Introduction to Finite-Element-Methods In Electromagnetism

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Main Goal for Today….

• Be able to understand, setup, and solve basic problems in electrostatics and magnetostatics using finite-element methods, e.g. “Finite Element Method Magnetics” (FEMM), on your local Windows/Linux/Mac computer
Outline

• Overview LaPlace’s equation and solutions
  - Cauchy-Riemann equations; Connection to electromagnetism (EM)
  - Conformal mappings
  - Shortcomings
• Finite-Element methods and codes (OPERA, COMSOL, FEMM, …)
• Getting FEMM setup on your local machine
• FEMM electric example
  - Problem setup: Electric dipole/quadrupole
  - How to extract the potential/E-field from FEMM
• FEMM magnetic example (next time)
  - Problem setup: H-dipole, quadrupole (more challenging)
  - How to extract the potential/B-field from FEMM
• Pro Tips And Tricks (next time)
  - Scripting in `lua` and `python`; Jupyter notebooks; WebPlotDigitizer; etc.
• Conclusions
But first, A Few Quick Words About Me….

- I am Prof. Syphers’ postdoc. I am stationed at Fermilab, and I work closely with Accelerator Division (AD) and the Muon g-2 Experiment (E989)
I’ll Also Try To Talk About The Bigger Picture Along The Way

- [Accelerator] Physics has many applications elsewhere, e.g. medicine

- The ability to perform high-level data analysis is always sought after!
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Laplace’s Equation

• In accelerator physics, most beamline elements (e.g. dipoles and quads) are static or slowly varying, so time-dependent terms in Maxwell’s equations are zero or negligible

\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \]

\[ \nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}. \]

• Additionally, we try **not** to pass the beam through other material (beam pipe, pole tip, electrode), so Maxwell’s equations simplify further:

\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = 0, \]

\[ \nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{B} = 0. \]
Laplace’s Equation

• The curl equations imply the field may be derived from a potential,

\[ \nabla \times \{E, B\} = 0 \quad \Rightarrow \quad \{E, B\} = -\nabla\{V_e, V_m\}. \]

• The divergence equations then imply Laplace’s equation,

\[ \nabla \cdot \{E, B\} = \nabla \cdot (-\nabla\{V_e, V_m\}) = -\nabla^2\{V_e, V_m\} = 0. \]

• Also, don’t forget about these 3 copies of Laplace’s equation,

\[ \nabla \times B = \nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A = 0 \quad \Rightarrow \quad -\nabla^2 A = 0. \]

(Coulomb gauge)

• Laplace’s equation is all over the place! We should spend some time trying to understand how to solve it....
Laplace’s Equation

- EM fields are derivable from potentials. The complex plane really helps with the analysis…

\[ f(z) = u(z) + iv(z), \quad z \equiv x + iy \in \mathbb{C}. \]

\[ f'(z) \equiv \frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \]

Suppose we have a particle here. We need to know the field (since \( F = q(E + v \times B) \)), so we need to calculate a derivative of the potential
Laplace’s Equation

- EM fields are derivable from potentials, and the complex plane really helps with the analysis:

\[ f(z) = u(z) + iv(z), \quad z \equiv x + iy \in \mathbb{C}. \]

\[
f'(z) = \frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}
\]

Approach 1: Let \( \Delta y = 0 \), and approach \( z_0 \) purely along the x-direction....
Laplace’s Equation

- EM fields are derivable from potentials, and the complex plane really helps with the analysis:

\[ f(z) = u(z) + iv(z), \quad z \equiv x + iy \in \mathbb{C}. \]

\[ f'(z) \equiv \frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x = 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \left( \begin{array}{c} \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \end{array} \right) \]

Approach 2: Now let \( \Delta x = 0 \), and approach \( z_0 \) purely along the y-direction
Laplace’s Equation

- EM fields are derivable from potentials, and the complex plane really helps with the analysis:

\[ f(z) = u(z) + iv(z), \quad z \equiv x + iy \in \mathbb{C}. \]

\[
\frac{f'(z)}{\partial z} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x=0} \Delta x + i \Delta y = \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}
\]

If the derivative of \( f(z) \) exists, the two approaches outlined above must give the same result!

Equating the real and the imaginary parts of the equations on the last two slides, we get the celebrated Cauchy-Riemann equations!

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]
Laplace’s Equation

- The Cauchy-Riemann equations are deeply connected to Laplace’s equation

\[ f(z) = u(z) + iv(z), \quad z \equiv x + iy \in \mathbb{C}. \]

\[
\frac{\partial u}{\partial x} = + \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}.
\]

**Cauchy-Riemann (CR)**

- The Cauchy-Riemann (CR) equations imply two separate copies of Laplace’s equations, one for both the real and the imaginary parts of \( f(z) \! \):

\[
\partial_x^2 u = \partial_x (\partial_x u) = \partial_x (\partial_y v) = \partial_y (\partial_x v) = -\partial_y^2 u \quad \Rightarrow \quad -(\partial_x^2 + \partial_y^2)u = 0
\]

\[
\partial_y^2 v = \partial_y (\partial_y v) = \partial_y (\partial_x u) = \partial_x (\partial_y u) = -\partial_x^2 v \quad \Rightarrow \quad -(\partial_x^2 + \partial_y^2)v = 0.
\]

- On Thursday, we’ll talk more about multipole expansions, \( f(z) = \sum c_n \cdot z^n \)
Conformal Transformations

- Conformal transformations of the complex plane are basically functions that preserve orientation and angles locally ("conformal")

Example from Mathematica

Example: Electric Potential / Field
Conformal Transformations $\Rightarrow$ Finite-Element Methods

- Mathematical expressions quickly get out of control. Limited to all but this simplest cases. Nowadays, with computers, we can do much more.…

Conformal transformations (i.e. math equations on paper) were basically superseded by numerical methods on computers c. 1960-1970
Finite-Element Methods (FEM)?

- Essentially, (FEM) “puts differential equations on a grid/mesh” and solves numerically with computers (fluid flow, heat flow, stress/strain, electromagnetism, etc.)
Finite-Element Methods (FEM): Laplace’s equation

- A simple example of solving Laplace’s equation via “grid relaxation”

Problem domain (“mesh”)

Laplace’s equation on the grid

\[
\begin{align*}
\left( \frac{\partial^2 u}{\partial x^2} \right) & \approx \frac{u(x-\Delta x, y) - 2u(x, y) + u(x+\Delta x, y)}{(\Delta x)^2} , \\
\left( \frac{\partial^2 u}{\partial y^2} \right) & \approx \frac{u(x, y-\Delta y) - 2u(x, y) + u(x, y+\Delta y)}{(\Delta y)^2} .
\end{align*}
\]

(quick derivation on the board)
Finite-Element Methods (FEM): Laplace’s equation

- We can rewrite the last slide using “index notation”

\[
\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta y)^2} = 0
\]

(this is the same equation as on the last slide)
Finite-Element Methods (FEM): Laplace’s equation

- Let $\Delta x = \Delta y = 1$ for simplicity. Then the solution at $u(i, j)$ is just the average value of its neighbors!

$$-4u_{z,z} + u_{1,z} + u_{z,1} + u_{z,3} + u_{3,2} = 0$$

known from B.C.

Ask to see Prof. Syphers’ notebook :)
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Getting FEMM Setup On Your Local Windows/Linux/Mac

• Instructions at the PHYS 790D course website (here)
• Windows installation is very easy
• Linux/Mac installation requires one additional step: Wine

FEMM (a Windows application)

WINE allows you to run Windows Apps on your Linux/Mac!

What is Wine?

Wine (originally an acronym for "Wine Is Not an Emulator") is a compatibility layer capable of running Windows applications on several POSIX-compliant operating systems, such as Linux, macOS, & BSD. Instead of simulating internal Windows logic like a virtual machine or emulator, Wine translates Windows API calls into POSIX calls on-the-fly, eliminating the performance and memory penalties of other methods and allowing you to cleanly integrate Windows applications into your desktop.
FEMM Tutorial: Electrostatics

- Electric problems are easier to setup in FEMM, so that’s where we’ll start
- Two simple examples: (1) electric dipole, (2) electric quadrupole

![Electric storage ring: EDM measurements](image1)

Example electric dipole FEMM simulation

We’ll look at straight/curved longitudinal axes
FEMM Tutorial: Electrostatics

- Here we go! Start FEMM and create a new “Electrostatics Problem”
FEMM Tutorial: Electrostatics

- Can define global properties of the FEMM simulation here
FEMM Tutorial: Electrostatics

• Always save your work! (I chose filename = “femm-electric-dipole”)
FEMM Tutorial: Electrostatics

• Build a “vacuum chamber” shape. Start by defining corners & edges

Add 4 corner points at \(\{\pm 10, \pm 10\}\)
by pressing <TAB> and entering
\((x, y)\) coordinates
Now add edges by first selecting “segment” tool, then double-clicking any two points.

Selecting these two points will draw a segment between them.
FEMM Tutorial: Electrostatics

- The vacuum chamber is at 0V — add a boundary condition
FEMM Tutorial: Electrostatics

• The vacuum chamber is at 0V — add a boundary condition

Select line segments, press <SPACE BAR>, add boundary condition
FEMM Tutorial: Electrostatics

• Add material, “Vacuum”

Choose a name for your material, and choose $\epsilon_x/\epsilon_0 = \epsilon_y/\epsilon_0 = 1$ (vacuum)
FEMM Tutorial: Electrostatics

• Add material, “Vacuum”

1. Select block-label tool. Press <TAB> and choose coordinates \((x, y) = (0, 0)\)
2. Select your newly created block label at \((x, y) = (0, 0)\)
3. Press <SPACE BAR> and apply your vacuum material definition from earlier
FEMM Tutorial: Electrostatics

• Create the “outer electrode” by first drawing a line segment

I chose electrode vertices

\((x, y) = (5, \pm 5) \text{ cm}\)
FEMM Tutorial: Electrostatics

- Define the potential for the outer electrode, e.g. +1kV
FEMM Tutorial: Electrostatics

- Apply potential to outer electrode, e.g. +1kV

Select line segments, press <SPACE BAR>, add boundary condition.
FEMM Tutorial: Electrostatics

• Now create the “inner electrode” by drawing a line, as before

Same steps as before. I chose \((x, y) = (-5, \pm 5)\) cm for vertices
FEMM Tutorial: Electrostatics

- Define the potential for the inner electrode, e.g. -1kV
FEMM Tutorial: Electrostatics

• Apply potential to inner electrode, e.g. -1kV

Select line segments, press <SPACE BAR>, add boundary condition
FEMM Tutorial: Electrostatics

- Mesh, analyze, and see the results!

1. Run mesh generator (mesh icon, yellow)
2. Run analysis (gear icon)
3. View results (glasses icon)
FEMM Tutorial: Electrostatics

• This is what you should see. If not, I can help!
FEMM Tutorial: Electrostatics

• Suppose we want the E-field in the horizontal midplane....

Same steps as before: Define a line via 2 points (press <TAB>). Can hover mouse to see coordinates listed at lower left.
FEMM Tutorial: Electrostatics

• Now make a plot by clicking the “plot” icon. Can export to text file….
FEMM Tutorial: Electrostatics

• Our first results! (Does the plot make sense?)

\[ \mathbf{E} = - \nabla V \]
FEMM Tutorial: Electrostatics

• Now we’ll modify the dipole to make a quadrupole. Always save your work!
FEMM Tutorial: Electrostatics

• Add another couple of electrodes to make quadrupole

Add 2 new electrodes. I chose to place the 2 new electrodes at \( y = \pm 6 \text{ cm} \) so they don’t touch the old electrodes.
FEMM Tutorial: Electrostatics

Move old electrodes slightly for symmetry using the “move” tool.
FEMM Tutorial: Electrostatics

• Set electrode voltages for vertical focusing of positively-charged particles

Apply the correct voltages for vertical focusing of positively charged particles
FEMM Tutorial: Electrostatics

• This is what your solution should look like. If not, I can help!
FEMM Tutorial: Electrostatics

- Plot the potential in the horizontal midplane. (Does it make sense?)

Same steps as before:
Define a line via 2 points (press <TAB>). Can hover mouse to see coordinates listed at lower left.
FEMM Tutorial: Electrostatics

• Plot the potential in the horizontal midplane. (Does it make sense?)
FEMM Tutorial: Electrostatics

• What about a curved longitudinal axis, i.e. cylindrical symmetry?
FEMM Tutorial: Electrostatics

• What about a curved longitudinal axis, i.e. cylindrical symmetry?
FEMM Tutorial: Electrostatics

• Does the new solution (with cylindrical symmetry) look any different?
FEMM Tutorial: Electrostatics

• Plot the potential in the horizontal midplane

Same steps as before:
Define a line via 2 points (press <TAB>). Can hover mouse to see coordinates listed at lower left.
FEMM Tutorial: Electrostatics

- Plot the potential in the horizontal midplane
FEMM Tutorial: Electrostatics

- Compare the two cases, i.e. Cartesian vs. Cylindrical coordinates

\[ E = - \nabla V \]

Curved longitudinal axis

\[ E = - \nabla V \]

Straight longitudinal axis
That’s All For Today — See You On Thursday!

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