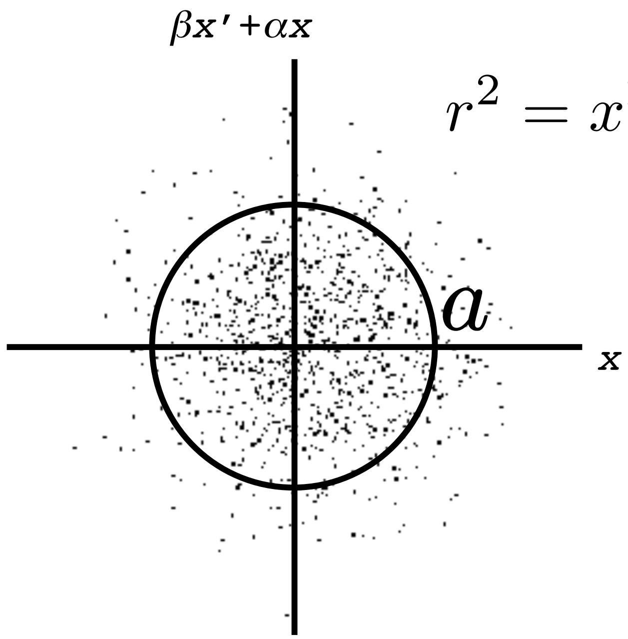
- Hence, as particles are accelerated, the area in *x-x'* phase space is not preserved, while area in x-p_x **is** preserved. Thus, we define a "normalized" beam emittance, as

$$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)_{Lorentz}$$

- In principle, the *normalized* beam emittance should be preserved during acceleration, and hence along the chain of accelerators from source to target. Thus it is a measure of beam quality, and its preservation a measure of accelerator performance.
- In practice, it is not preserved -- non-adiabatic acceleration, especially at the low energy regime; non-linear field perturbations; residual gas scattering; charge stripping; field errors and setting errors; *etc.* -- all contribute at some level to increase the beam emittance. Best attempts are made to keep the emittance as small as possible.

Emittance of a Stationary Gaussian Distribution

- Imagine a transverse distribution of particles with a Gaussian profile in transverse coordinate *x* with zero mean and standard deviation σ.
  - The distribution can be described as follows:



$$r^2 = x^2 + (\beta x' + \alpha x)^2$$

$$\rho(r,\theta) r dr d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\, dr d\theta$$

$$\int_0^{2\pi} \int_0^a \rho\, r dr d\theta = f$$

Radius, *a*, containing fraction, *f*, of particles, corresponding to phase space area with emittance, ϵ:

$$a^2 = -2\sigma^2 \ln(1-f) = \epsilon\beta/\pi$$

Emittance of a Stationary Gaussian Distribution

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- So, the normalized emittance that contains a fraction *f* of a Gaussian beam is:

$$\epsilon_N = \frac{-2\pi \ln(1-f)\sigma^2(s)}{\beta(s)} (\beta\gamma)$$

Lorentz!

- Common values of *f* :

f	$\epsilon_N/(\beta\gamma)$
95%	$6\pi\sigma^2/\beta$
86.5%	$4\pi\sigma^2/\beta$
39%	$\pi\sigma^2/\beta$
15%	σ^2/β

Perhaps most commonly used, typically called the "rms" emittance; but, always ask if not clear in context!

more typical for light sources, e- colliders