Emittance Preservation

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- Liouville's Theorem: the volume enclosed by surface in phase space is invariant under conservative forces
- Another theorem from classical dynamics: integration over a time period of the "action variables" is an adiabatic invariant

$$J = \int_0^T p_x(t) \frac{dx}{dt} dt$$

  - transverse: (x, p_x), (y, p_y) are action variables
  - longitudinal: ΔE and Δt are also action variables

- "normalized" transverse phase space emittances,
  - $\epsilon_N = (\beta\gamma)\epsilon = (p/mc)\int x' dx = \int p_x \, dx / mc$

Protons vs. Electrons

- When dealing with a beam line or along a linac, the same issues affecting beam emittance exist for both electron and proton (or heavier ion) beams.
- In the case of circular accelerators, there is a distinct difference: charged particles radiate as they are accelerated, and electrons will radiate much more than protons and, as we have seen, the final emittance of electron beams in a ring will be defined by the optics of the ring.
- This is not true for a proton beam. If the emittance is increased due to errors or mismatches, the damage is done and cannot be undone without ***much*** effort

Sources of Emittance Growth

- Will discuss two classes of disruptive processes…

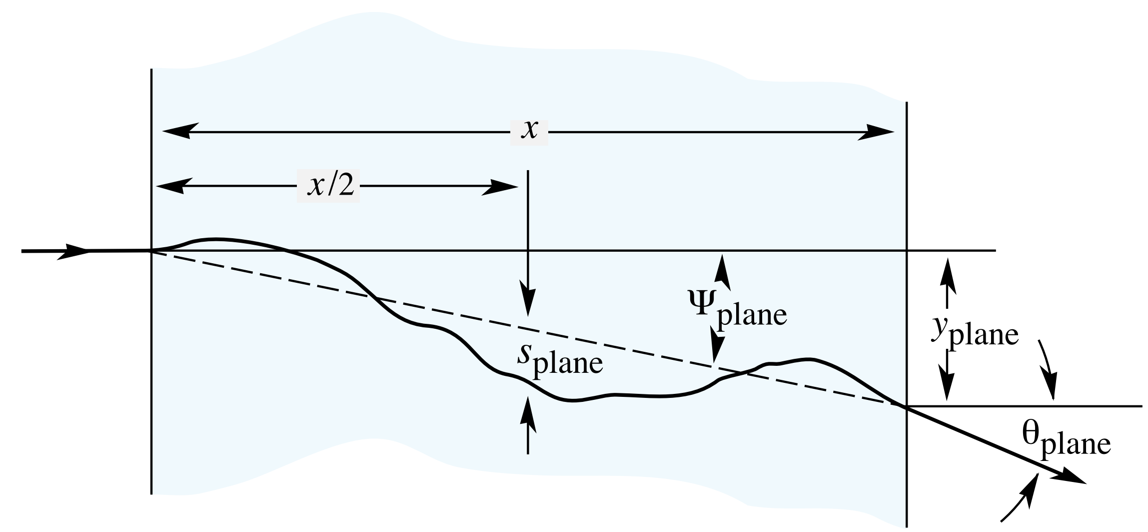
  - **Single non-adiabatic disturbance** of the distribution
    - *examples*: injection errors (steering, focusing); electrostatic "spark"; single pass through a vacuum window; a pinger/kicker excitation; intrusive diagnostic measurement; …

  - **Repetitive random disturbances** of individual particles, leading to diffusion
    - *examples*: RF noise; beam-gas scattering; power supply noise; mechanical vibrations; …

Non-adiabatic Disturbances Example: single pass through a thin object (vacuum window)

- Example: single pass through a vacuum window
  - multiple Coulomb scattering through material

The beam particles interact with the atoms in the material and scatter, primarily from Coulomb interactions. In either plane — x or y — the distribution of scattering angles emerging from the material is given by:



$$\theta_{rms} \approx \frac{13.6 \text{ MeV}}{\beta pc}\sqrt{\frac{\ell}{L_{rad}}}$$

where L_{rad} is the "radiation length" of the material:

$$\frac{1}{L_{rad}} \approx 2\alpha\frac{N_A}{A}\rho Z^2 r_e^2 \ln\frac{a}{R}$$

N_A = Avogadro's No., A = atomic mass, Z = charge state, r_e = "classical electron radius", a = radius of target atom, R = radius of target nucleus, α = fine structure constant

for more accurate estimates, see *Particle Data Booklet*, http://pdg.lbl.gov

Side Note: The Bethe Formula

- Radiation Length is related to the stopping power of material as charged particles pass through

H. Bethe und J. Ashkin in "Experimental Nuclear Physics, ed. E. Segré, J. Wiley, New York, 1953, p. 253

  - mean distance e⁻ travels before losing all but 1/e of its energy

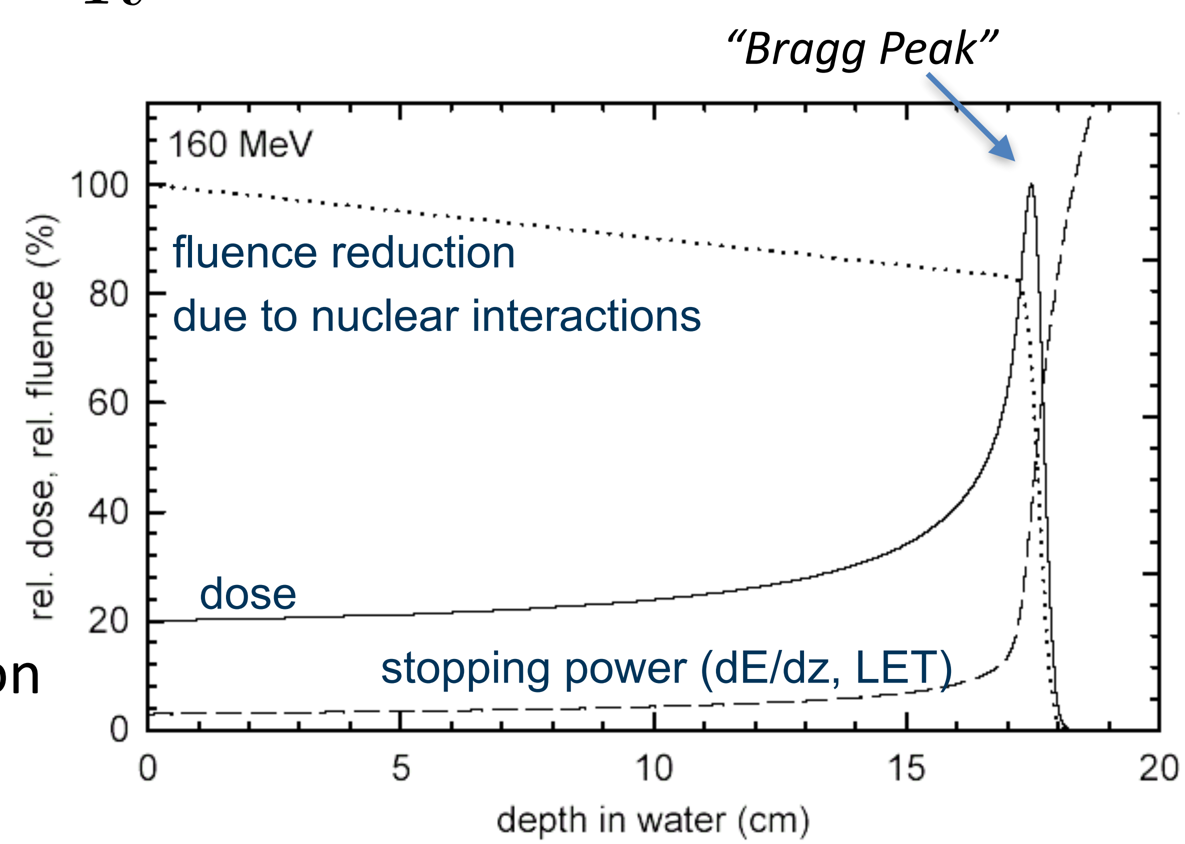
- The average energy loss rate is given by the Bethe formula:

$$\frac{1}{L_{rad}} \approx 2\alpha \frac{N_A}{A} \rho Z^2 r_e^2 \ln \frac{a}{R}$$

$$-\left\langle \frac{dE}{dx} \right\rangle = \frac{4\pi}{m_e c^2} \cdot \frac{nz^2}{\beta^2} \cdot \left(\frac{e^2}{4\pi\varepsilon_0} \right)^2 \cdot \left[\ln\left(\frac{2m_e c^2 \beta^2}{I \cdot (1-\beta^2)} \right) - \beta^2 \right]$$

$$n = \frac{N_A \cdot Z \cdot \rho}{A \cdot M_u}$$

Used to determine depth of energy deposition for proton or ion therapy, for instance:



Non-adiabatic Disturbances Example: single pass through a thin object (vacuum window)

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- As we saw earlier, the emittance and Courant-Snyder parameters describing a distribution can be written as:

$$\epsilon_N = (\beta\gamma)\epsilon \qquad\qquad \epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

- From the scattering, the angular distribution will be altered:

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon} \qquad \alpha = -\frac{\pi\langle xx'\rangle}{\epsilon} \qquad \gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

- We then average over the distribution to see the effect on the CS parameters and emittance…

$$x' = x'_0 + \Delta\theta$$

$\Delta\theta$ is random, with $\langle\Delta\theta^2\rangle \equiv \theta_{rms}^2$

Non-adiabatic Disturbances Example: single pass through a thin object (vacuum window)

- As we saw earlier, the emittance and Courant-Snyder parameters describing a distribution can be written as:

$$x = x_0$$
$$x' = x'_0 + \Delta\theta$$

$$\begin{aligned}\langle x^2\rangle &= \langle x_0^2\rangle \\ \langle x'^2\rangle &= \langle (x'_0+\Delta\theta)^2\rangle \\ &= \langle {x'_0}^2 + \Delta\theta^2 + 2x'_0\Delta\theta\rangle \\ &= \langle {x'_0}^2\rangle + \langle\Delta\theta^2\rangle \\ \langle xx'\rangle &= \langle x_0(x'_0+\Delta\theta)\rangle = \langle x_0x'_0\rangle\end{aligned}$$

assuming that the scattering process is uncorrelated with the phase space variables. The new emittance after scattering is then given by,

$$(\epsilon/\pi)^2 = \langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2 = \langle x_0^2\rangle(\langle {x'_0}^2\rangle + \theta_{rms}^2) - \langle x_0x'_0\rangle^2$$

or,

$$\epsilon = \epsilon_0\sqrt{1 + \frac{\langle x_0^2\rangle\theta_{rms}^2}{(\epsilon_0/\pi)^2}}$$

where $\theta_{rms} = \langle\Delta\theta^2\rangle^{1/2}$. Since $\beta_0 = \pi\langle x_0^2\rangle/\epsilon_0$, then we can write

$$\epsilon = \epsilon_0\sqrt{1 + \theta_{rms}^2\frac{\beta_0}{(\epsilon_0/\pi)}}.$$

And, using this new emittance, we can describe the resulting distribution after the scattering by new Courant-Snyder parameters

$$\begin{aligned}\beta &= \frac{\pi\langle x^2\rangle}{\epsilon} = \beta_0\ \frac{\epsilon_0}{\epsilon} \\ \alpha &= \alpha_0\ \frac{\epsilon_0}{\epsilon} \\ \gamma &= \left(\gamma_0 + \frac{\theta_{rms}^2}{(\epsilon_0/\pi)}\right)\ \frac{\epsilon_0}{\epsilon}\end{aligned}$$

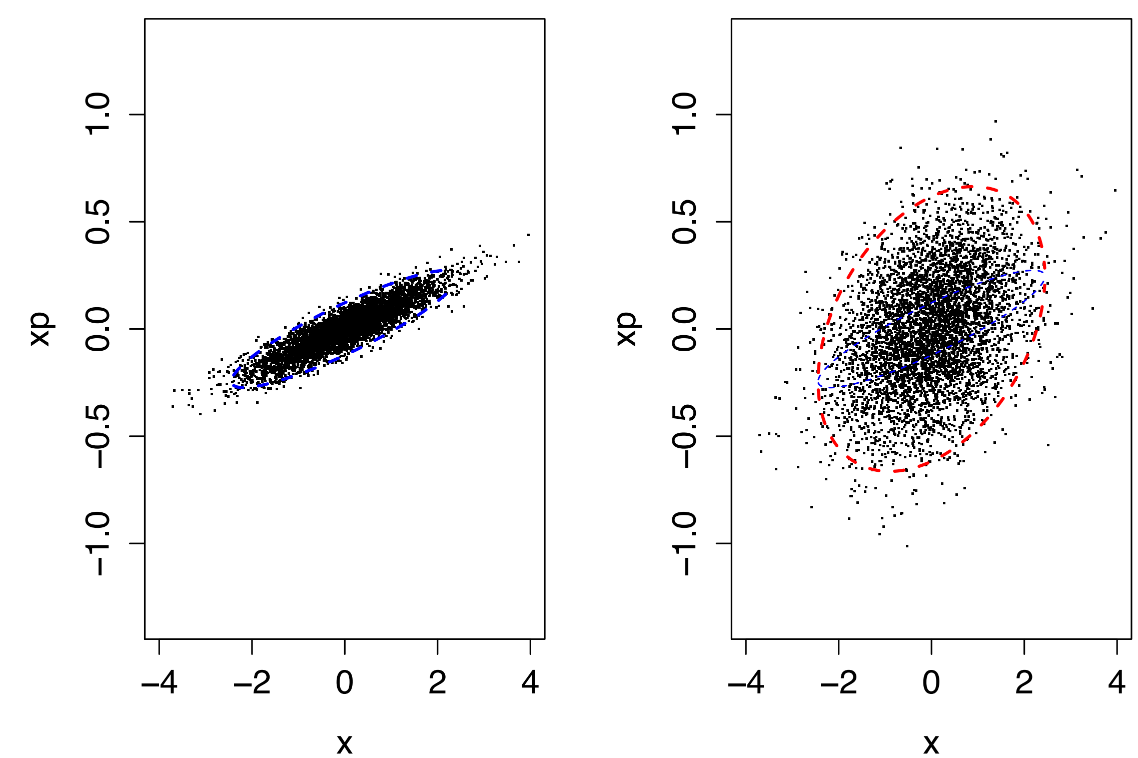

Figure 1: Phase space before/after (left/right) scattering. Blue/red lines indicate 95% emittances before/after scattering.

If the scattering is inherent in the system, then can imagine re-tuning the downstream optical system to match to the new conditions; but will still have an overall emittance growth

Non-adiabatic Disturbances
Example: single pass through a linac/beamline

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$$\theta_{rms} \approx \frac{13.6 \text{ MeV}}{\beta p c}\sqrt{\frac{\ell}{L_{rad}}}$$

(*beam line at Fermilab*)

- Suppose we pass through a long "evacuated" tube of length $L = 230$ *m*. As an example, consider a tube which started with air, and has been evacuated to an average pressure of 10^{-6} torr (760 torr = 1 atm)
- From the PDG report, find L_{rad} of air (dry; 1am):
  - density = 1.205 g/l , L_{rad} = 36.6 g/cm^2
  - so, L_{rad} = (36.6 g/cm^2)/(1.205 g/ml)(l/1000 cm^3)
  - = 30373 cm = 304 m
  - at 10^{-6} torr, through *PV=nRT*, L_{rad} = 231x10^9 m
- Estimate the rms scattering angle of a typical particle, just due to this effect:

$$\theta_{rms} \approx \frac{13.6 \text{ MeV}}{27 \text{ MeV}}\sqrt{\frac{230 \text{ m}}{23 \times 10^{10} \text{ m}}} \approx 16\ \mu\text{rad}$$

Side Note: Kicker Magnet

- Want to induce an angular deflection of a particle bunch, or bunch train, without affecting other particles outside of the bunch/train
- Need significant B fields that turn on/off on the scale of, say, μs
  - ex: bunches @ 1 MHz = 1 μs
- Ex: discharge large current into an inductive load (magnet) with a resistance,   gives time constants on the scale of ~ L/R: μH/Ohm = μs

fast means low inductance,
thus low fields and low turns,
high currents in the magnets

Side Note: Kicker Magnet

- Kickers typically used to deflect beam into and out of beam lines and accelerators

Ex: injection into a synchrotron:

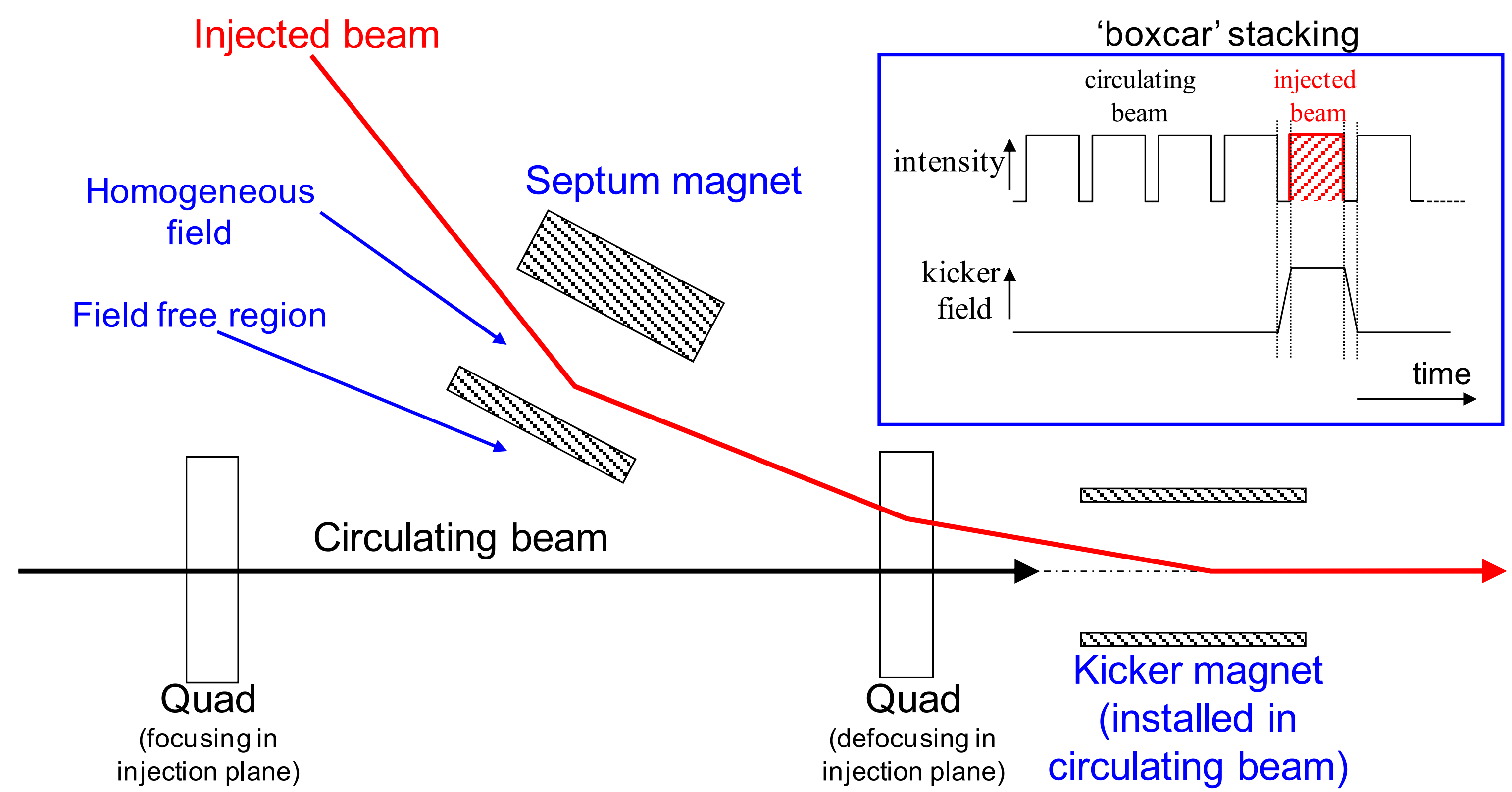


figure courtesy of M.J. Barnes, et al., CERN

- Can also be used for diagnostic purposes, by intentionally inducing a betatron oscillation in the beam and observing downstream reaction

Non-adiabatic Disturbances
Example: Discharge of a beam kicker in a synchrotron

- Initially, the distribution is simply "displaced" by the action of the kick:

- Nonlinearities will yield:
  - tune vs. amplitude
  - decoherence
  - *filamentation*
  - emittance growth

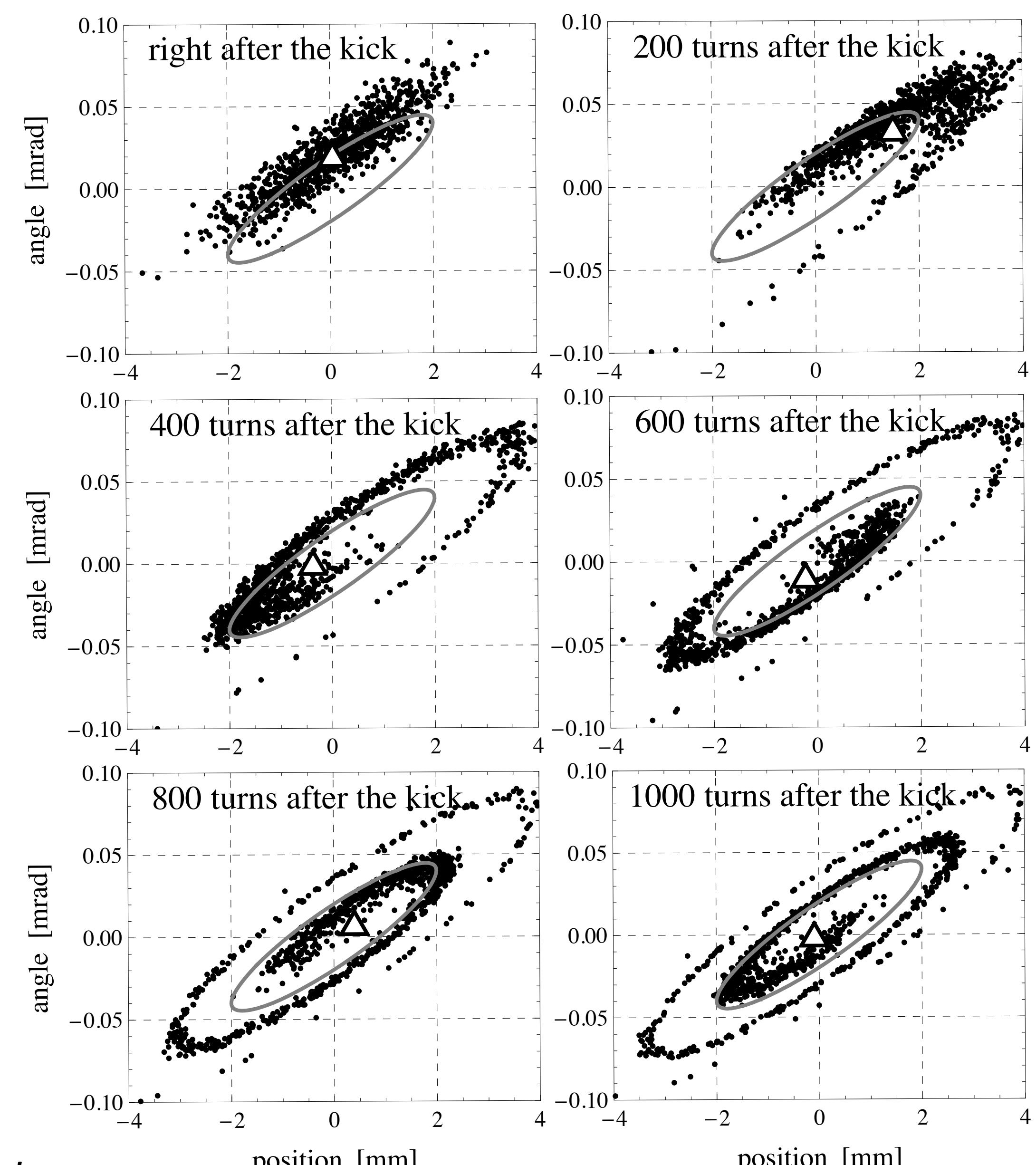
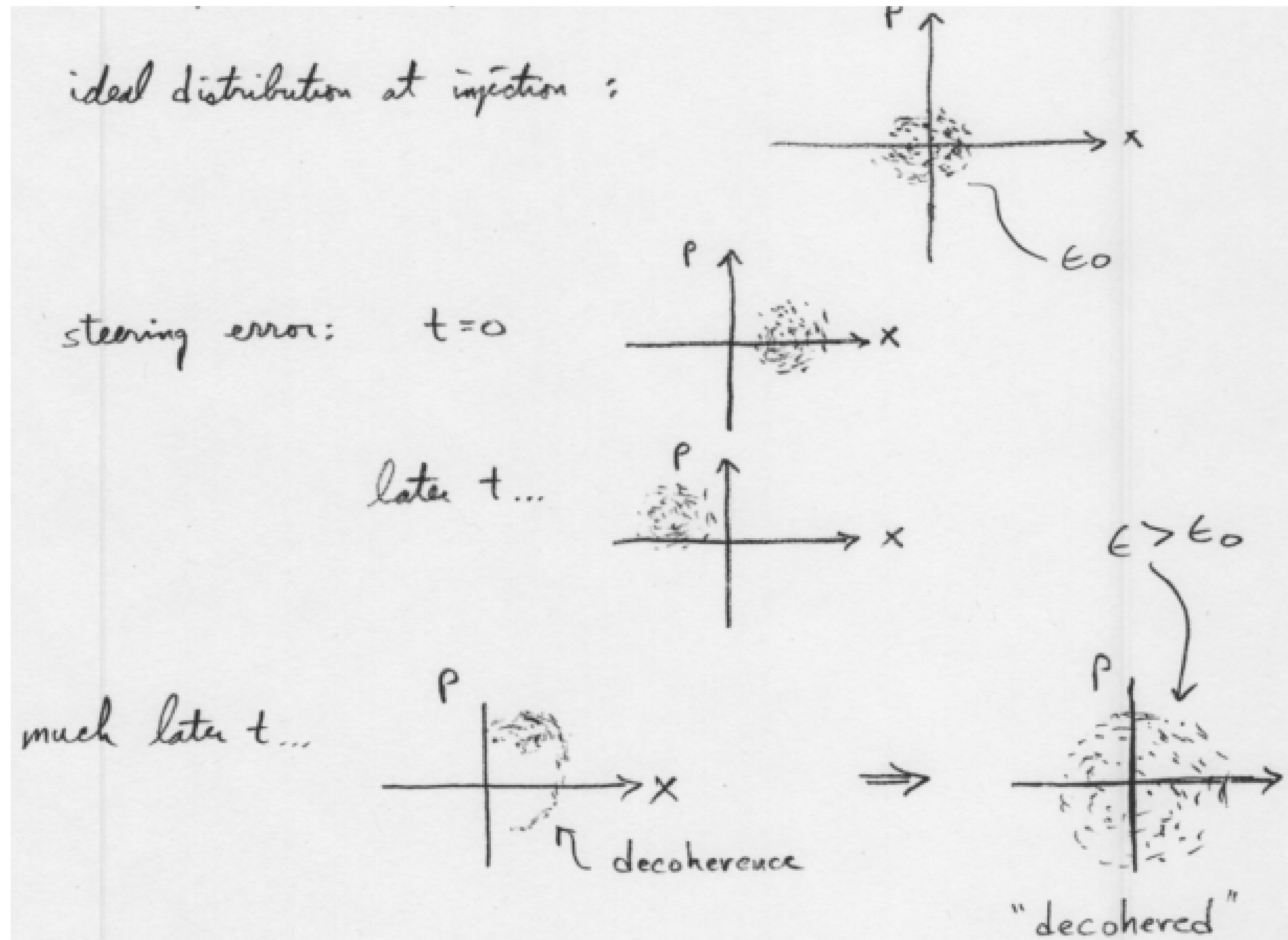


figure by R. Miyamoto

Accelerator Model

- So we will model these effects by assuming the distribution will oscillate about the closed orbit, and that the oscillation frequencies of the particles will depend upon the amplitude of their oscillations
  - typically: $\nu \approx \nu_0 + ka^2$
  - coherent at first,
  - then "decoheres"
    - » leads to filamentation
  - eventually: larger emittance

nonlinear tune shift



Example: Injection Steering Mismatch

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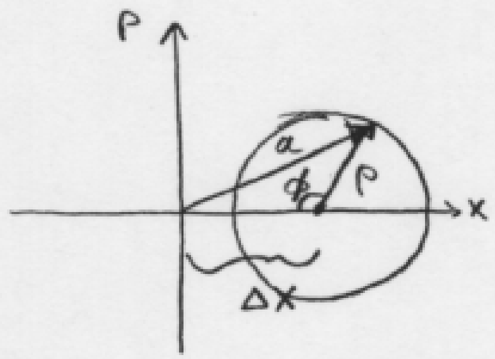
- Example: injection steering error

how does rms evolve ?



if $\Delta x = 0$, would have

$$\langle x^2 \rangle = \tfrac{1}{2}\rho^2 \equiv \sigma_0^{\,2}$$

but here,

$$a^2 = \rho^2 + \Delta x^2 - 2\Delta x\,\rho\cos\phi$$

average over all particles .. $\langle a^2 \rangle = \langle \rho^2 \rangle + \Delta x^2 - 2\Delta x \underbrace{\langle \rho\cos\phi \rangle}_{\sim\phi}$

$\Rightarrow \langle a^2 \rangle = \langle \rho^2 \rangle + \Delta x^2$ after decoherence

$\Rightarrow \langle x^2 \rangle = \sigma_0^{\,2} + \tfrac{1}{2}\Delta x^2$ "

if $\epsilon \equiv \frac{6\pi\sigma^2}{\beta}\cdot(\gamma\beta)$ $\longrightarrow$ $\boxed{\Delta\epsilon_N = \frac{3\pi\,\Delta x^2}{\beta}(\gamma\beta)}$ or, $\boxed{\epsilon/\epsilon_0 = 1 + \tfrac{1}{2}\left(\frac{\Delta x}{\sigma_0}\right)^2}$

β @ location of Δx

95% emittance; no "6" if rms emittance

Note:

injection errors - $\Delta\epsilon_N \propto (\gamma\beta)$, $\therefore$ more important @ higher energies

let $\Delta x = 1\,\text{mm}$ @ injection to

	$\hat{\beta}$	$\gamma\beta$	$\Delta\epsilon_N$	
800	30m	1.017	0.10 π mm-mr	
MI	50 m	9.48	0.57 π mm-mr	
TEV	100m	160	4.8 π mm-mr	☆

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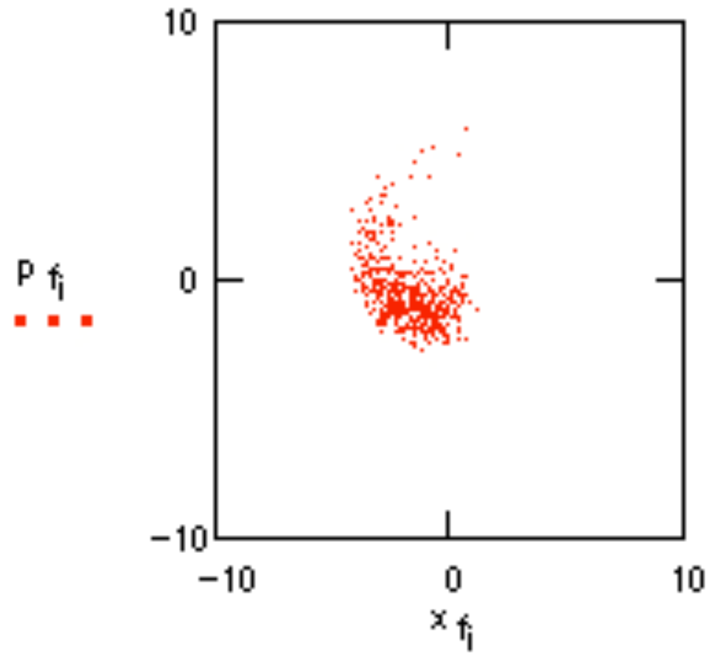
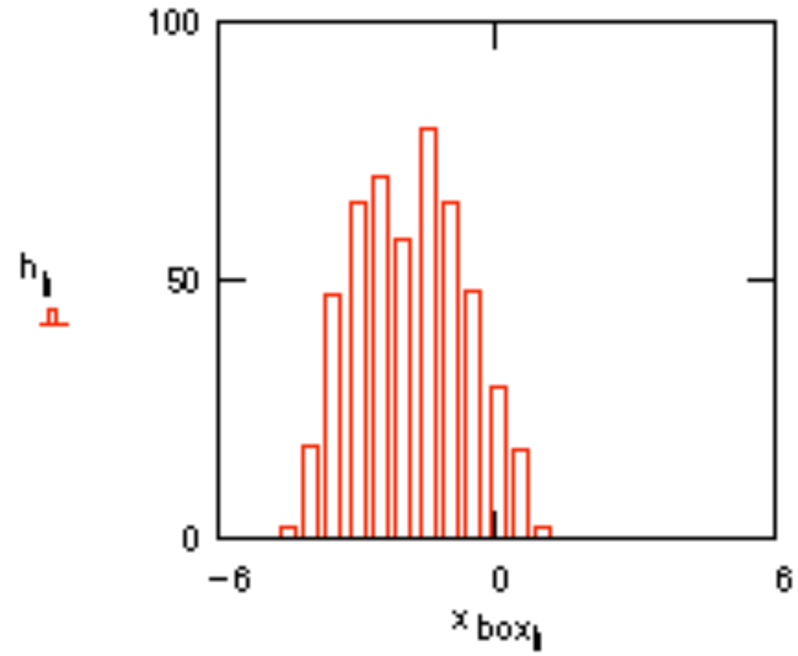
Injection Mismatch

Phase Space



x Profile



$\mathrm{mean}(x_f) = -1.609$ $\mathrm{stdev}(x_f) = 1.179$

Emittance Increase: $\mathrm{stdev}(x_f)^2 = 1.39$

Predicted "typical" values:

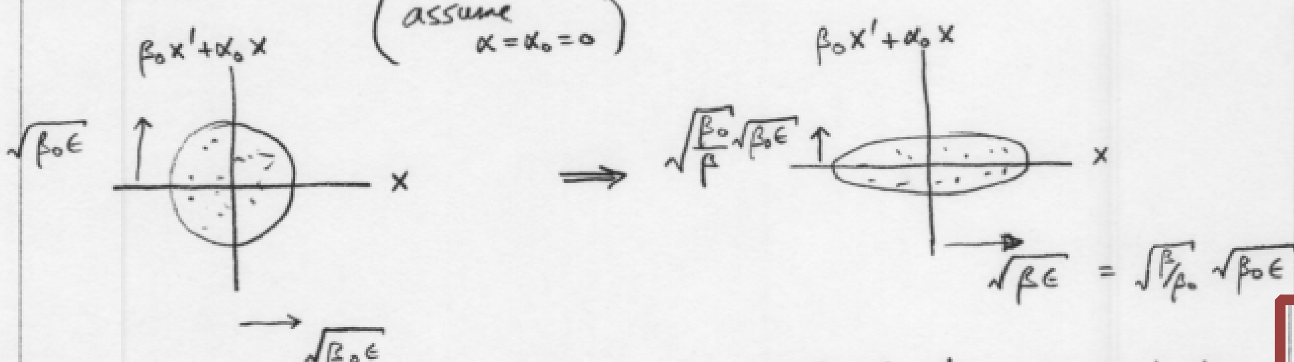
(Steering Mismatch) $1 + \frac{1}{2}\cdot\Delta x^2 = 3$

FRAME = 8 (Amplitude function Mismatch) $\frac{r_\beta^2 + 1}{2\cdot r_\beta} = 1$

Injection “Beta” Mismatch

- We imagine a ring with an ideal amplitude function, β, at an injection point. But, suppose the beam line transporting beam from an upstream injector delivers the wrong β function:

∴ in "circular" phase space ... (assume $\alpha = \alpha_0 = 0$)



β_0, α_0 : periodic functions of synchrotron

β, α : delivered by the beamline

then, after the distribution tumbles and filaments in phase space, the emittance will grow...

$$a^2 = x^2 + (\beta_0 x' + \alpha_0 x)^2 \equiv x^2 + p^2$$

let $b \equiv \beta/\beta_0$ $\Rightarrow$ $\langle a^2 \rangle = \langle x^2 \rangle + \langle p^2 \rangle$

$$= (\sqrt{b}\,\sigma_0)^2 + \left(\frac{1}{\sqrt{b}}\,\sigma_0\right)^2$$

$$\Rightarrow \quad 2\sigma^2 = \frac{b^2+1}{b}\,\sigma_0^2 \quad \Rightarrow \quad \boxed{\epsilon/\epsilon_0 = \frac{b^2+1}{2b}}$$

Injection "Beta" Mismatch

- Can write a more general result in terms of the "mismatch" invariant:
  - det(ΔJ) = | $\Delta\beta\Delta\gamma$ - $\Delta\alpha^2$ | = *invariant*
- If inject with *"beam"* parameters α , β, γ, whereas the ring has periodic parameters α_0 , β_0, γ_0, then…
- … after filamentation, the final emittance will be given by

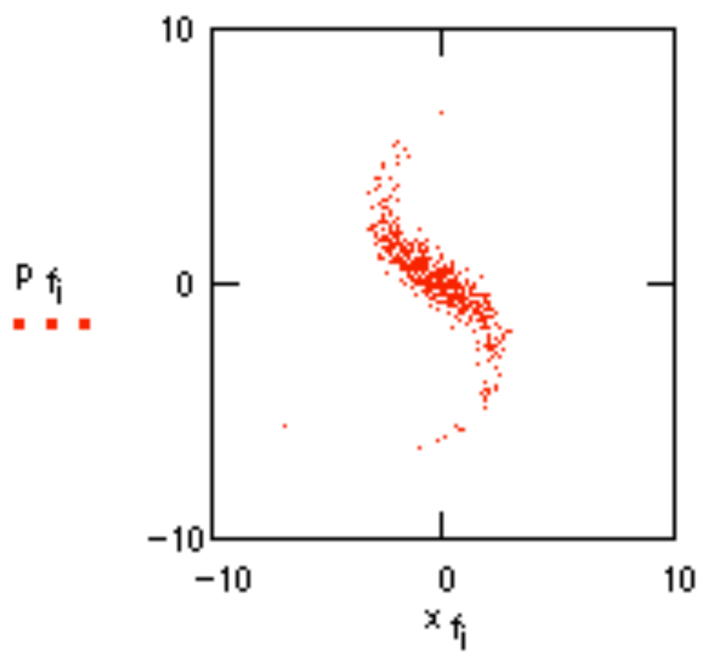
$$\boxed{\epsilon/\epsilon_0 = 1 + \frac{1}{2}|\det \Delta J|}$$

$$\Delta J = \begin{pmatrix} \Delta\alpha & \Delta\beta \\ -\Delta\gamma & -\Delta\alpha \end{pmatrix}$$

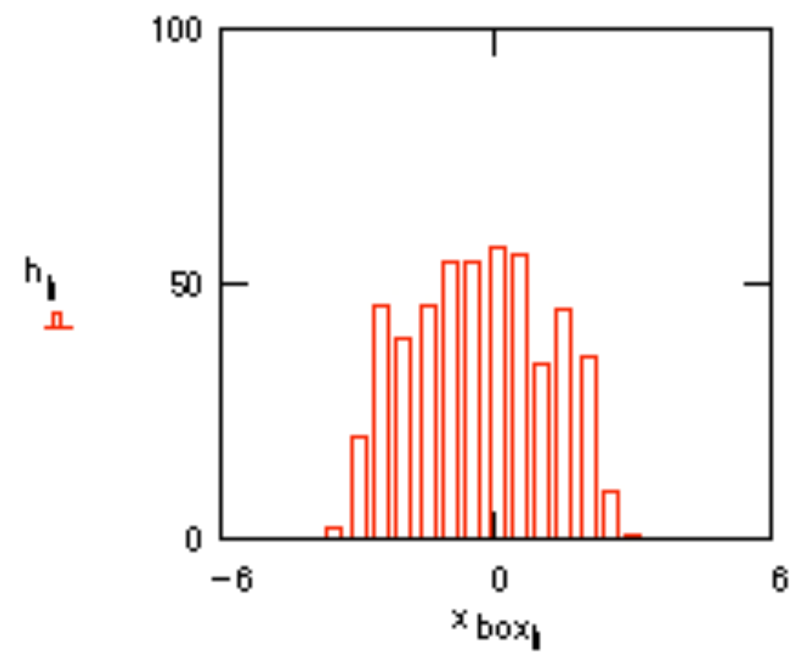
Injection Mismatch

- movie…

Phase Space



x Profile



$\mathrm{mean}(x_f) = -0.12$ $\mathrm{stdev}(x_f) = 1.515$

Emittance Increase: $\mathrm{stdev}(x_f)^2 = 2.295$

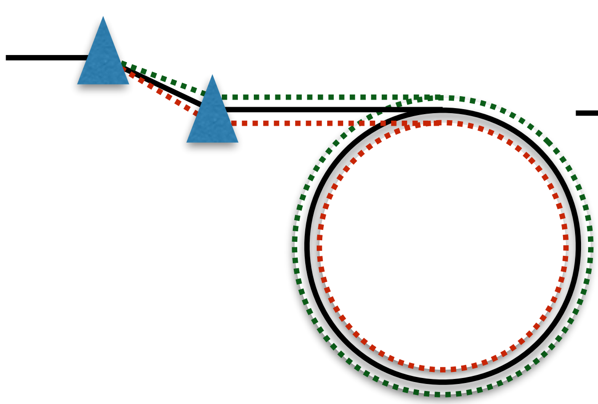
Predicted "typical" values:

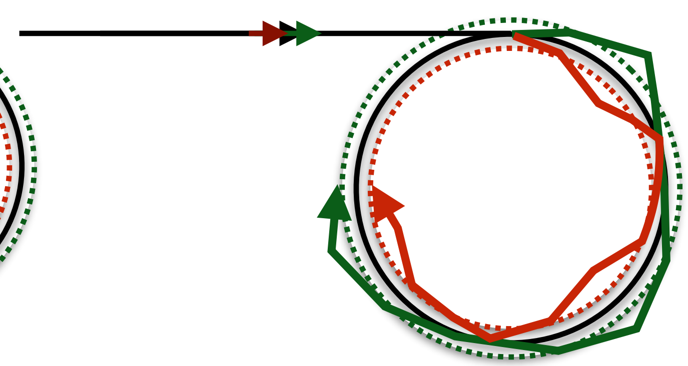
(Steering Mismatch) $1 + \frac{1}{2}\cdot\Delta x^2 = 1$

FRAME = 2 (Amplitude function Mismatch) $1 + \frac{1}{2}\cdot\frac{\delta\beta^2}{1+\delta\beta} = 2.6$

Mismatch of the Dispersion Function

- Can also imagine having the dispersion function entering the accelerator from a beam line having the wrong value
  - amounts to an injection steering error for an off-momentum particle
    - — similar analysis as before



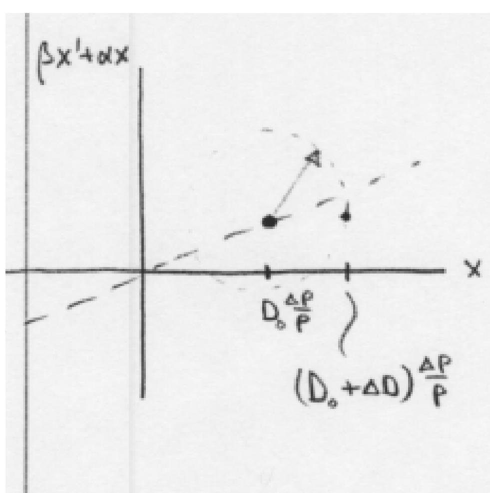


÷ 6 for rms emittance growth

and so,

$$\boxed{\Delta\epsilon_N = \frac{3\pi\,\Delta D^2(s)}{\beta(s)}\left(\frac{\sigma_p}{p}\right)^2(\gamma\beta)}$$

worse for larger σ_p/p (coalesced!)



each particle of momentum $\frac{\Delta p}{p}$ will see an injection steering error of amplitude $x_p = \Delta D \frac{\Delta p}{p}$

$$\Rightarrow \quad a^2 = \rho^2 + \Delta D^2 \left(\frac{\Delta p}{p}\right)^2$$

$$\Rightarrow \quad \langle a^2 \rangle = \langle \rho^2 \rangle + \Delta D^2 \langle (\tfrac{\Delta p}{p})^2 \rangle$$

ex: coalesced beam, w/ $\frac{\sigma_p}{p} = 5\times10^{-4}$, and $\Delta D \approx 1$ m

$$\Rightarrow \quad \Delta\epsilon_N \approx \frac{3\pi\,(1\,\mathrm{m})^2}{100\,\mathrm{m}}(0.5\times10^{-3})^2\,(160) = 1.2\ \pi\ \text{mm-mrad}$$

important if the incoming beam has a high momentum spread

Emittance Growth from Diffusive Processes

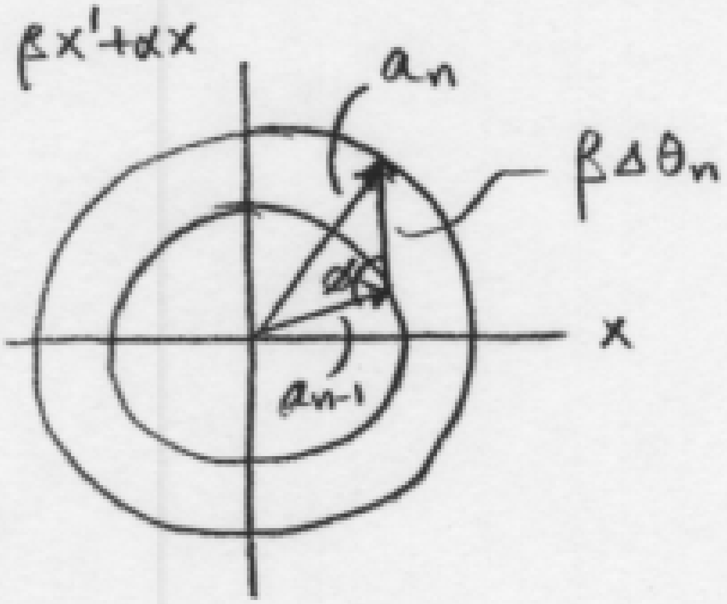
- So far have looked at single, non-adiabatic disturbances of our initial particle distribution
- Next, we look at the effect of repetitive random disturbances of individual particles, leading to diffusion
  - » examples: scattering of particles off of the residual gas in the vacuum chamber; power supply noise; RF noise; continuous mechanical vibrations, …
- This amounts to continuous, random events taking place to alter the transverse amplitudes of the motion of individual particles

Repetitive Random Disturbances

- Estimations from a phase space perspective…

Next, consider diffusive mechanisms…

each particle's amplitude is changed "continuously" by a random process…



$$a_1^2 = a_0^2 + \beta^2 \Delta\theta_1^2 - 2 a_0 \beta \theta_1 \cos\phi_1$$

$$a_n^2 = a_{n-1}^2 + \beta^2 \Delta\theta_n^2 - 2 a_{n-1} \beta \theta_n \cos\phi_n$$

average over all particles…

$$\langle a^2 \rangle_n = \langle a^2 \rangle_{n-1} + \beta^2 \langle \Delta\theta^2 \rangle_n$$

$$\cancel{- 2\beta \langle a \rangle \langle \theta \rangle_n \langle \cos\phi \rangle_n} \to 0$$

$$\therefore\ d\langle a^2 \rangle = \beta^2 \theta_{rms}^2 \, dn$$

Repetitive Random Disturbances

- Look again at "vacuum" problem examined earlier
- Suppose circulating in a synchrotron w/ P=10^{-6} torr:
  - suppose $<\beta>$ = 20 m around the circumference, and that E = 13.6 GeV
  - $d<a^2> = 2\, d<x^2> = 2 <\beta>\, d(\epsilon/\pi) = <\beta>^2\, \theta_{rms}^2\, dn$
  - $d\epsilon/dt = \pi/2 <\beta>\, \theta_{rms}^2\, f_0 = \pi/2 <\beta>\, (0.0136/E)\, v/L_{rad}$
  - $= \pi/2\, (20\ \text{m})\, (10^{-3})\, (3\text{x}10^8/2.3\text{x}10^{11})$
  - $= \pi\, (13\text{x}10^{-6}\ \text{m/s}) = 13\, \pi$ mm-mr/s !
  - so, might need much better vacuum here!

Repetitive Random Disturbances

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- So, we see that in repetitive systems such random scattering events and other similar disturbances can cause emittance growth over time
- Wish to analyze such conditions
  - analytical approaches
  - simulations

The Diffusion Equation

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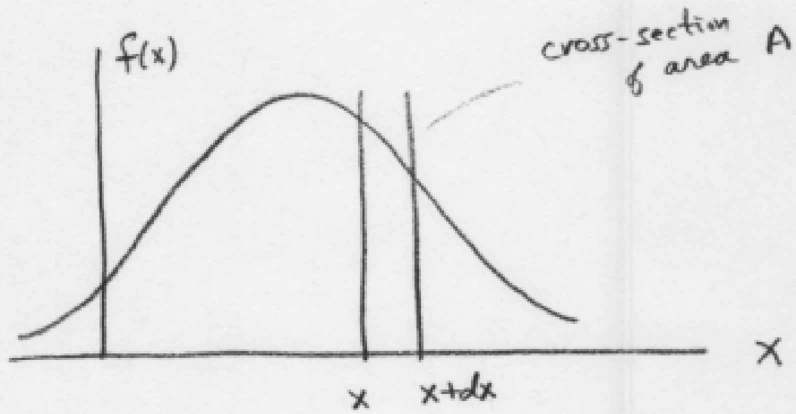
Diffusion Equation

Look @ 1-D first:



$$\frac{\partial}{\partial t}(f \cdot A \cdot dx) = \underbrace{A \cdot J(x)}_{\#\text{ into}} - \underbrace{A \cdot J(x+dx)}_{\#\text{ out of}}$$

J = average # particles per unit time passing position x

$$\Rightarrow \quad \frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x}$$

if f uniform, then $J = 0$; otherwise $J \propto -\frac{\partial f}{\partial x}$

$$\therefore \quad \boxed{\frac{\partial f}{\partial t} = C \cdot \frac{\partial^2 f}{\partial x^2}} \qquad C = \text{constant}$$

more general 3-D case : $\boxed{\frac{\partial f}{\partial t} = C \, \nabla^2 f}$

- particle velocities are randomly altered
- particles will move from one region into another
- the rate at which particles cross into or out of a region depends on the *slope* of the distribution function

can solve analytically

The Diffusion Equation

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- think of betatron motion in terms of coordinates:
  - *x, αx+βx'*   *(circular)*
- use cylindrical coordinates for the Diffusion Equation
- re-cast in terms of an emittance ~ r^2/β
- with appropriate scaling, can write a dimensionless equation for the distribution function. Emittance is now scaled by the aperture acceptance
- apply boundary conditions

$$\frac{\partial f}{\partial t} = C \cdot \nabla^2 f$$

$$\epsilon_p \equiv r^2/\beta$$

$$\frac{\partial f}{\partial t} = C \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right)$$

$$R \equiv \frac{\partial}{\partial t}\langle r^2/\beta \rangle$$

$$\frac{\partial f}{\partial t} = R \frac{\partial}{\partial \epsilon_p}\left(\epsilon_p \frac{\partial f}{\partial \epsilon_p}\right)$$

$$W \equiv a^2/\beta \ \sim \ \text{“acceptance”}$$

$$\tau \equiv (R/W)t$$

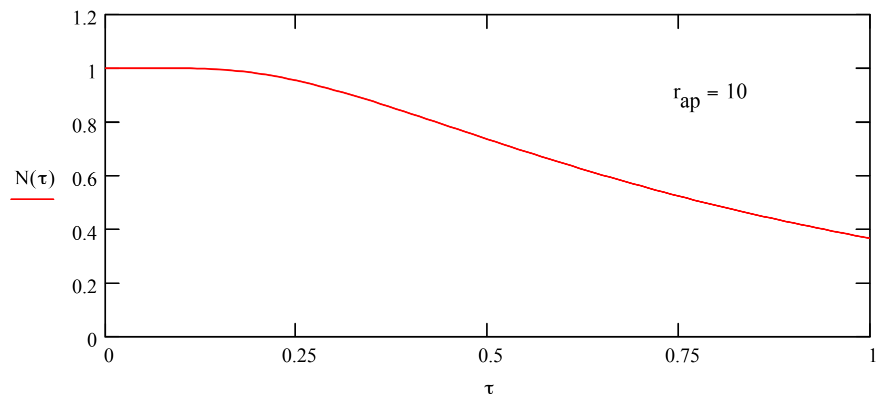
$$Z \equiv \epsilon_p/W$$

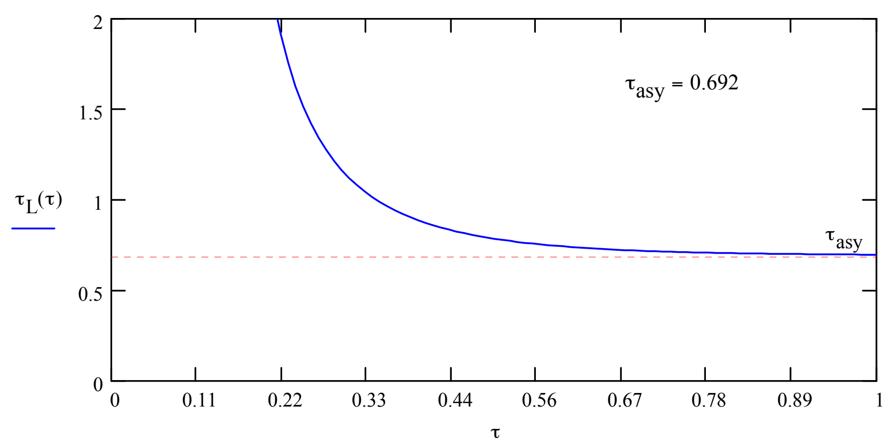
$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial Z}\left(\frac{\partial f}{\partial Z}\right)$$

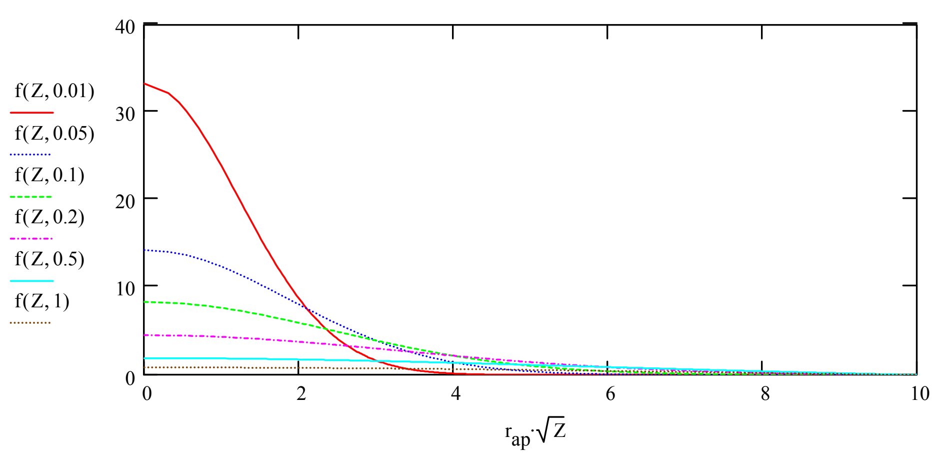
$$f(Z,0) = f_0(Z)$$
$$f(1,\tau) = 0$$

The Diffusion Equation

- Analytical Calculations:
  - solve and make plots:







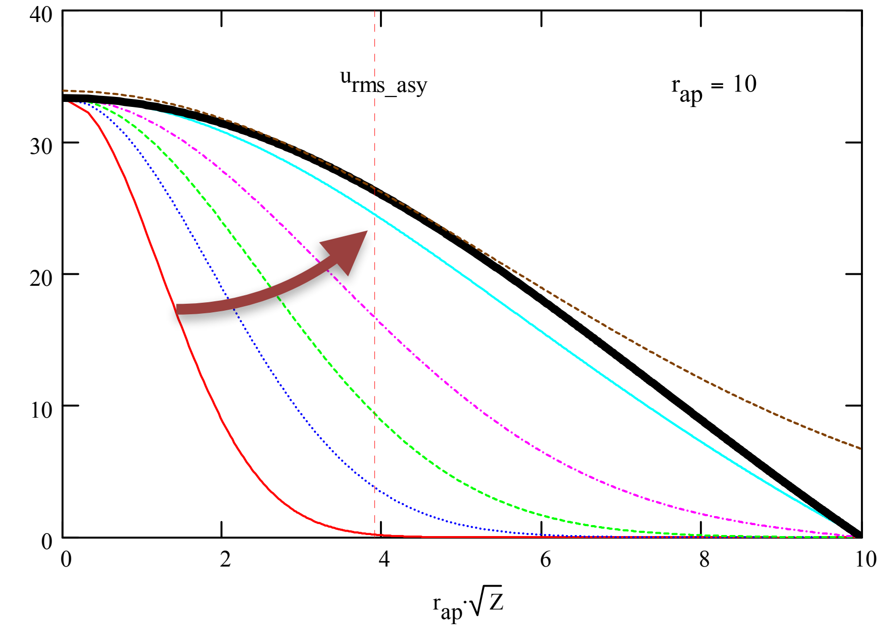
- Numerical Simulations
  - give particles random kicks over time, track in phase space, and plot distribution, etc.

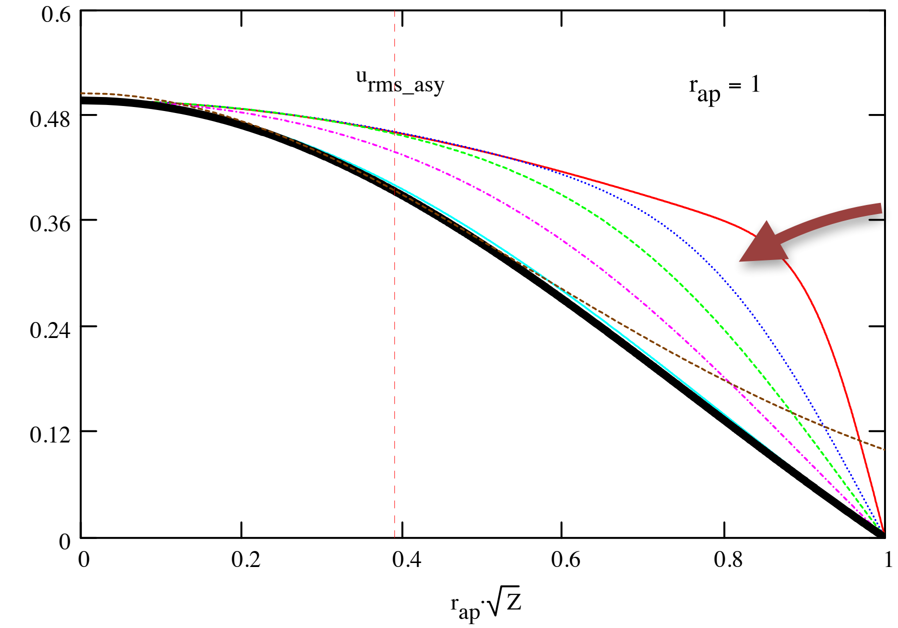
$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = M_{2\pi\nu} \begin{pmatrix} x_n \\ x'_n + \Delta\theta_n \end{pmatrix}$$

random number each time, determined by the process

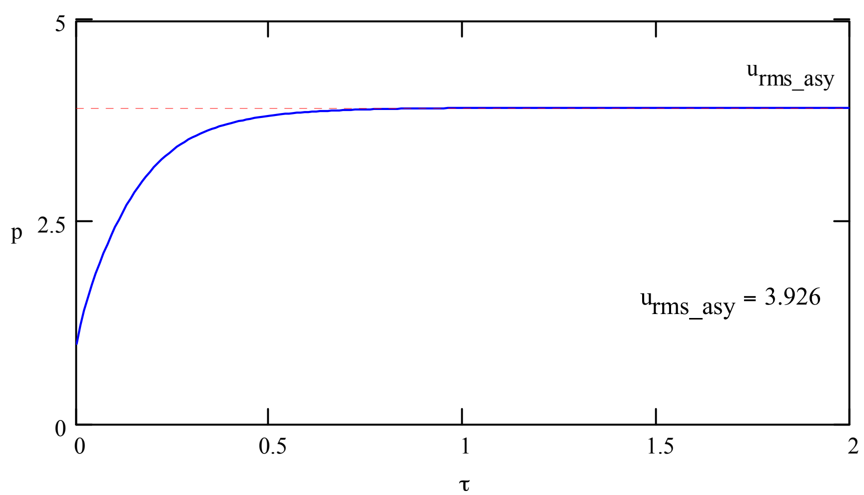
Analytical Solutions

ap = 10 σ_0 | ap = 1 σ_0

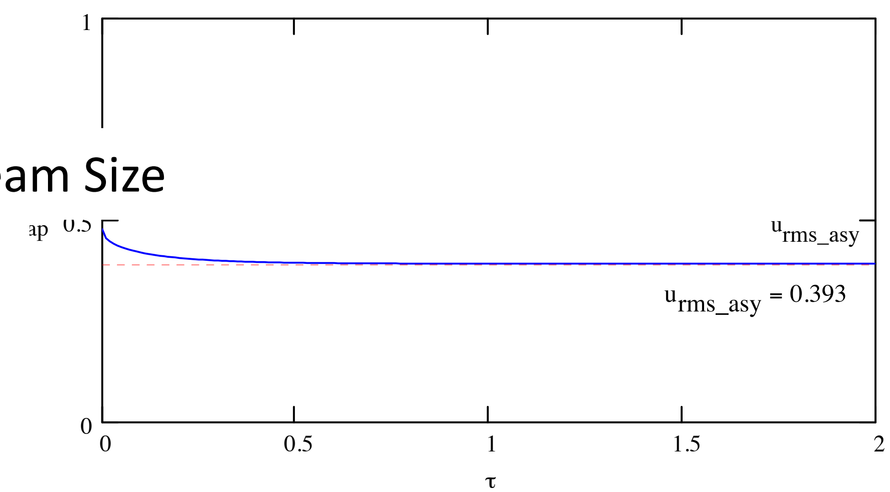




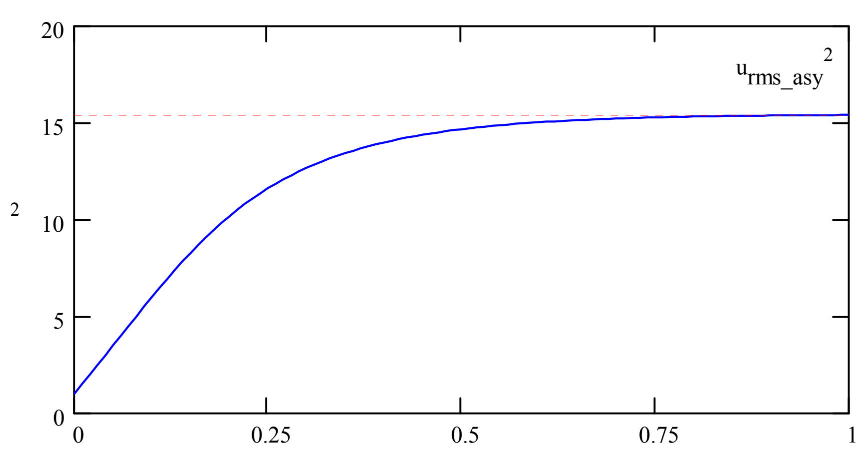
Beam Size



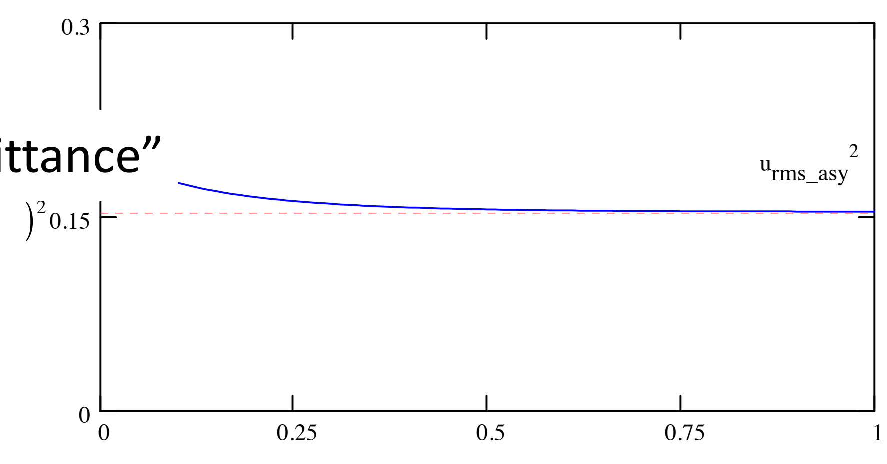
Beam Size



“Emittance”



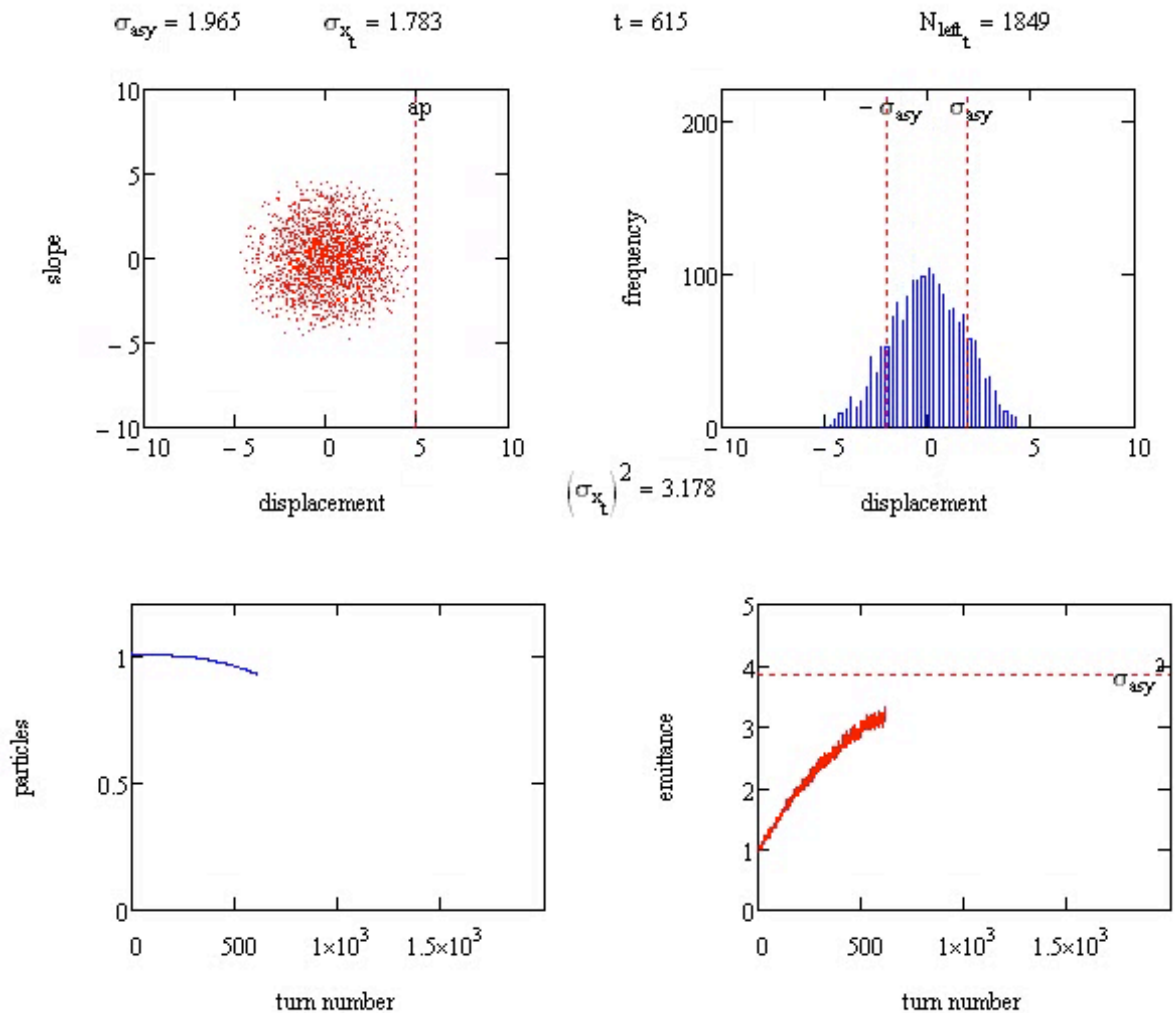
“Emittance”



Transverse Diffusion — Scattering

- movie

each particle gets a random "kick" in x' each turn, taken from a Gaussian distribution with rms value of θ_{rms}



Transverse Diffusion

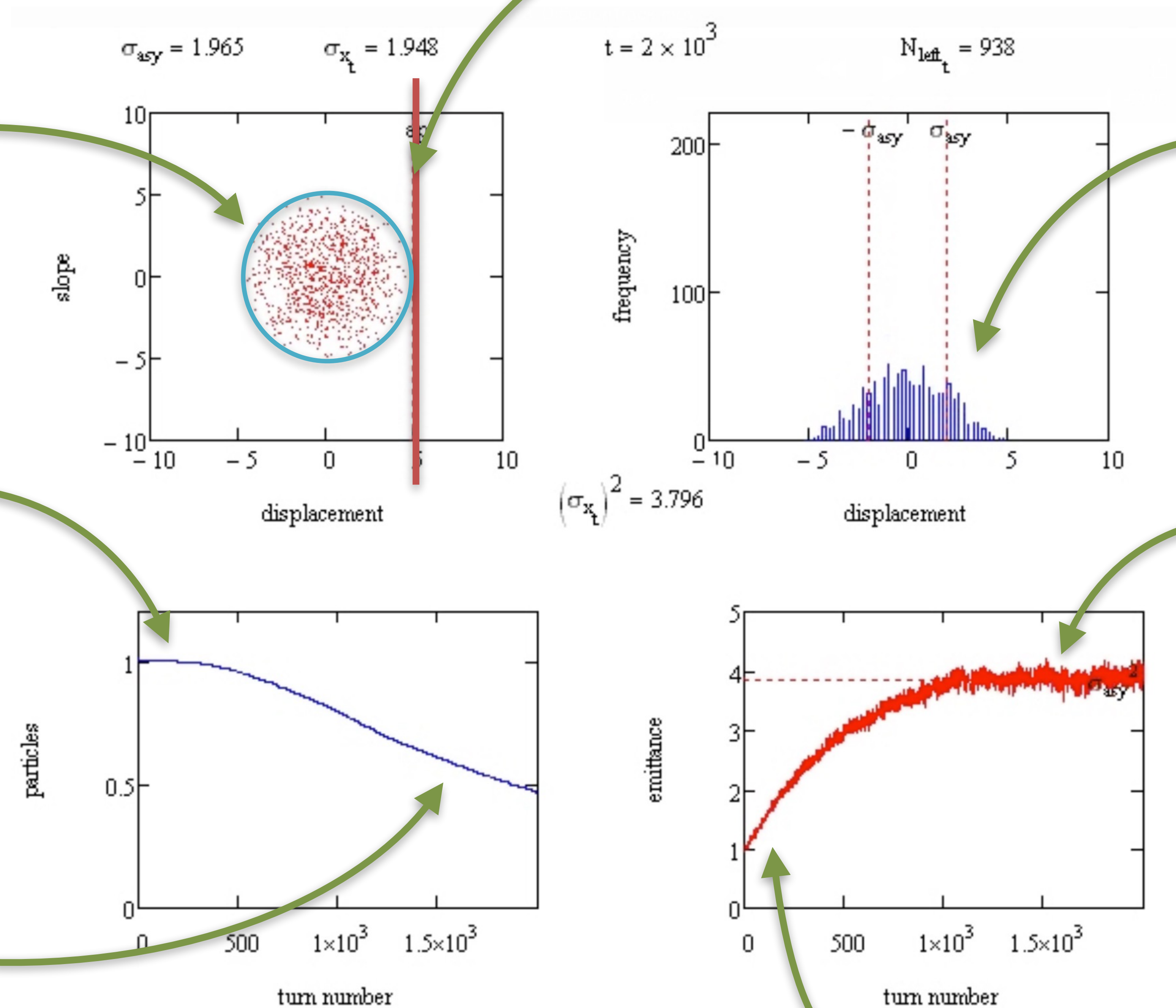
- movie

aperture at x = a

$W = \pi a^2/\beta$

final equilibrium distribution is well-defined

beam has not reached the aperture yet



no *apparent* emittance growth, but particle amplitudes are indeed growing, and particles are being lost

final $N(t) \sim e^{-t/\tau}$

$R = d\epsilon/dt$ = constant *(given by mechanism)*

$\tau = (4/2.405^2)(W/R)$

Some Comments

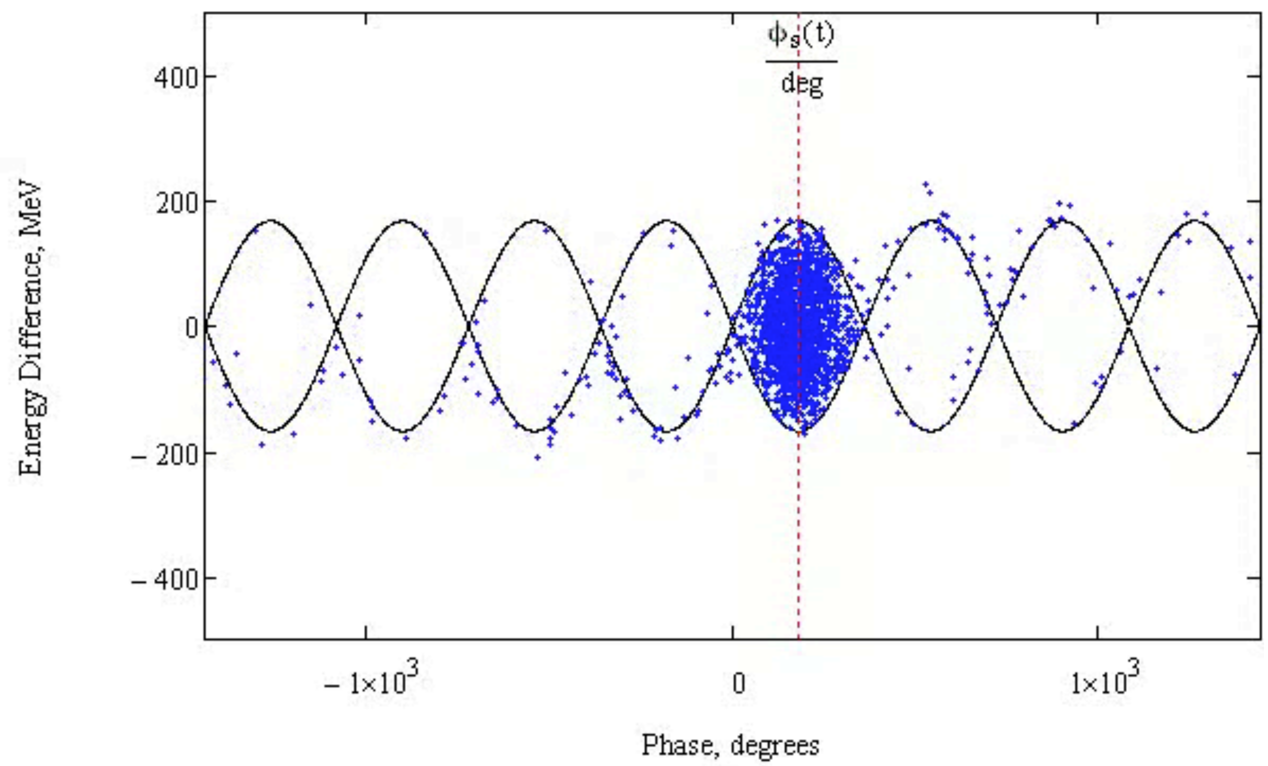
- The emittance may be growing, but the intensity will not decrease until the beam reaches an aperture
- The beam size may stop growing, but that does not mean that the individual particle amplitudes are no longer growing — just that the aperture was reached
- The long-term exponential decay of the beam intensity can tell you what the emittance growth rate is, if you know the transverse acceptance
- A beam with an initially more uniform distribution can actually have its “rms” value decrease until equilibrium is reached — it is NOT being “cooled”

Comments on “Beam Cooling”

- Stochastic Beam Cooling
- Electron Cooling of Hadron Beams
- Ionization Cooling

Example:Longitudinal Diffusion due to RF Noise

- movie



Here, a random phase error is given to each particle every turn