Northern Illinois University

Acceleration and Longitudinal Dynamics

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Operator Lectures

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Longitudinal Focusing

- sometimes referred to as “phase focusing” or “time focusing”
- particles of different energy (momentum) move at different speeds, so tend to “spread out” relative to the “ideal” particle which is assumed to exist traveling with perfect synchronism with respect to the oscillating fields
- wish to study the (longitudinal) motion of particles relative to this “synchronous particle”

Longitudinal Focusing

- time of flight — the "slip factor"

- Evolution due to *dp/p* or *dW/W*

- Longitudinal focusing, time of arrival:
  - bunchers, rebunchers, debunchers

The Slip Factor

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$$t = \frac{L}{v}$$

$$\frac{dt}{t} = \frac{dL}{L} - \frac{dv}{v}$$

$$\frac{dv}{v} = \frac{1}{\gamma^2}\frac{dp}{p}$$

$$\frac{dt}{t} = \left(\alpha_p - \frac{1}{\gamma^2}\right)\frac{dp}{p}$$

$$\frac{dt}{t} = \eta\,\frac{dp}{p}$$

Momentum Compaction Factor:

$$\alpha_p \equiv \left(\frac{dL/L}{dp/p}\right)$$

The Slip Factor: $$\eta \equiv \alpha_p - \frac{1}{\gamma^2}$$

For a straight section, or a linac, $\eta = -1/\gamma^2 < 0$

For a region with bending, α_p might not be zero

A Simple Example

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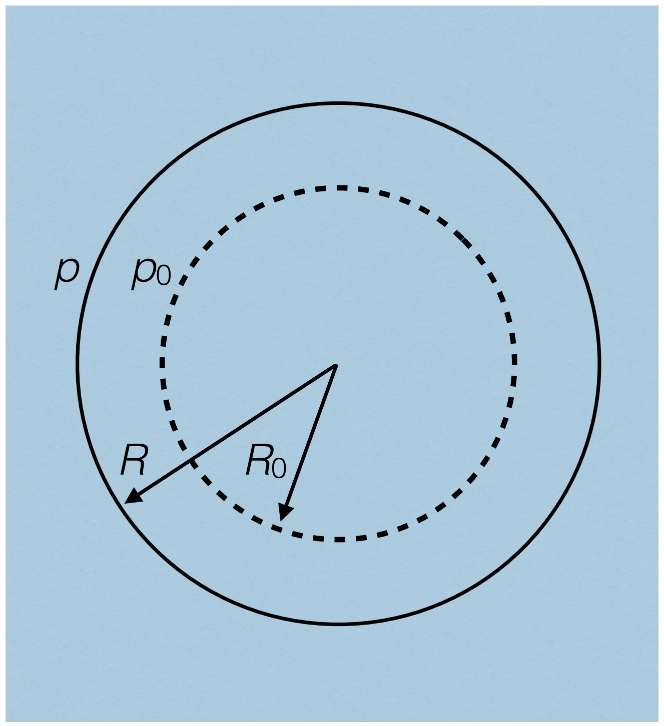
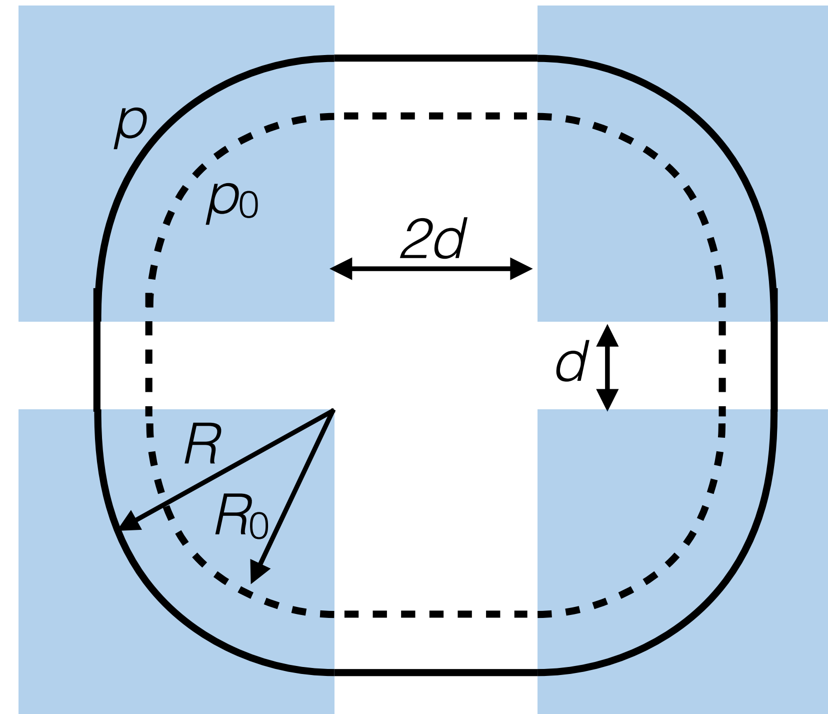


$$R/R_0 = p/p_0$$

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p}$$



$$\tau = 2\pi R/v$$

$$\tau = (2\pi R + 6d)/v$$

$$\frac{\Delta L}{L_0} = \frac{\Delta R}{R_0} = \frac{\Delta p}{p_0}$$

$$\frac{\Delta L}{L} = \frac{2\pi(R - R_0)}{2\pi R_0 + 6d} = \frac{1}{1 + 3d/\pi R_0}\frac{\Delta p}{p_0}$$

$$\frac{\Delta \tau}{\tau_0} = (1 - \frac{1}{\gamma_0^2})\frac{\Delta p}{p_0}$$

$$\frac{\Delta \tau}{\tau_0} = \left(\frac{1}{1 + 3d/\pi R_0} - \frac{1}{\gamma_0^2}\right)\frac{\Delta p}{p_0}$$

Implications of the Slip Factor

- Suppose no bending in the line (e.g., linac), or, perhaps have bending yet $\gamma^2 < 1/\alpha_p$
  - then, the slip factor is negative, and particles of higher momentum take less time to traverse the same distance as the ideal particle

$$\eta = \alpha_p - \frac{1}{\gamma^2}$$

- If the energy of the particles is high enough in the presence of bending, then can have $\gamma^2 > 1/\alpha_p$
  - in this case, the slip factor is positive — the changes in path length outweigh the changes in speed when determining the time of flight difference
  - here, a higher-momentum particle will actually take ***longer*** to traverse the same distance as the ideal particle, even though it's moving faster

Linear Motion Very Near the Ideal Particle

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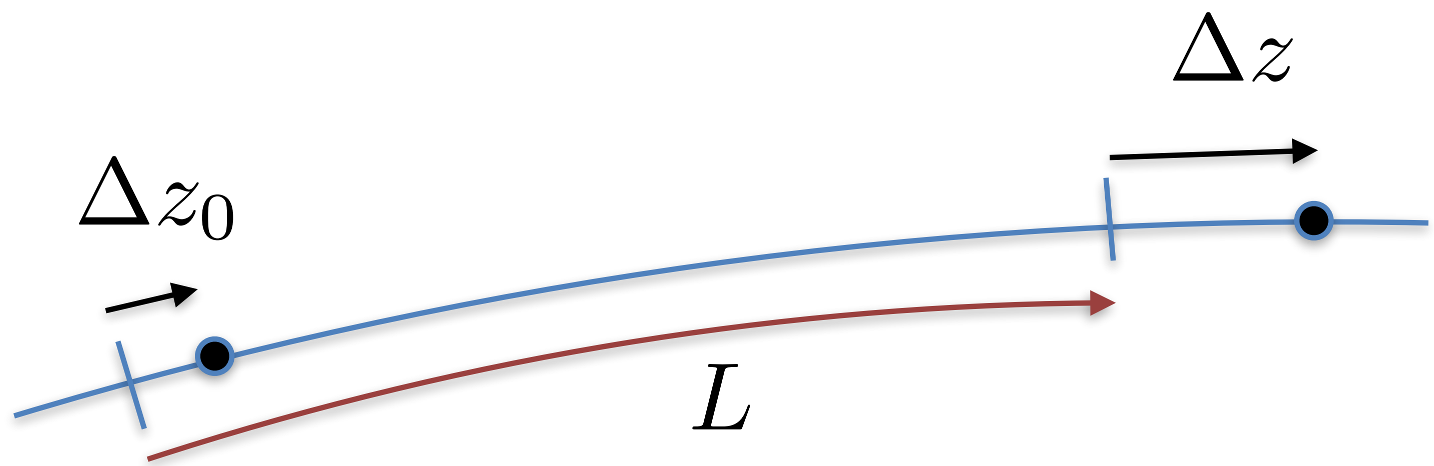
- Particles moving along the ideal trajectory move toward or away from the ideal particle according to their speed (momentum/energy) and path length differences

Δt = arrival time relative to the ideal arrival time ($\Delta t = 0$)

$$\Delta z = -\beta c \, \Delta t$$



$$\tau = L/v = L/(\beta c)$$

$$\Delta z = \Delta z_0 - \eta L \frac{\Delta p}{p}$$

$$\Delta t = \Delta t_0 + \eta \frac{L}{\beta c} \frac{\Delta p}{p}$$

$$\text{Note}: \quad \frac{\Delta p}{p} = \frac{1}{\beta^2} \frac{\Delta E}{E} = \frac{1}{\beta^2} \frac{\gamma - 1}{\gamma} \frac{\Delta W}{W}$$

Linear Motion Very Near the Ideal Particle [2]

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- Imagine a particle on the ideal trajectory and that has the ideal energy, W_s. A second particle on the ideal trajectory, but with a different energy, W, may be ahead of or lagging behind the ideal particle.
- We will use **radio frequency (RF) cavities** to provide an accelerating voltage to the particles as they pass by.
- The ideal particle will arrive at the cavity at the "ideal" time or, equivalently, at an ideal phase, ϕ_s, to receive an appropriate increase in its energy (which might be an increase of "0").
- We will keep track of the "difference" in energy between our test particle and the ideal particle:

$$W_s = \text{``ideal'' energy}$$

$$\Delta W \equiv W - W_s$$

Acceleration using AC Fields

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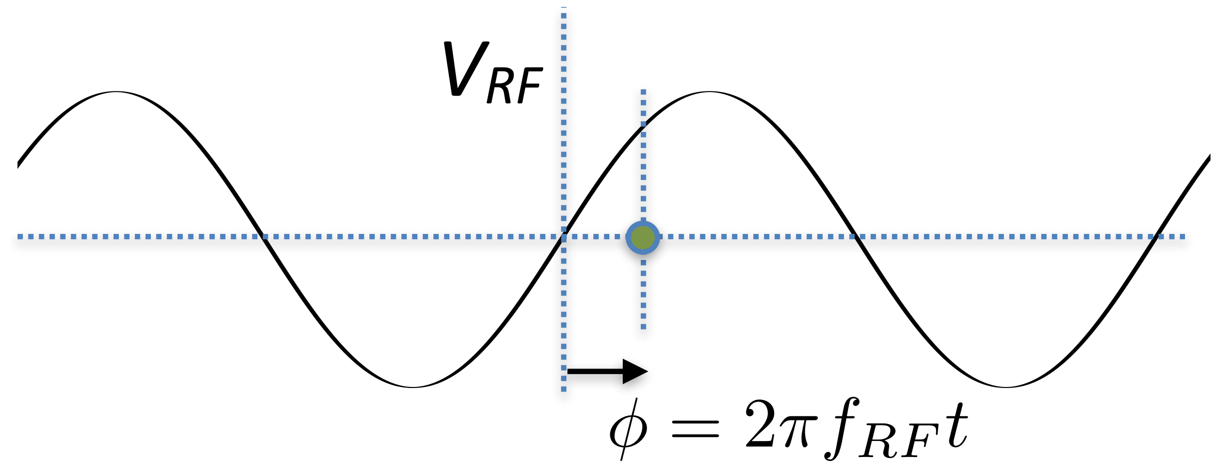
- Pass through a gap with an oscillating field, particle gains energy …

$$W = W_0 + q\,E\,g\,\sin(\phi)$$
$$= W_0 + qV\sin(\phi)$$



- Here, ϕ is the "phase" of the oscillating field at the time of arrival

- But here, V is an "average" or "effective" potential; depends upon the frequency of the field in the gap, the incoming speed of the particle (due to the field varying with time), and the phase of the oscillation relative to the particle arrival time:

$$W = W_0 + q\,T(\beta)\,V_0\sin(\phi)$$

- For our purposes today, we will lump the transit time factor, *T*, and the peak voltage, V_0, into a single "effective voltage", *V*

Linear Motion Very Near the Ideal Particle

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- Let the ideal particle receive energy gain according to:

$$W_s = W_{s,0} + qV \sin(\phi_S) \qquad W_s = \text{“ideal” energy}$$

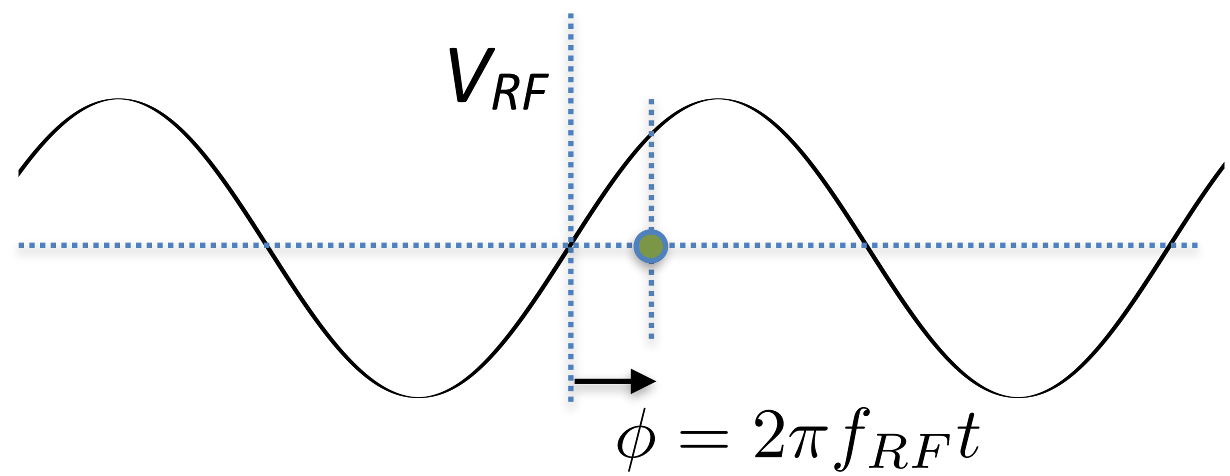
- As nearby particles pass through the cavity, will give particles that are ahead/behind a decrease/increase in energy

a particle's energy difference from the ideal: $\Delta W \equiv W - W_s$

$$\Delta W = \Delta W_0 + qV(\sin\phi - \sin\phi_s)$$

$$= \Delta W_0 + qV[\sin(\phi_s + \Delta\phi) - \sin\phi_s]$$

$$\Delta W \approx \Delta W_0 + qV \cos\phi_s \Delta\phi = \Delta W_0 + qV \cos\phi_s (2\pi f_{RF})\Delta t_0$$



- Can use matrix techniques to propagate the longitudinal motion

Linear Motion through Cavities and Drifts

- Keep track of time differences and energy differences…

drift:

$$\begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix} = \begin{pmatrix} 1 & \eta \frac{L}{c} \frac{1}{\beta^3 \gamma} \frac{1}{mc^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix}_0$$

through cavity:
longitudinal focusing

$$\begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (2\pi f_{RF}) qV \cos\phi_s & 1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta W \end{pmatrix}_0$$

remember —

$$\eta \equiv \alpha_p - \frac{1}{\gamma^2}$$

Note: this "linearization" valid when $\sin(2\pi f_{RF}\,\Delta t) \approx 2\pi f_{RF}\,\Delta t$

Linear Motion through Cavities and Drifts

- So, with this in mind, can create a system to transport beams with large momentum spread that keeps the particles "together" in time along the path

$$M = \ldots \;\; M_{drift} M_{cavity} M_{drift} M_{cavity} M_{drift}$$

focus

Most important for low-energy beams, such as high-charge-state ion beams

Bunchers, Re-bunchers, Debunchers

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- If start with continuous stream of particles (DC current, with no strong "AC" component), can create bunches (AC beam) using a single cavity (buncher)

- If already have bunched beam that is allowed to travel a certain distance, the particles within the bunch will begin to spread out due to the inherent spread in momentum
  - re-buncher: mitigate this effect (last slide)
  - debuncher: enhance this effect
    - for example, to spread beam out when injected into a storage ring or synchrotron

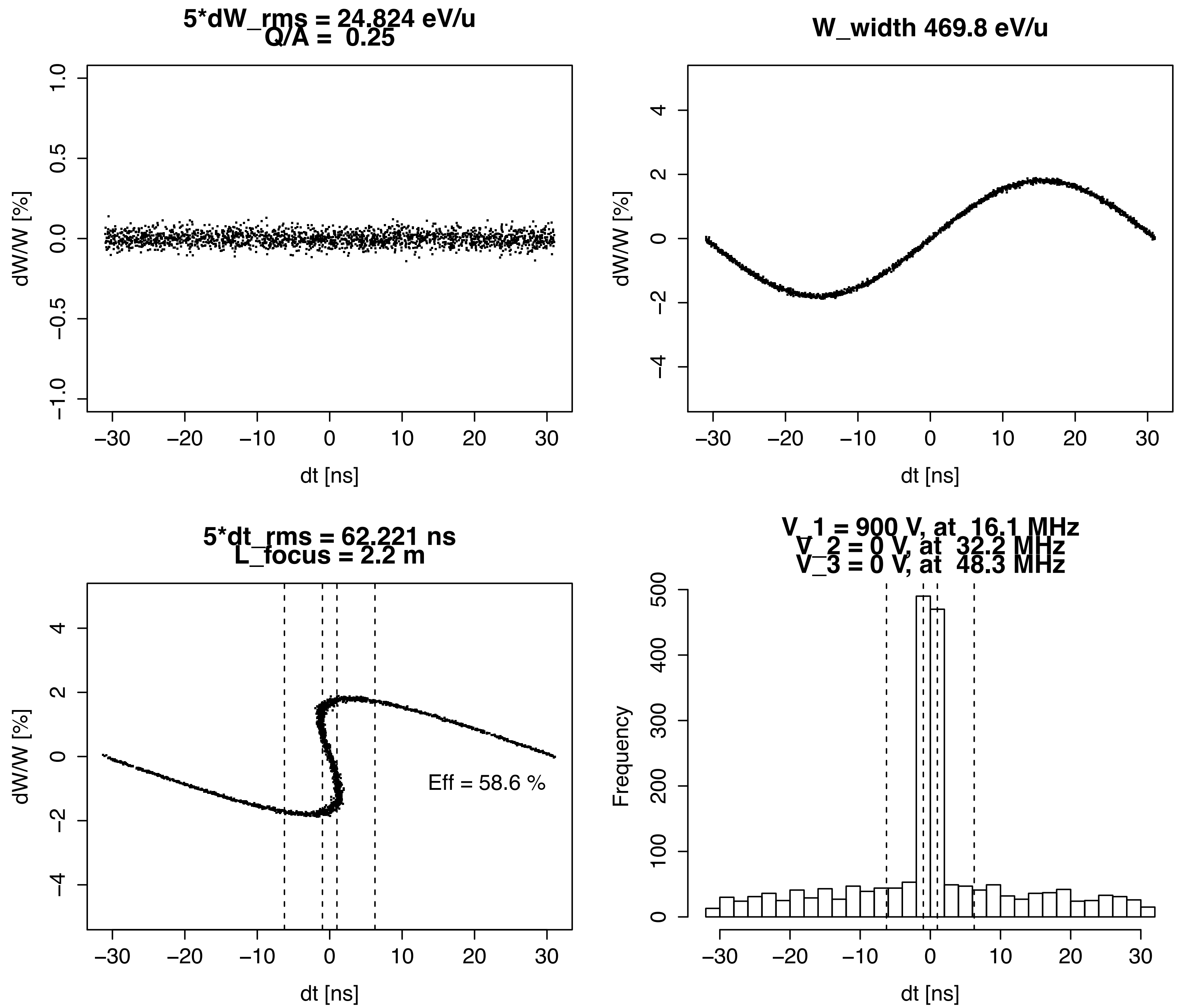
Beam Buncher

incoming DC beam

after the cavity

downstream of cavity

resulting time profile



Beam Buncher

incoming DC beam

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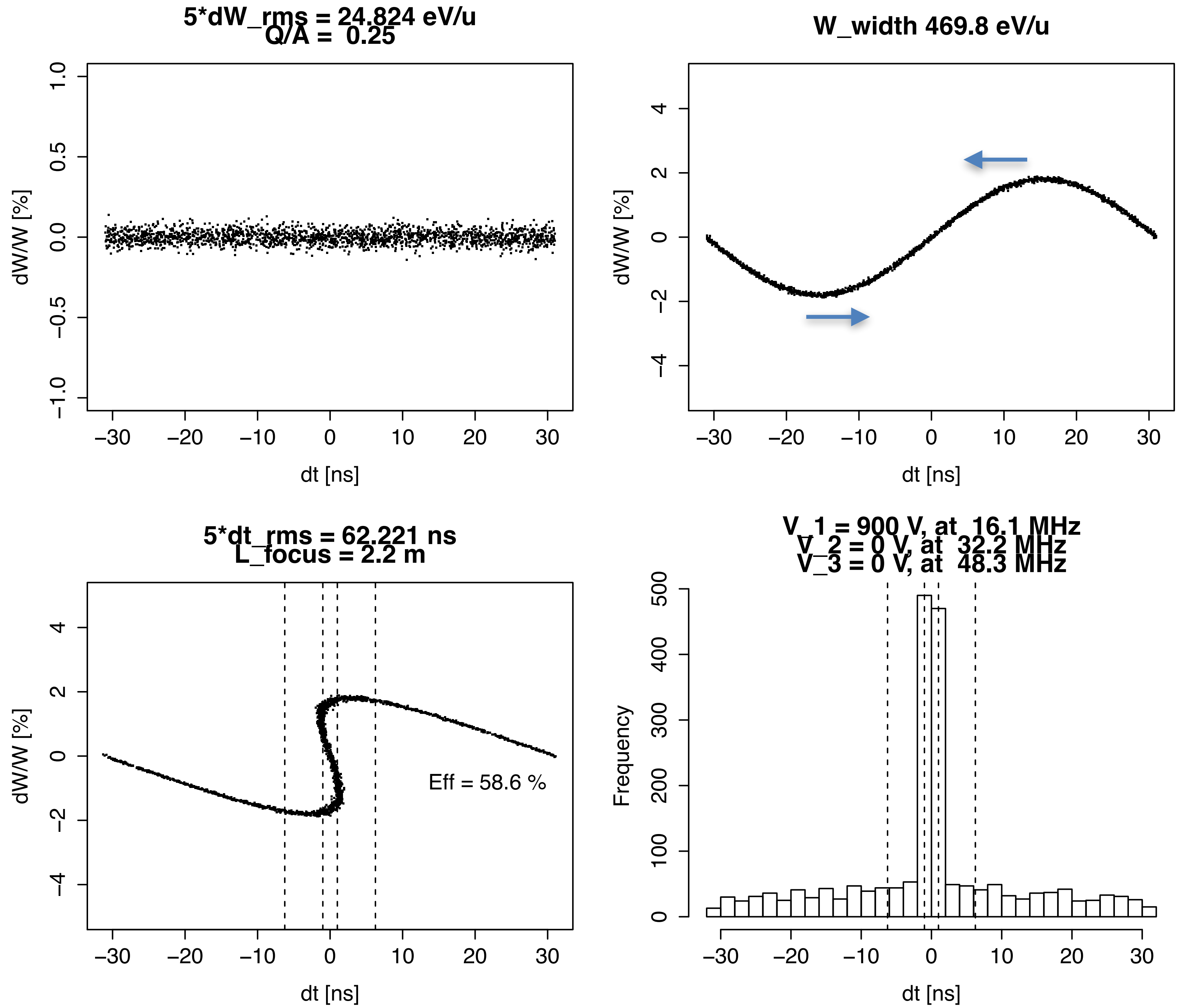
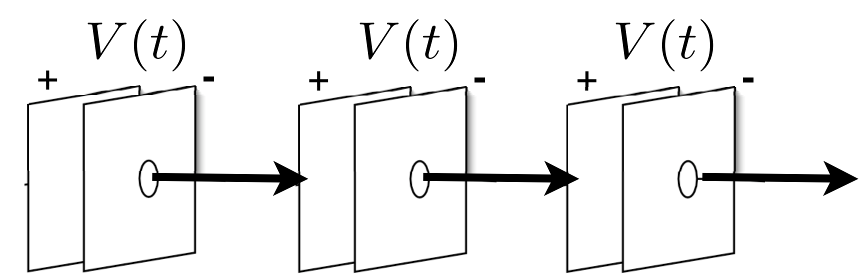


Linacs and Synchrotrons

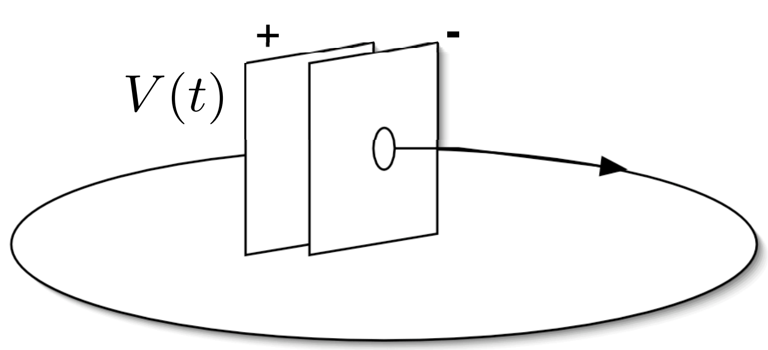
- Essential difference:



Linear Accelerator

- pass N cavities 1 time each

— or —



Circular Accelerator

- pass 1 cavity N times

- otherwise, essentially the same longitudinal dynamics

Acceleration of Ideal Particle

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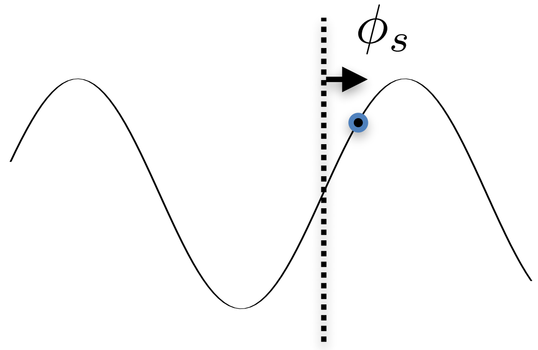
Wish to accelerate the ideal particle. As this particle exits the *(n+1)-th* RF cavity/station we would have



$$E_s^{(n+1)} = E_s^{(n)} + QeV \sin\phi_s$$

If we are considering a synchrotron, we can consider the above as the total energy gain on the *(n+1)-th* revolution. The ideal energy gain per second would be:

$$dE_s/dt = f_0 QeV \sin\phi_s$$

$f_0 = \text{revolution frequency}$

Next, look at (longitudinal) motion of particles near the ideal particle: ϕ = phase w.r.t. RF system

$\Delta E \equiv E - E_s$ = energy difference from the ideal

Difference Equations of Motion

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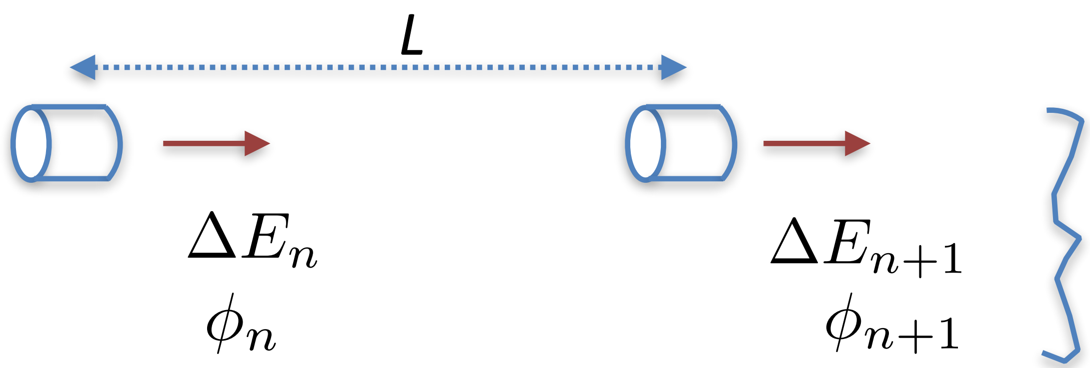
- Assume accelerating system of cavities is set up such that ideal particle arrives at each cavity when the accelerating voltage V is at the same phase (called the "synchronous phase"); consider at "test" particle:



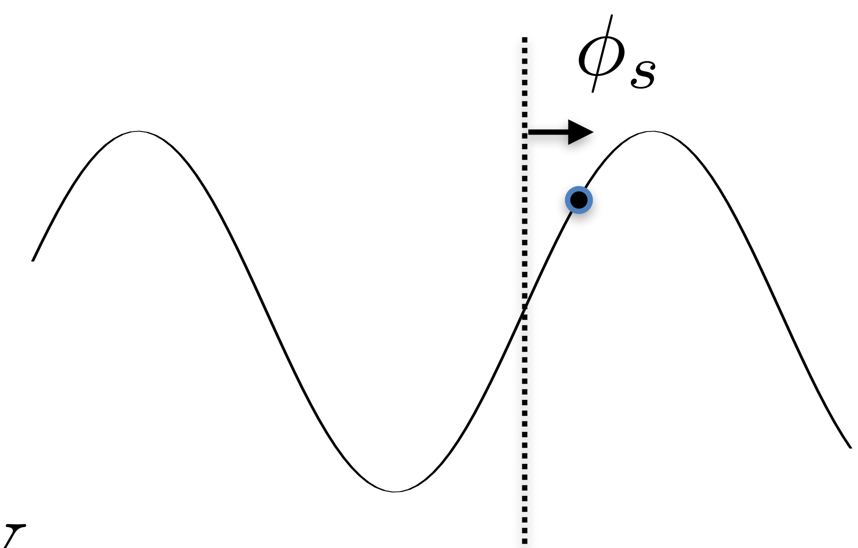
$$\phi_{n+1} = \phi_n + \frac{2\pi h \eta}{\beta^2 E} \Delta E_n$$

$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

(*difference equations*)

Notes:

$$h = L/\beta\lambda, \quad \lambda = c/f_{\rm rf} \quad \text{or,} \quad h = f_{\rm rf} L/v$$



Desire *h* to be an integer.
If *L* is circumference of a synchrotron then: $h = f_{\rm rf}/f_0$
where f_0 is the revolution frequency,
In this case, *h* is called the "harmonic number"

$$E = mc^2 + W; \qquad \Delta E \Leftrightarrow \Delta W$$

Applying the Difference Equations

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```
while (i < Nturns+1) {
  phi = phi + k*dW
  dW  = dW + QonA*eV*(sin(phi)-sin(phis))
  points(phi*360/2/pi, dW, pch=21,col="red")
  i = i + 1
 }
```

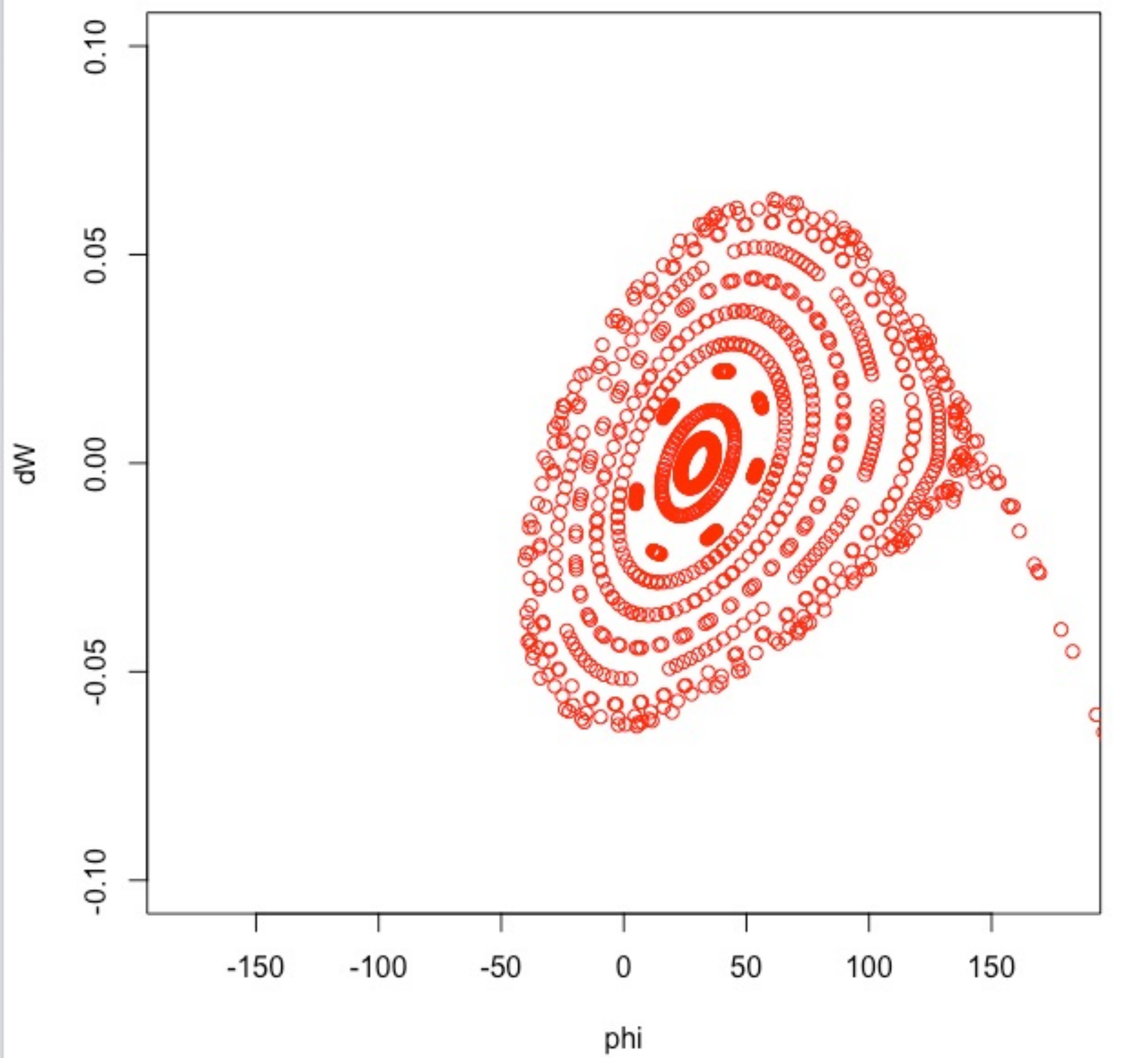
Let’s run a code…

v0_RFtrack.R

```r
# Program to plot longitudinal phase space motion
#  through a system of cavities (just an example...)

Nturns = 100

# Some Parameters
Ws = 1.0 # MeV/u
phis = 30*pi/180  # synchronous phase angle
eV = 0.2 # MeV/u
QonA = 0.25
gamma = (931+Ws)/931
beta = sqrt(1-1/gamma^2)
eta = -1/gamma^2
h = 1/(beta*3e8/80.5e6)
k = 2*pi*h*eta/beta^2*(gamma-1)/gamma/Ws

# initialize the phase space plot
phi   = 0
dW    = 0
plot(phi, dW, xlim=c(-180,180), ylim=c(-0.1,0.1), typ="n")

trk = 1
while (trk < 16) {
# initialize particle positions in phase space
  u0 <- locator(1)
  phi   <- u0$x/180*pi
  dW    <- u0$y
# track the particle...
  i = 1
  while (i < Nturns+1) {
    phi = phi + k*dW
    dW  = dW + QonA*eV*(sin(phi)-sin(phis))
    points(phi*360/2/pi, dW, pch=21,col="red")
    i = i + 1
  }
trk = trk + 1
}
```

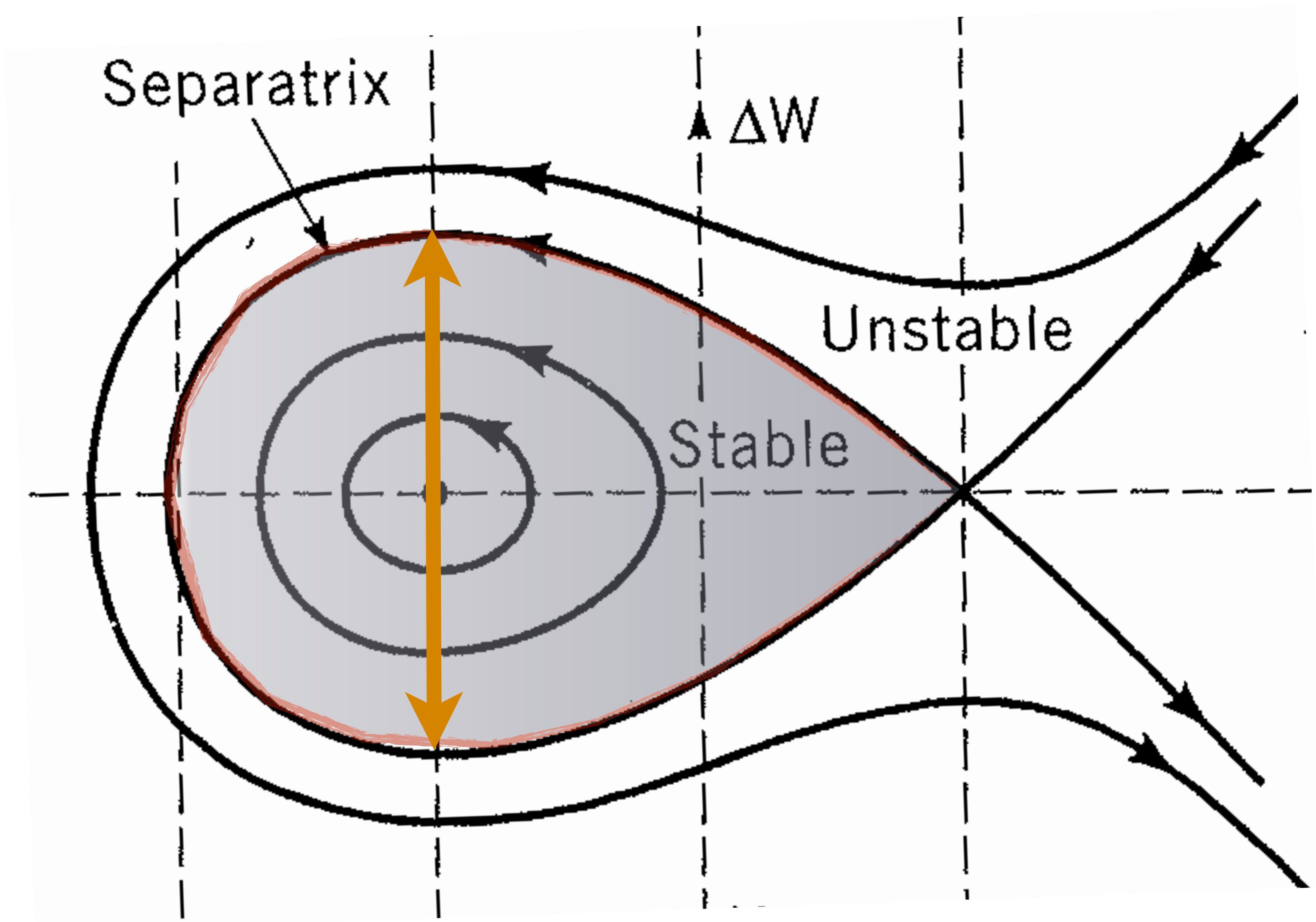
38:1 (Top Level) R Script

Files Plots Packages Help Viewer



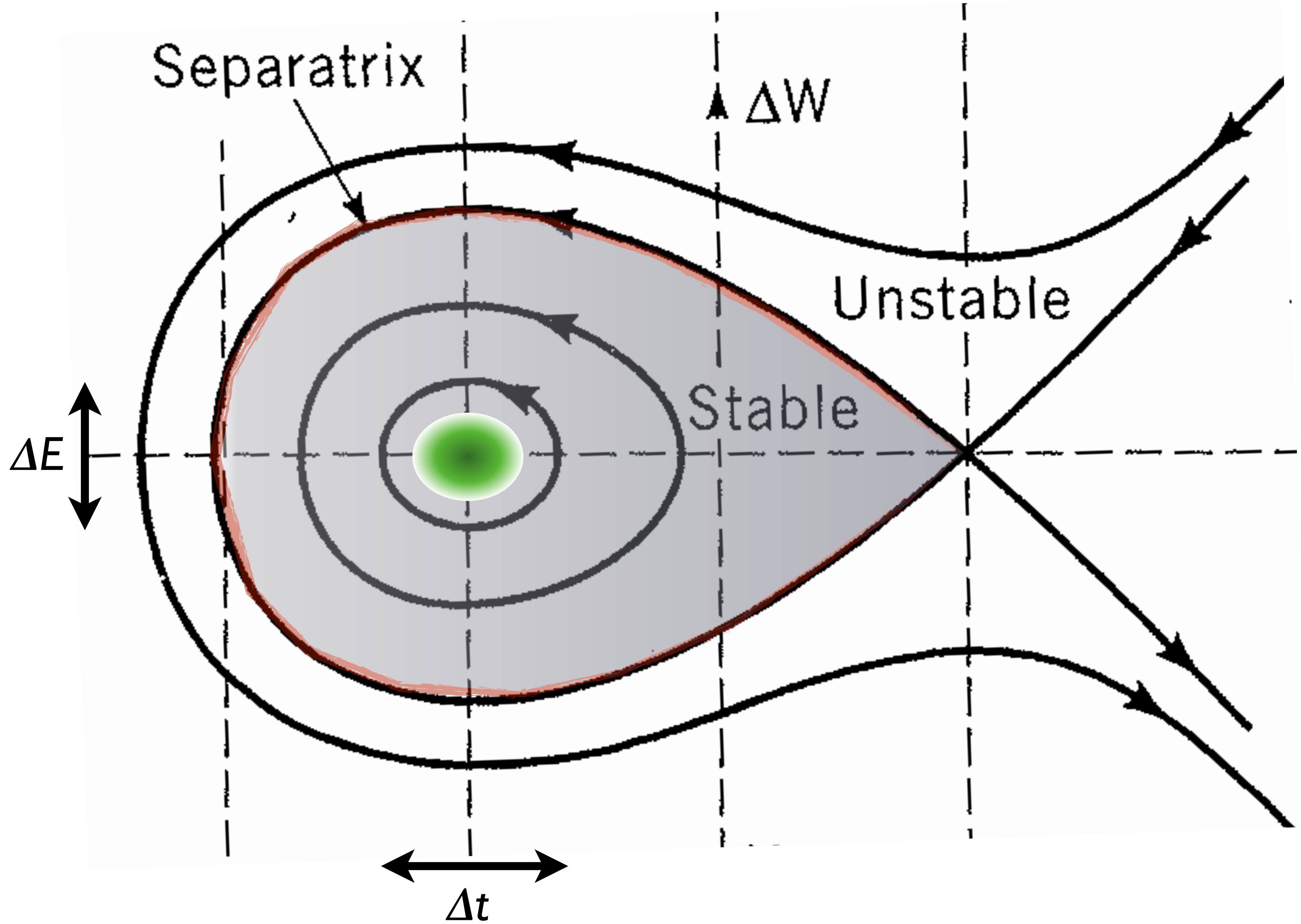
Acceptance and Emittance

- Stable region often called an RF “bucket”
  - “contains” the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system
-



Acceptance and Emittance

- Stable region often called an RF "bucket"
  - "contains" the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system
- Desire the beam particles to occupy much smaller area in the phase space



area: "eV-sec"
Note: *E*, *t* *canonical*

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

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differential approach…

start with above difference eqs $\rightarrow \frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E}\Delta E, \qquad \frac{d\Delta E}{dn} = QeV(\sin\phi - \sin\phi_s)$

(1)

(2)

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

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$$\rightarrow \frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E}\,\Delta E, \qquad \frac{d\Delta E}{dn} = QeV(\sin\phi - \sin\phi_s)$$

$$\rightarrow \frac{d^2\phi}{dn^2} = \frac{2\pi h\eta}{\beta^2 E}\frac{d\Delta E}{dn} = \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) \qquad (1)$$

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$$\Rightarrow \frac{d^2\phi}{dn^2} - \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) = 0$$

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$$\Rightarrow \frac{d^2\phi}{dn^2} - \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) = 0$$

find 1st integral:

$$\int\left(\frac{d^2\phi}{dn^2}\right)\frac{d\phi}{dn}\,dn - \frac{2\pi h\eta}{\beta^2 E}QeV\int(\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn = 0$$

(2)

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

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$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{2\pi h\eta}{\beta^2 E}QeV(\cos\phi + \phi\sin\phi_s) = constant$$

(2)

$$\phi_{n+1} = \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n$$

$$\Delta E_{n+1} = \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s)$$

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differential approach...

start with above difference eqs $\rightarrow \frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E}\Delta E, \qquad \frac{d\Delta E}{dn} = QeV(\sin\phi - \sin\phi_s)$

$$\rightarrow \frac{d^2\phi}{dn^2} = \frac{2\pi h\eta}{\beta^2 E}\frac{d\Delta E}{dn} = \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) \qquad (1)$$

$$\Rightarrow \frac{d^2\phi}{dn^2} - \frac{2\pi h\eta}{\beta^2 E}QeV(\sin\phi - \sin\phi_s) = 0$$

find 1st integral:

$$\int \left(\frac{d^2\phi}{dn^2}\right)\frac{d\phi}{dn}\,dn - \frac{2\pi h\eta}{\beta^2 E}QeV\int(\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn = 0$$

$$\frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{2\pi h\eta}{\beta^2 E}QeV(\cos\phi + \phi\sin\phi_s) = constant$$

or,

$$\boxed{\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta}QeV(\cos\phi + \phi\sin\phi_s) = constant} \qquad (2)$$

The equation of the ***trajectories*** in phase space!

Synchrotron Oscillations

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- Particles near the synchronous phase and ideal energy will oscillate about the synchronous particle with the "synchrotron frequency" (this is call synchrotron motion, even for a linac!) In a synchrotron, …

- "synchrotron tune" == # of synch. osc.'s per revolution

compute small oscillation frequency:

in (1), let $\phi = \phi_s + \Delta\phi \quad \rightarrow \quad \sin\phi - \sin\phi_s = \sin\phi_s \cos\Delta\phi + \cos\phi_s \sin\Delta\phi - \sin\phi_s$

$$\approx \Delta\phi \ \cos\phi_s$$

$$\Rightarrow \frac{d^2\Delta\phi}{dn^2} - \left(\frac{2\pi h\eta}{\beta^2 E} QeV \cos\phi_s\right) \Delta\phi = 0$$

$$(2\pi\nu_s)^2$$

$$\Rightarrow \quad \nu_s = \sqrt{-\frac{h\eta QeV}{2\pi\beta^2 E}\cos\phi_s}$$

if $\eta > 0$, choose $\cos\phi_s < 0$

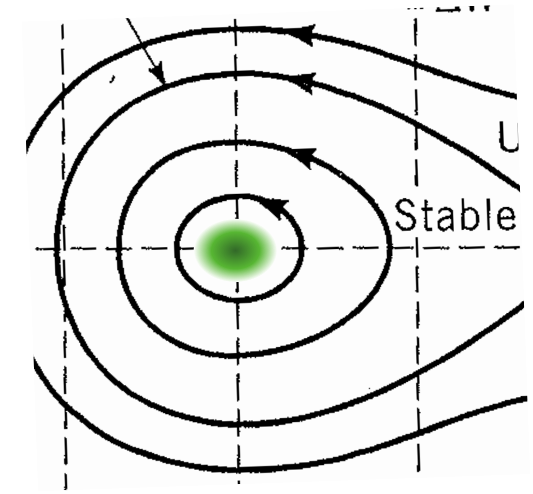
Comment on Frequencies of the Motion

- From what we've just seen today, the synchrotron motion in a circular accelerator takes many (perhaps hundreds of) revolutions to complete one synchrotron period
- On the other hand, in the transverse plane, a particle will typically undergo many betatron oscillations during one revolution
- Thus, transverse/longitudinal dynamics typically occur on very different time scales — this actually justifies us studying them independently

Motion Near the Ideal Particle

Linearize the motion, and write in matrix form…

$$\phi = \phi_s + \Delta\phi$$



$$\begin{aligned}
\phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= \Delta E_n + QeV(\sin\phi_{n+1} - \sin\phi_s) \\
&= \Delta E_n + QeV(\sin\phi_s\cos\Delta\phi_{n+1} + \sin\Delta\phi_{n+1}\cos\phi_s) - \sin\phi_s) \\
&= \Delta E_n + QeV\cos\phi_s\;\Delta\phi_{n+1} \\
&= \Delta E_n + QeV\cos\phi_s\left[\Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n\right]
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta\phi_{n+1} &= \Delta\phi_n + \frac{2\pi h\eta}{\beta^2 E}\Delta E_n \\
\Delta E_{n+1} &= QeV\cos\phi_s\Delta\phi_n + \left(1 + \frac{2\pi h\eta}{\beta^2 E}QeV\cos\phi_s\right)\Delta E_n
\end{aligned}$$

or,

$$\begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_{n+1} = \begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ QeV\cos\phi_s & \left(1 + \frac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s\right) \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_n$$

$$= \begin{pmatrix} 1 & 0 \\ QeV\cos\phi_s & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{2\pi h\eta}{\beta^2 E} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta\phi \\ \Delta E \end{pmatrix}_n$$

$$M = M_c \cdot M_d$$

"thin" cavity — *drift*

(*acts as longitudinal focusing element*)

Note: for $\eta < 0$, M_d is a "backwards" drift; i.e., $\Delta\phi$ decreases for $\Delta E > 0$

(when no bending)

$\eta = -1/\gamma^2$ in straight region (linac)

An aside (for interested parties…):

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Remember from transverse motion, $x \propto \sqrt{\beta} \sin \Delta\psi$
and when *M* was periodic,

$$M = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \qquad \text{and} \quad tr M = 2\cos\Delta\psi$$

$\Delta\psi$ = phase advance through periodic section

Can imagine "longitudinal" *β*, *α*, *γ*, *Δψ* parameters as well

Note: from *M* of previous page, if represents periodic structure (synchrotron or portion of linac), then

$$tr M = 2 + \frac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s = 2\cos\Delta\psi_s$$

longitudinal phase advance

$$\Delta\psi_s = 2\pi\nu_s$$

oscillation frequency
w.r.t. cavity number, "*n*"
(e.g., synchrotron *tune*)

$$\cos\Delta\psi_s \approx 1 - \frac{1}{2}(\Delta\psi_s)^2 = 1 + \frac{\pi h\eta}{\beta^2 E} QeV\cos\phi_s \left[= \frac{1}{2} tr M\right]$$

$$\Rightarrow \quad \nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV\cos\phi_s}$$

as found previously!

Some Numbers (finally!)…

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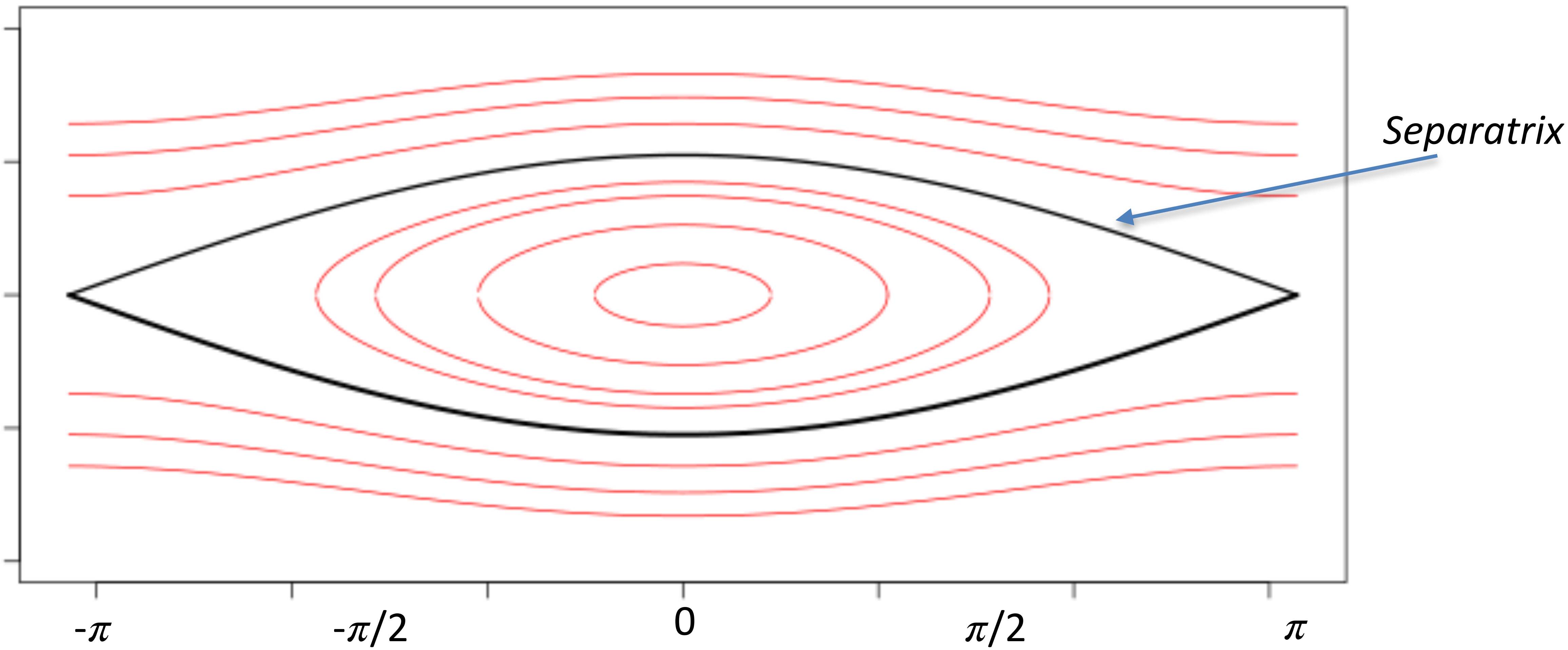
$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV \cos\phi_s}$$

- Suppose (~ Main Injector):
  - h = 588
  - $\eta = 1/\gamma_t^2 - 1/\gamma^2$ = 1/18^2 - 1/9.5^2 = -0.008
  - Q = 1; V = 1 MV; $\cos\phi_s$ = 1
  - E = 8.9 GeV, β = 1
  -
- then, …
- $$\nu_s = \sqrt{-\frac{588 \cdot (-0.008)}{2\pi(8.9 \times 10^9)} \cdot 10^6} = 0.009 = 1/109$$

takes ~100 turns to undergo a complete synchrotron oscillation

The Stationary Bucket

- Suppose do not wish to accelerate the ideal particle…
  - for lower energies, where the slip factor is negative, then need to choose $\phi_s = 0^o$



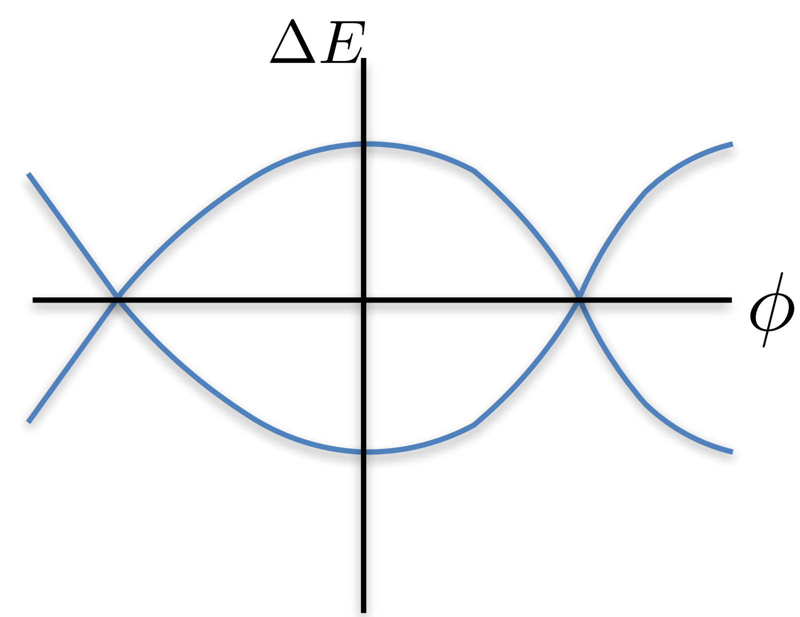
for 53 MHZ: -9.4 ns -4.7 ns 0 4.7 ns 9.4 ns

“stationary” bucket: $\phi_s = 0,\ 2\pi \quad (\sin\phi_s = 0)$ —> no average acceleration

anticipate stability: —> choose $\phi_s = 0, \quad \eta < 0$

then,

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV\cos\phi = constant$$



on the separatrix: $\Delta E = 0$ at $\phi = \pm\pi$

$$0 - 2\frac{\beta^2 E}{2\pi h\eta} QeV = constant$$

thus, solve for *constant* and hence find the Eq. of separatrix:

separatrix:

$$\Delta E = \pm\sqrt{-\frac{2\beta^2 E}{\pi h\eta} QeV}\cos(\phi/2)$$

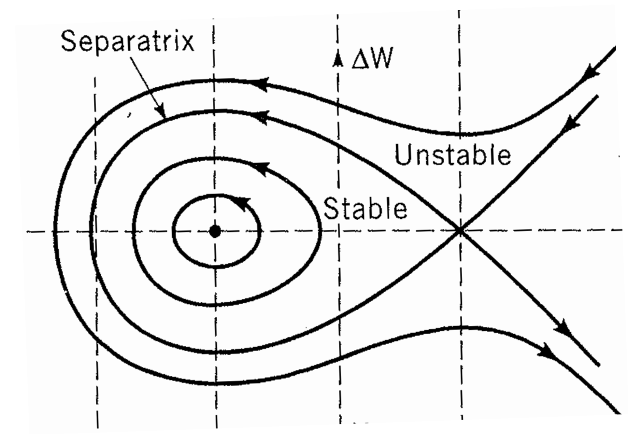
(for “stationary bucket”)

Ex: $\frac{\Delta\hat{E}}{E} = \sqrt{\frac{2}{\pi(588)(0.008)(8.9\times 10^9)}(10^6)}$ = 0.004

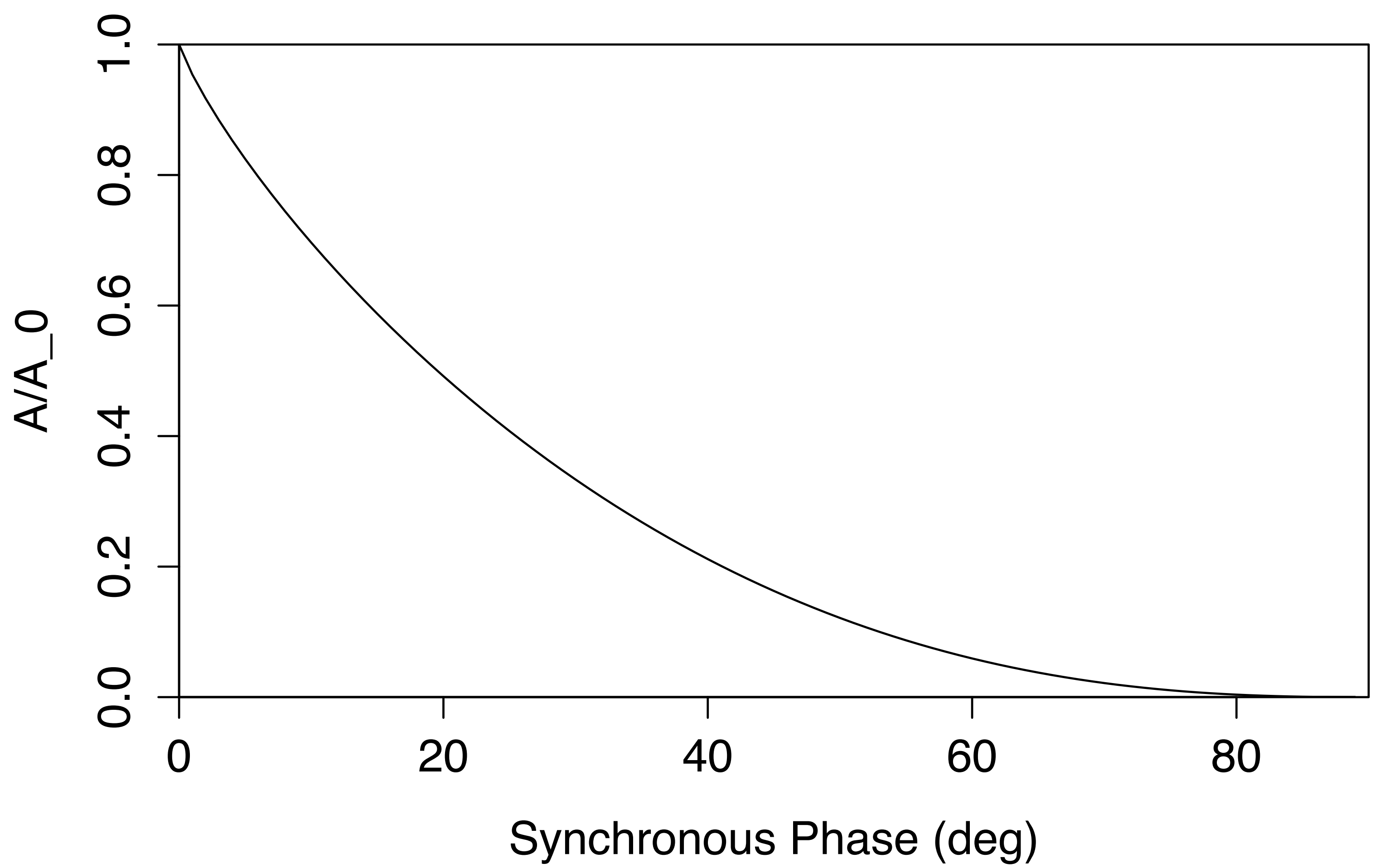
maximum energy spread that can be held by the system

Numerical Solution for Bucket Area

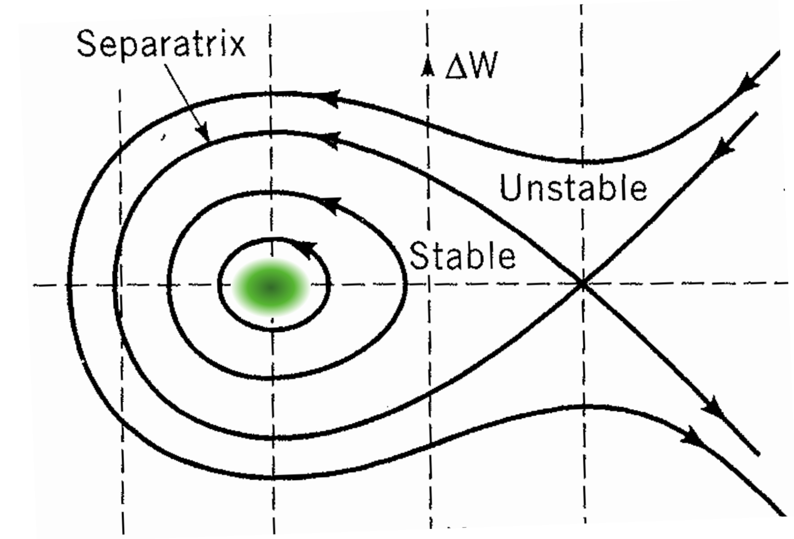
During acceleration, the stable phase space area will be less…




```r
# Solve for bucket area; phis = 0 is "stationary"

Xout <- array(0,dim=c(91,4))
phisDeg <- -1

for(i in (1:90)){
phisDeg <- phisDeg + 1
phis <- phisDeg*pi/180

f <- function(x){
   cos(x)+x*sin(phis)+cos(phis)-(pi-phis)*sin(phis) }
dE <- function(x){
   sqrt(cos(phis)-(pi-phis)*sin(phis)+cos(x)+x*sin(phis)) }

phi1 <- pi-phis
phi2 <- uniroot( f, c(-pi, 2*pi))$root
A <- -1/4/sqrt(2)*integrate(dE, phi1, phi2)$value

Xout[i,] = c(phis*180/pi, phi1*180/pi, phi2*180/pi, A)  }

plot(Xout[,1],Xout[,4],typ="l",
   xlab="Synchronous Phase (deg)", ylab="A/A_0",
   xaxs="i", yaxs="i",xlim=c(0,90))

Xout
```



Back to Small Oscillations…



from (2), $$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi + \phi\sin\phi_s) = constant$$

if $\phi = \phi_s + \Delta\phi$ (*small*), then …

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi_s\cos\Delta\phi - \sin\phi_s\sin\Delta\phi + (\phi_s + \Delta\phi)\sin\phi_s) = constant$$

$$\Delta E^2 + 2\frac{\beta^2 E}{2\pi h\eta} QeV(\cos\phi_s(1 - \frac{1}{2}\Delta\phi^2) - \sin\phi_s\Delta\phi$$
$$+\phi_s\sin\phi_s + \Delta\phi\sin\phi_s) = constant$$

$$\boxed{\Delta E^2 + \left(-\frac{\beta^2 E}{2\pi h\eta} QeV\cos\phi_s\right)\Delta\phi^2 = constant} \quad (3)$$

This Eqn. represents trajectories in longitudinal phase space of particles ***near*** the ideal particle.

Beam Longitudinal Emittance

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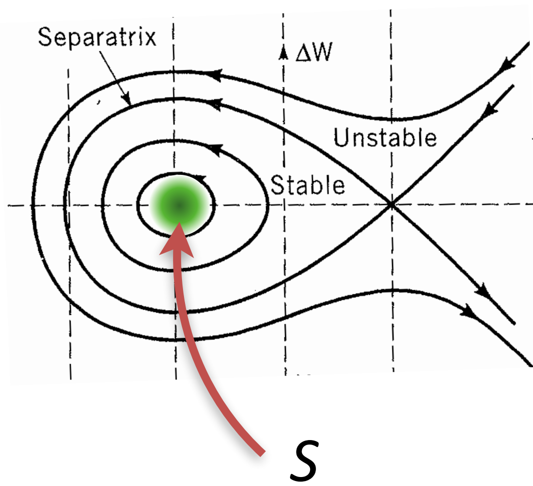
Suppose beam is well contained within an ellipse given by (3), and suppose we know either $\Delta\hat{E}$ or $\Delta\hat{\phi}$ (or, $\Delta\hat{t}$) of the distribution (i.e., maximum extent). Then, the *constant* is easily seen to be:

$$constant = \Delta\hat{E}^2 = -\frac{\beta^2 E}{2\pi h\eta} QeV\cos\phi_s \Delta\hat{\phi}^2$$

So, area of ellipse (the *longitudinal emittance*) is: $\pi\ \Delta\hat{E}\Delta\hat{\phi}$

or, in *E-t* coordinates, $S \equiv \pi\ \Delta\hat{E}\Delta\hat{t} = \pi\ \Delta\hat{E}\frac{\Delta\hat{\phi}}{2\pi f_{\rm rf}}$



$$S = \frac{1}{2f_{\rm rf}}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta}Q\cos\phi_S}\ \ \Delta\hat{\phi}^2$$

or,

$$S = 2\pi^2 f_{\rm rf}\sqrt{-\frac{\beta^2 EeV}{2\pi h\eta}Q\cos\phi_S}\ \ \Delta\hat{t}^2$$

units: "eV-sec"

Ex: $S = 2\pi^2(53\times10^6)\sqrt{\frac{(8.9\times10^9)(10^6)}{2\pi(588)(0.008)}}(10\times10^{-9})^2$ = 1.8 eV-sec

Transition Energy

- In a synchrotron, there can be an energy at which the slip factor changes sign — this is call the “transition energy”

$$\eta = \alpha_p - \frac{1}{\gamma^2} = \left\langle \frac{D}{\rho} \right\rangle - \frac{1}{\gamma^2}$$

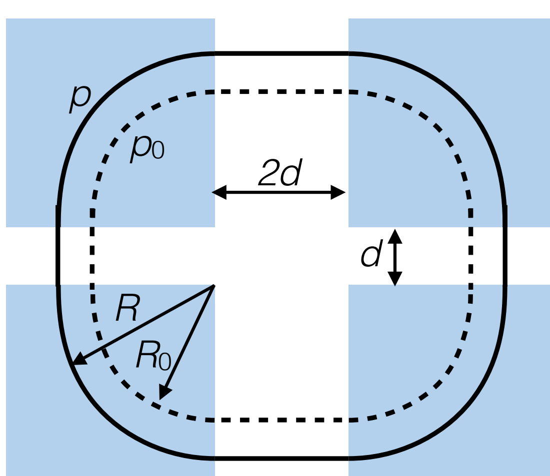
$$\eta = 0 = \alpha_p - \frac{1}{\gamma^2}$$

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

$$\boxed{\gamma_t \equiv \frac{1}{\sqrt{\alpha_p}}}$$

- In a typical FODO-style synchrotron, the transition gamma is roughly equal to the betatron tune

earlier:



$$\alpha_p \equiv \left(\frac{dL/L}{dp/p} \right)$$

$$\frac{\Delta\tau}{\tau_0} = \left(\frac{1}{1 + 3d/\pi R_0} - \frac{1}{\gamma_0^2} \right) \frac{\Delta p}{p_0}$$

Transition

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

We had... $\Rightarrow \dfrac{d^2\Delta\phi}{dn^2} - \left(\dfrac{2\pi h\eta}{\beta^2 E} QeV\cos\phi_s\right)\Delta\phi = 0$

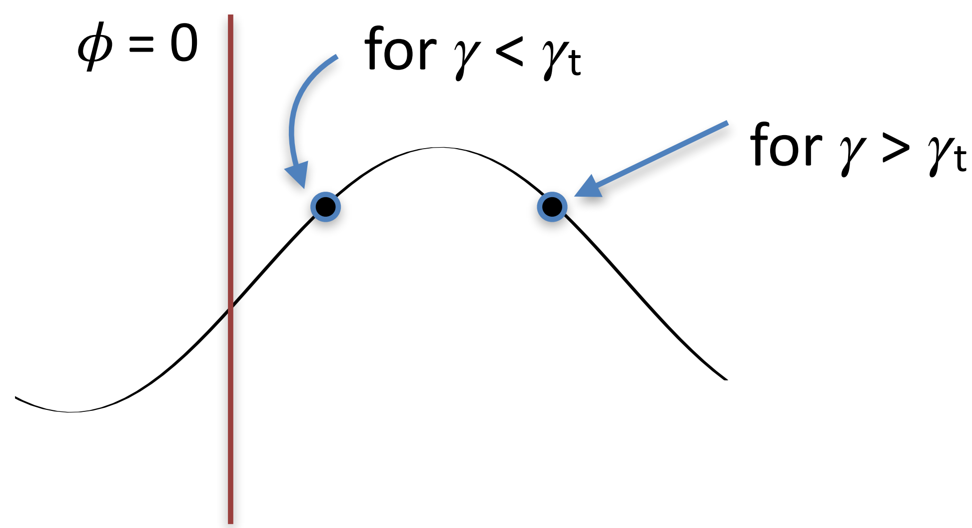
$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV\cos\phi_s}$$

$\text{if } \eta > 0, \text{ choose } \cos\phi_s < 0$

So,

when $\eta < 0$, we want $\cos\phi_s > 0$

when $\eta > 0$, we want $\cos\phi_s < 0$

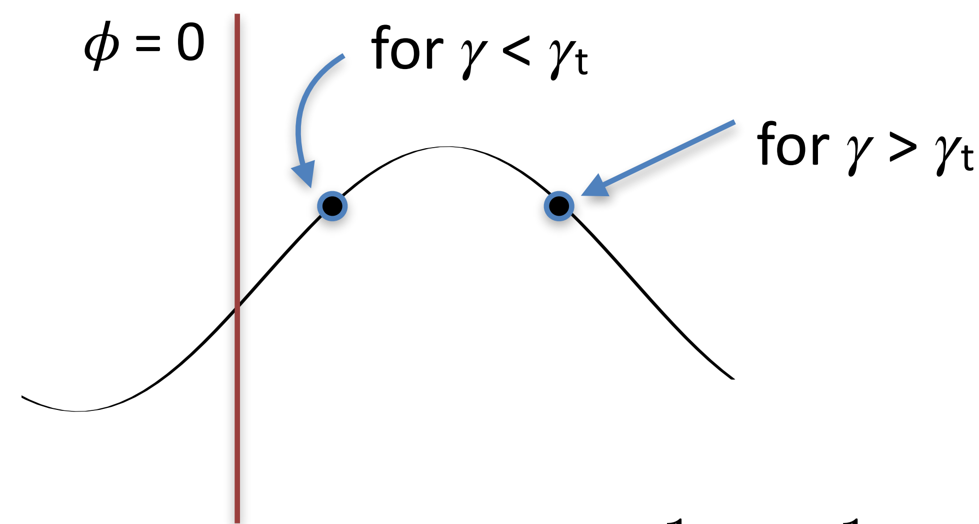


$$\therefore \text{ if } \gamma_t \text{ exists, need ``phase jump'' to occur at transition crossing}$$

$$\gamma_t mc^2 = \text{transition energy}$$

Transition Crossing

- If the synchrotron accelerates through its transition energy, then the phase of the RF system has to be shifted at the time of transition crossing
- The synchrotron motion slows down as approach transition — it would stop if the slip factor were exactly zero!
  - loss of phase stability!
  - momentum spread also gets larger near transition
- So, best to accelerate quickly through this energy region!
  - and change the phase quickly!



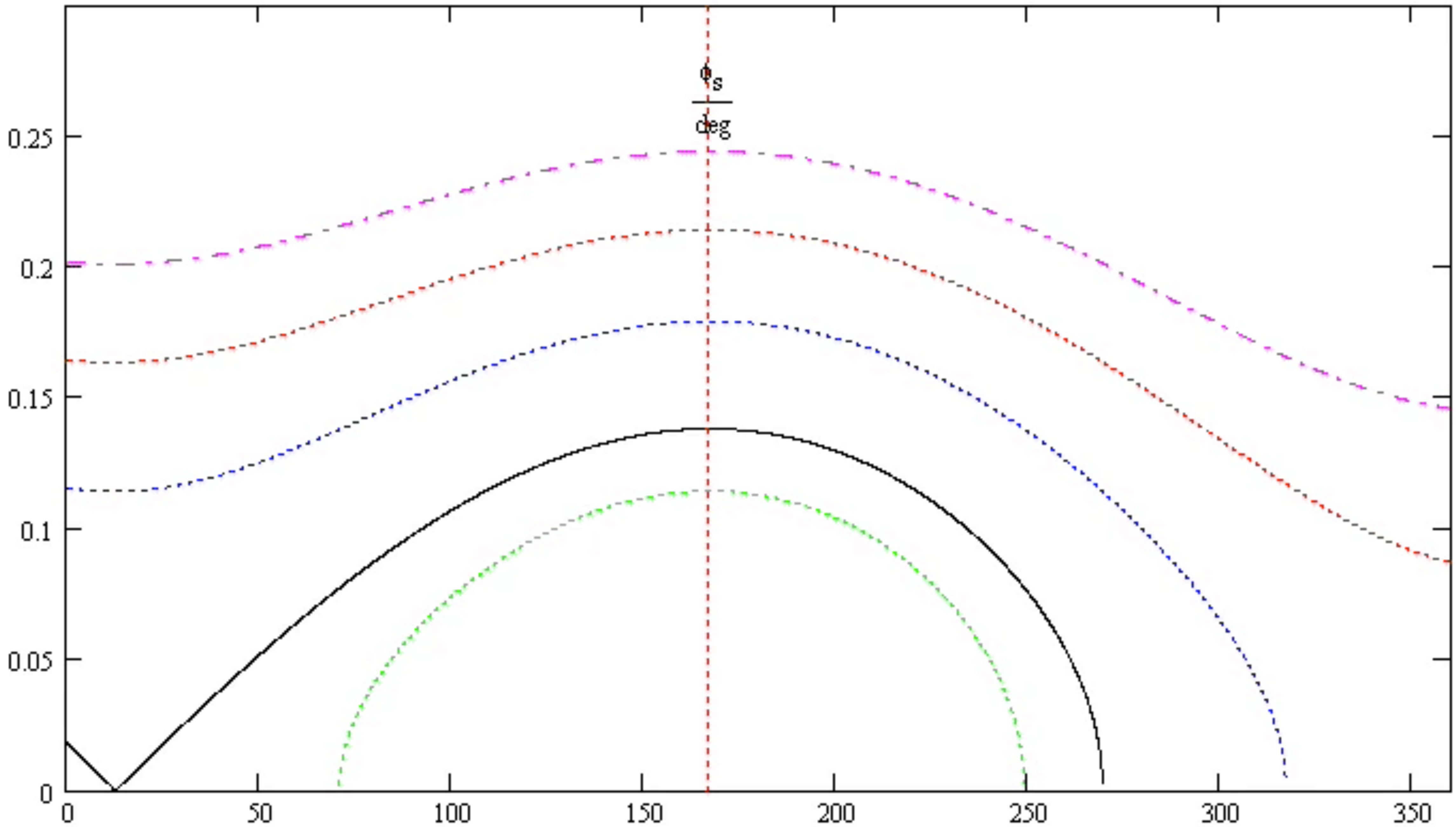
$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

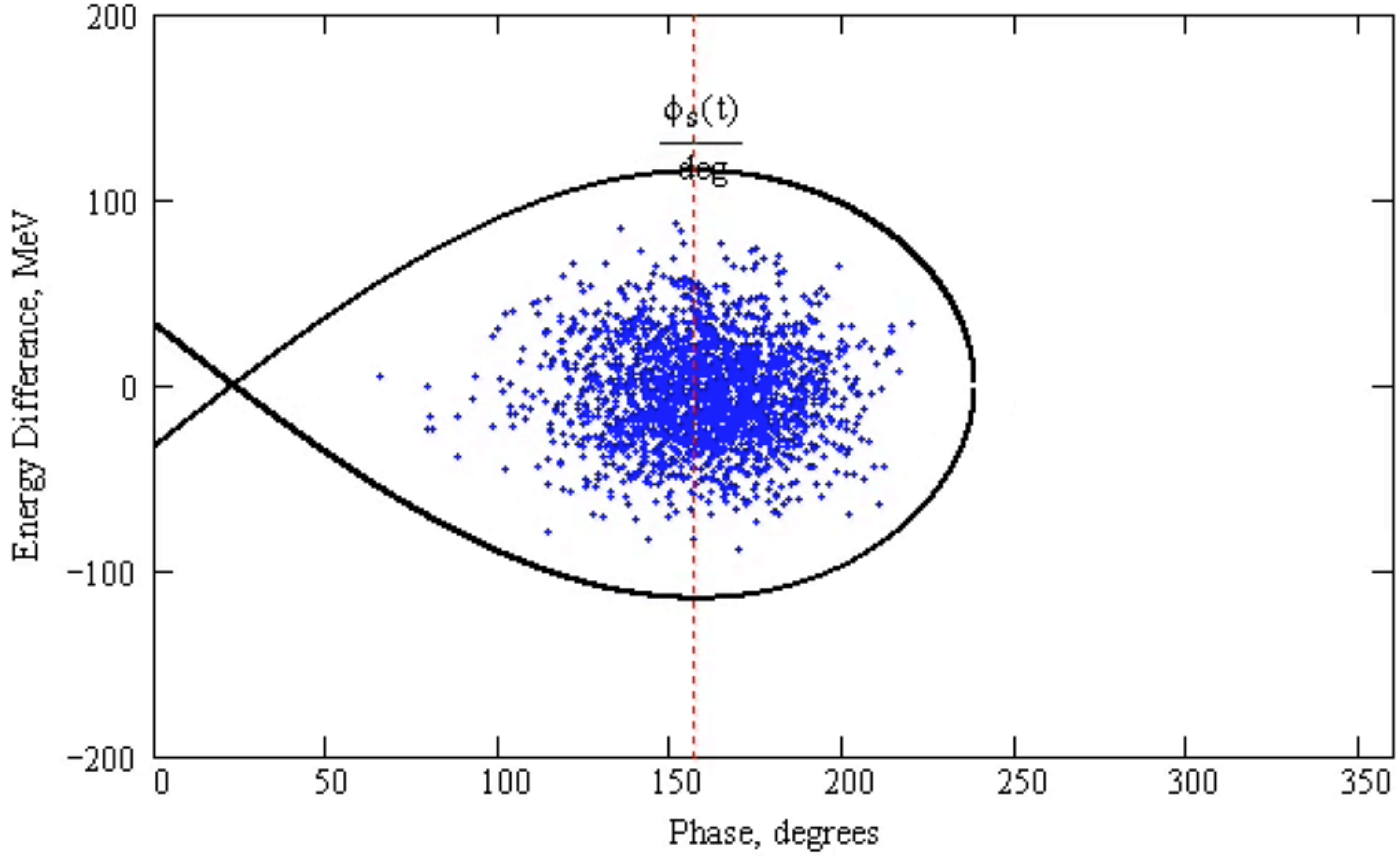
$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E} QeV \cos\phi_s}$$

Some Movies…

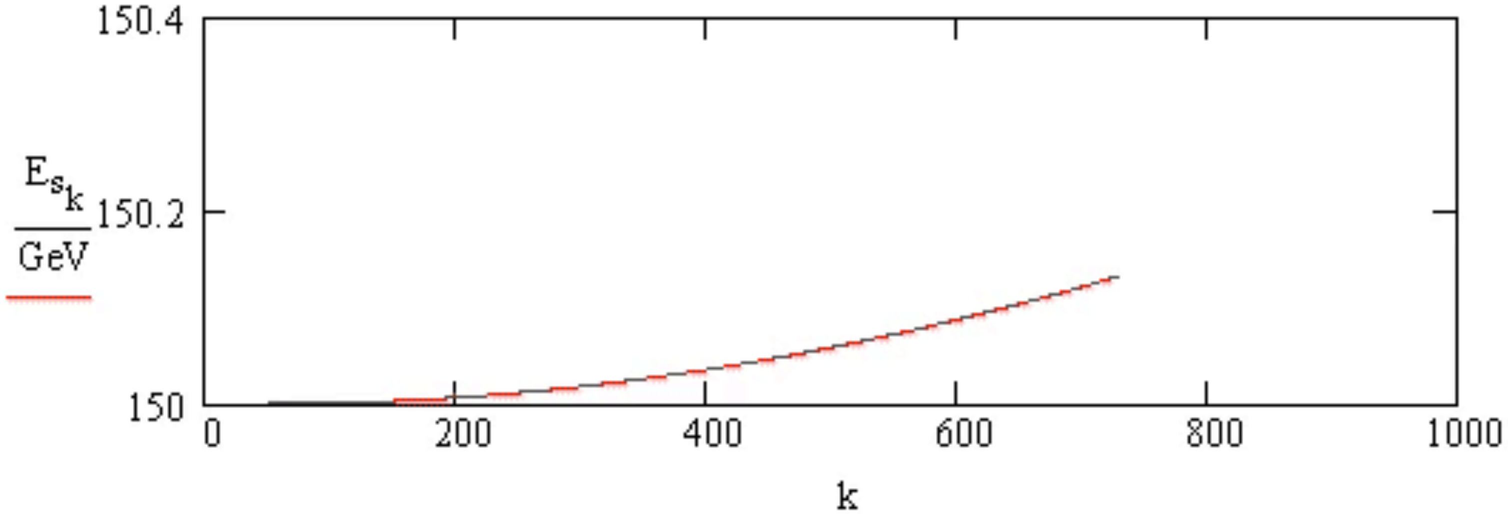
- Bucket Transformation
- Parabolic acceleration
- Parabolic acceleration — full bucket
- Snap Capture
- Adiabatic Capture
- Transition Crossing

Phase space contours, for various values of k. Synchronous phase: $\phi_s = 167.25\,\mathrm{deg}$

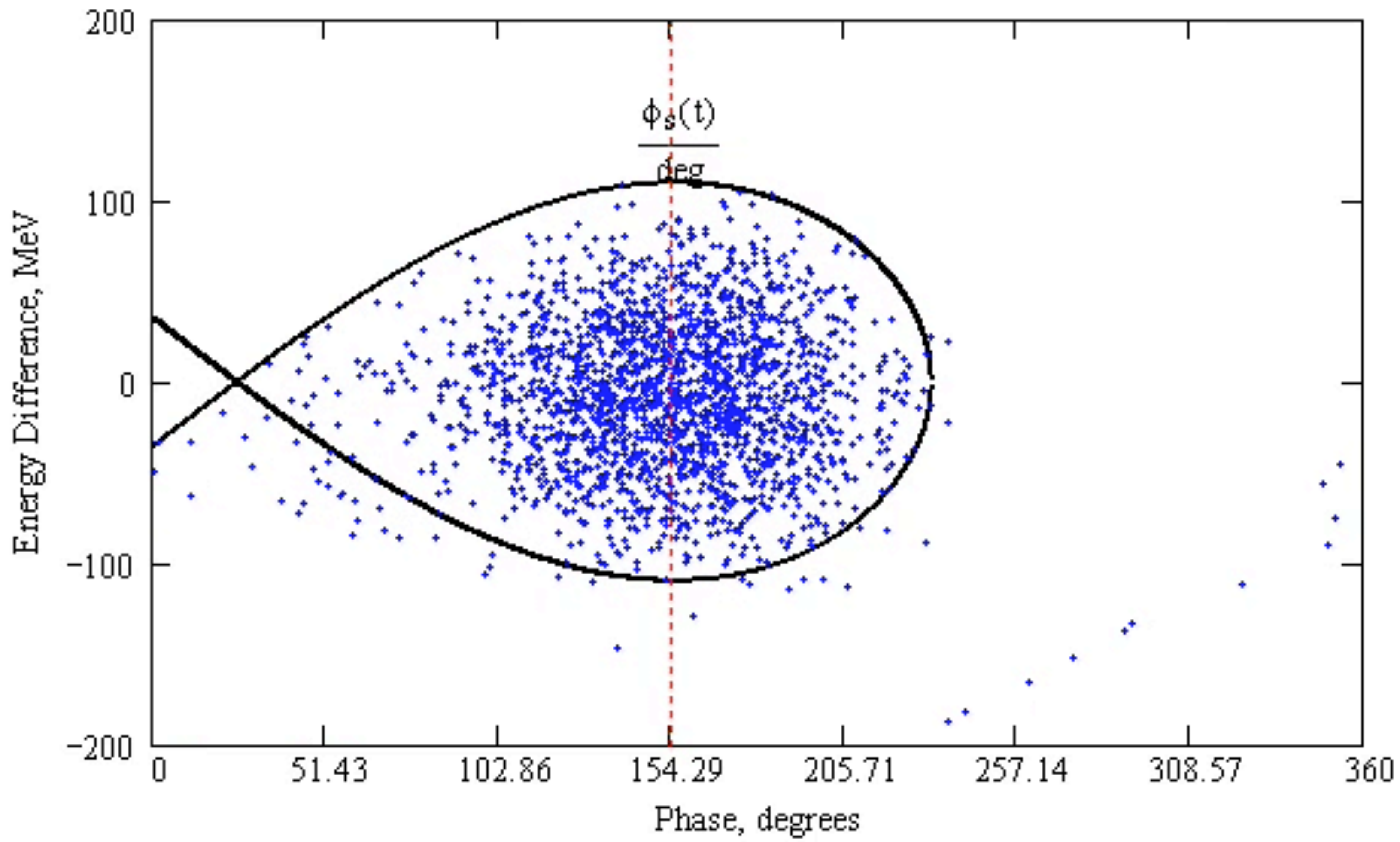




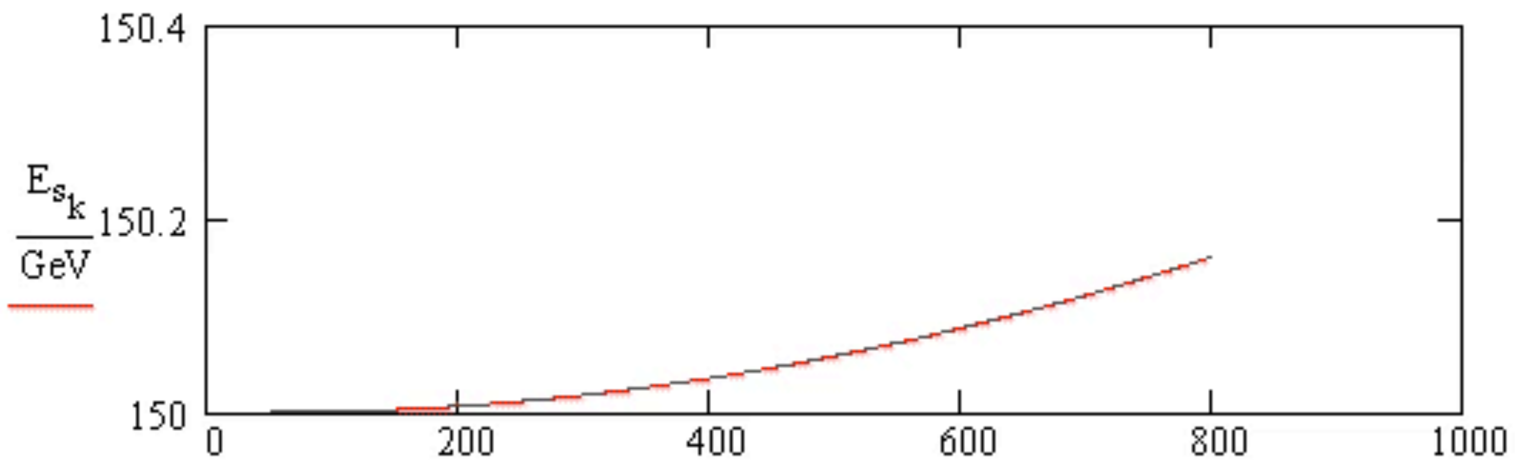
200
100
0
−100
−200
Energy Difference, MeV
0
50
100
150
200
250
300
350
Phase, degrees
$\frac{\phi_s(t)}{deg}$



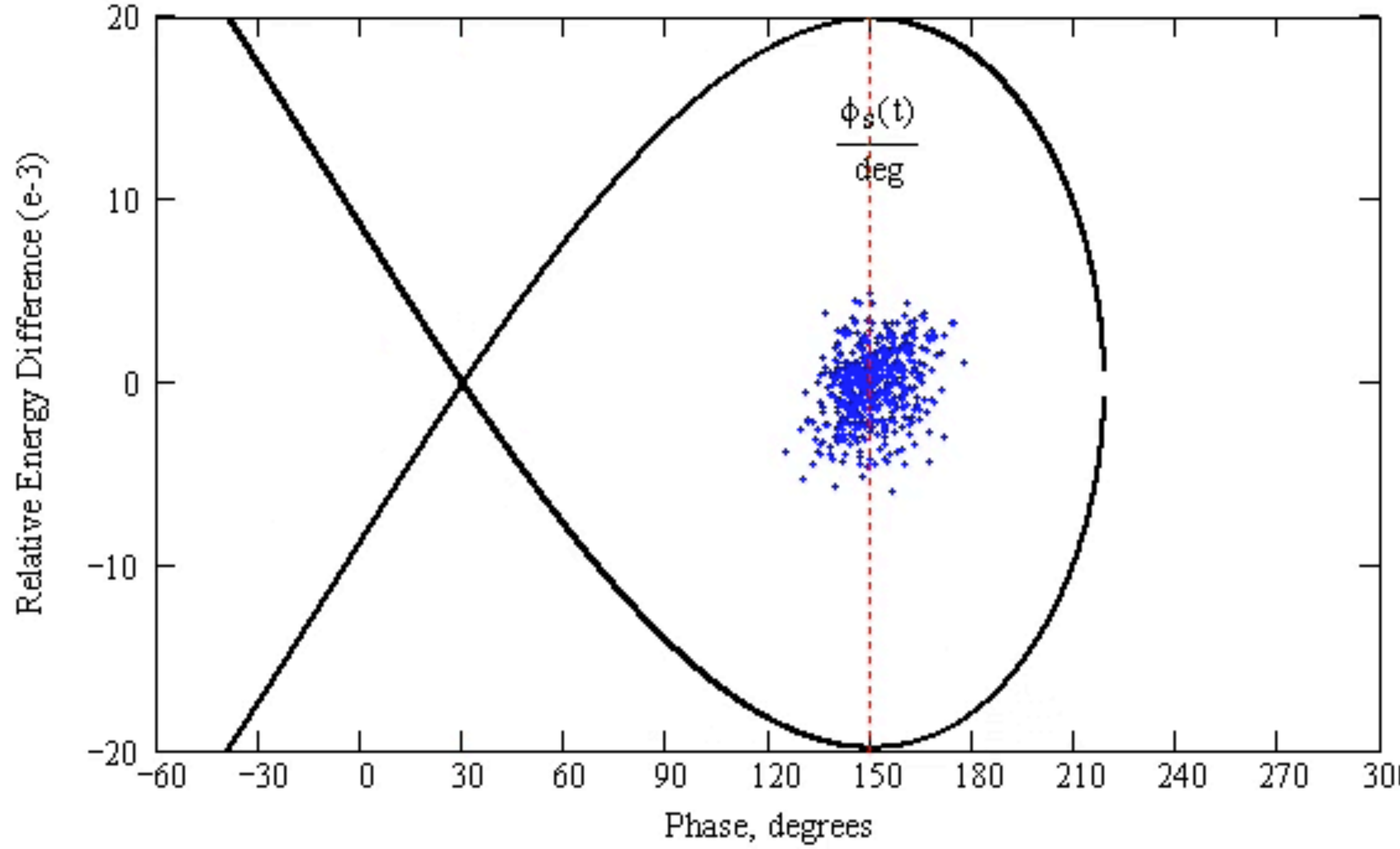
150.4
150.2
150
$\frac{E_{s_k}}{GeV}$
0
200
400
600
800
1000
k



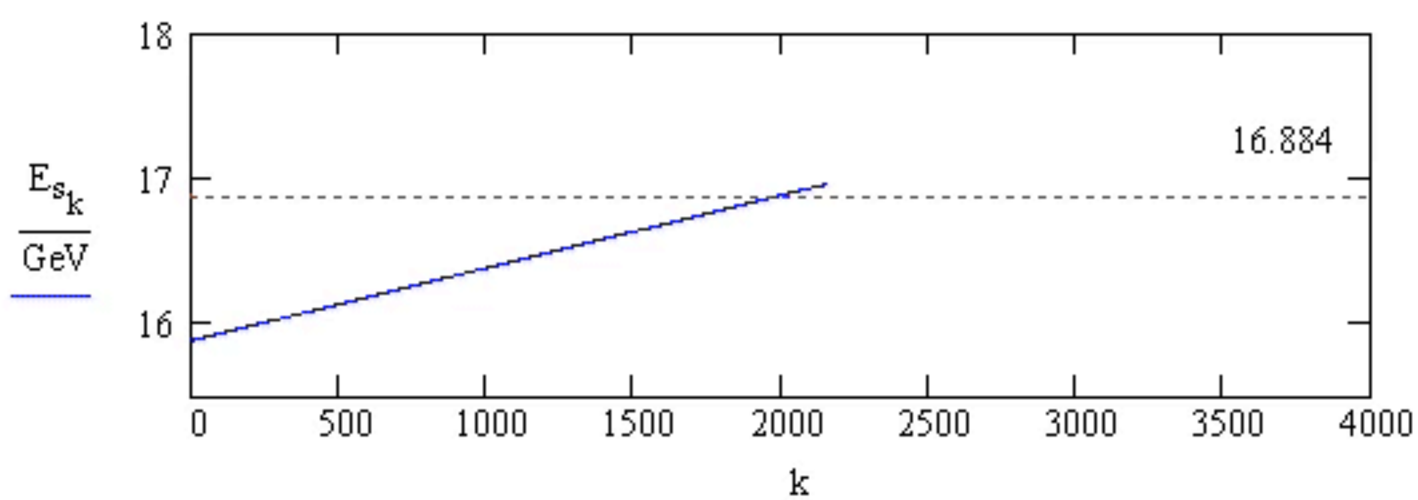
200
100
0
−100
−200
Energy Difference, MeV
0
51.43
102.86
154.29
205.71
257.14
308.57
360
Phase, degrees
ϕs(t)
deg

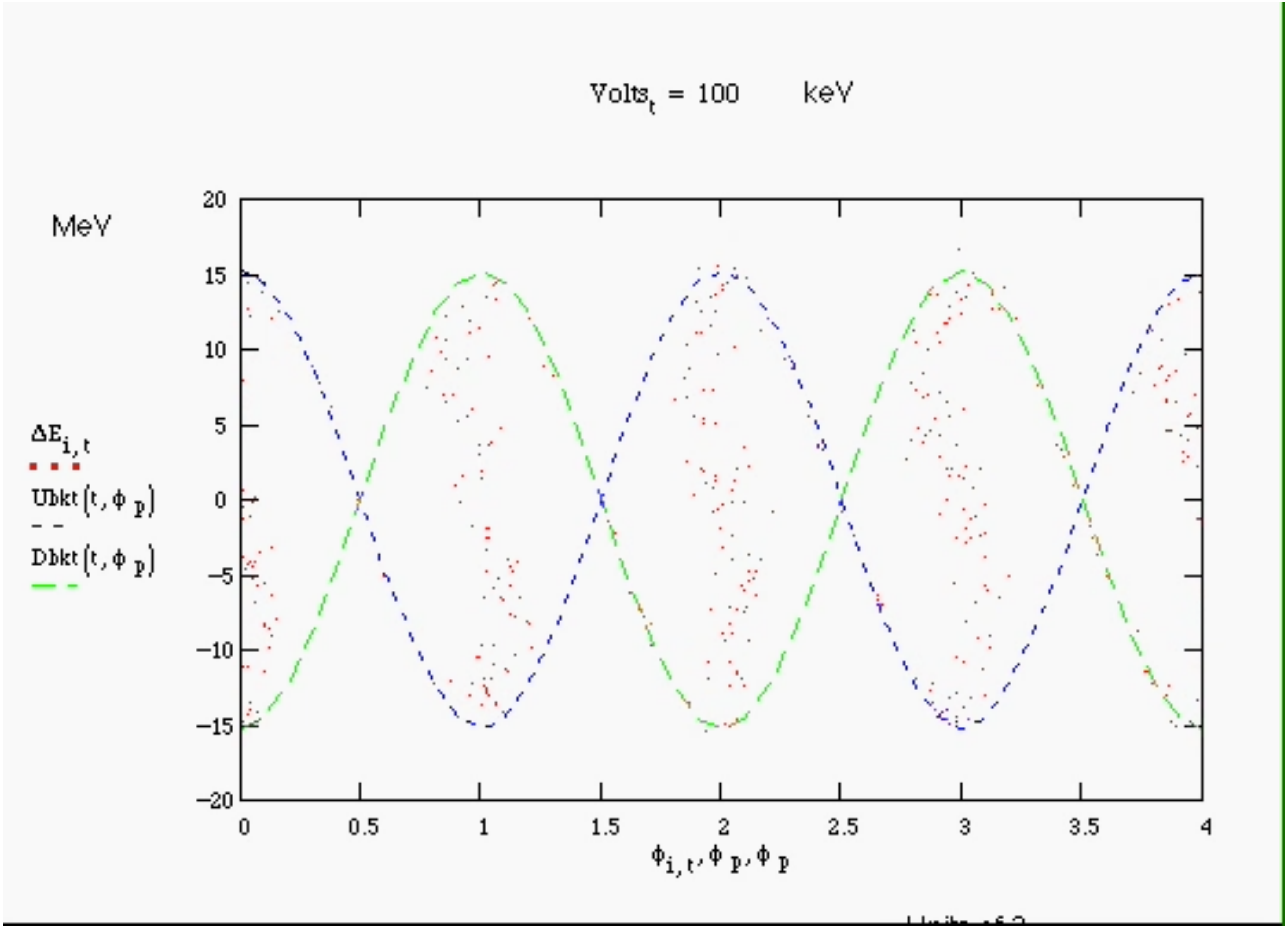


150.4
150.2
150
Esk
GeV
0
200
400
600
800
1000

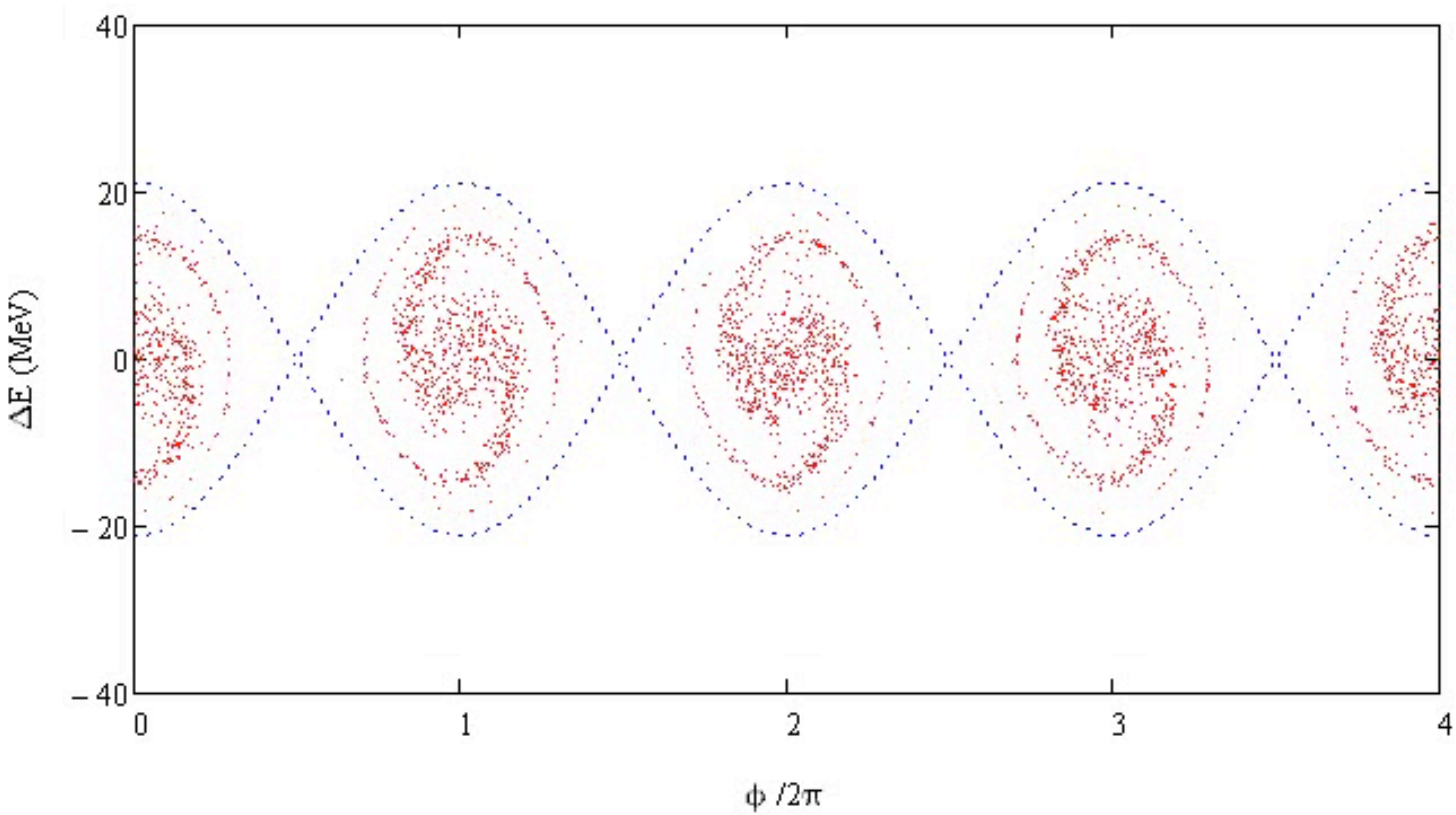


$\sigma_{EonE_t} = 1.958 \times 10^{-3}$ $t = 2.161 \times 10^{3}$




Volts_t = 100 keV
MeV
ΔE_i,t
Ubkt(t, φ_p)
Dbkt(t, φ_p)
20
15
10
5
0
−5
−10
−15
−20
0
0.5
1
1.5
2
2.5
3
3.5
4
φ_i,t, φ_p, φ_p

$eV(n) = 193.334\,\text{keV}$



Bunch Manipulations in Synchrotrons

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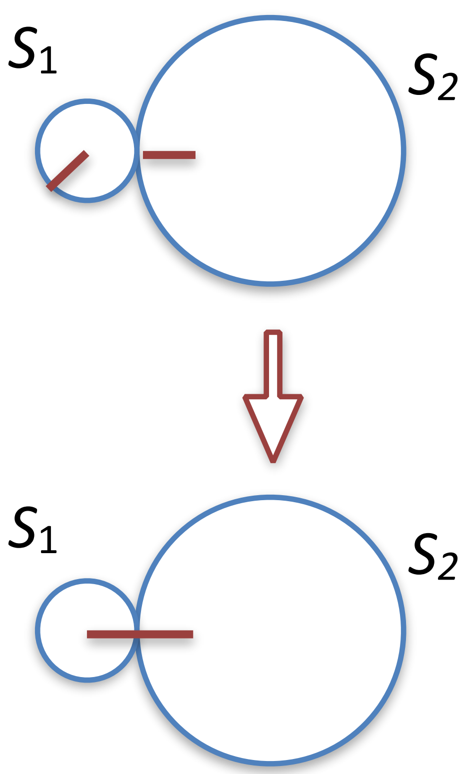
- Cogging
- Slip Stacking
- Bunch Rotation
- Bunch Coalescing
- Barrier Buckets

Cogging

- Essentially, phase slippage by changing the relative momeutm
- Ex: beam transfers between two synchrotrons



Suppose $C_2 = 2C_1$; want to inject bunch in synchrotron S_1 into a particular "bucket" location in synchrotron S_2

need to adjust the revolution frequency of one ring (pick S_1, say) until the two revolving "markers" line up

if $C_2 = 2C_1$ <<==>> $f_1 = 2f_2$, and may *never* line up!

So, make $\Delta\tau_1/\tau_1 = \eta\ \Delta p/p$ such that, after N turns,

$$N\ |\Delta\tau_1|\ =\ \Delta C_1/v$$

Cogging [2]

Suppose want to "cog" beam by one RF bucket in S_1 … then $\Delta C_1 = C_1/h$

$$\longrightarrow \qquad \frac{\Delta\tau}{\tau} = -\frac{\Delta f}{f} = -\frac{\Delta f_{\rm rf}}{f_{\rm rf}} = \eta\frac{\Delta p}{p}$$

adjust Δf_{RF} which yields $\Delta\tau_1$ each turn; leave on for N turns; N = (time between buckets)/$\Delta\tau_1$

to cog by one bucket, $N\,|\Delta\tau_1| = 1/f_{rf} \quad \Rightarrow \quad N\,(\tau_1\,\eta\,\Delta p/p) = 1/f_{rf} \quad \Rightarrow \quad N\,\Delta p/p = 1/((\tau_1\eta h\,f_1)$

or, $N\,\Delta p/p = 1/(\eta h)$

Note: when generate an average $\Delta p/p$, the average horizontal displacement in the synchrotron at a particular position where there is ***dispersion*** will be $\Delta x = D\,\Delta p/p$.

Thus, $N\,\Delta x = D/(\eta h)$

Ex: Suppose we can accommodate radial motion on the scale of 10 mm where the dispersion function has value 2.5 m in a synchrotron with $\eta = 0.05$ and h=100.

Then, to cog by one RF bucket would take

N = (2.5 m / 0.01 m) / (0.05 * 100) = 50 revolutions.

Ex: Slip Stacking (ex: FNAL Main Injector)

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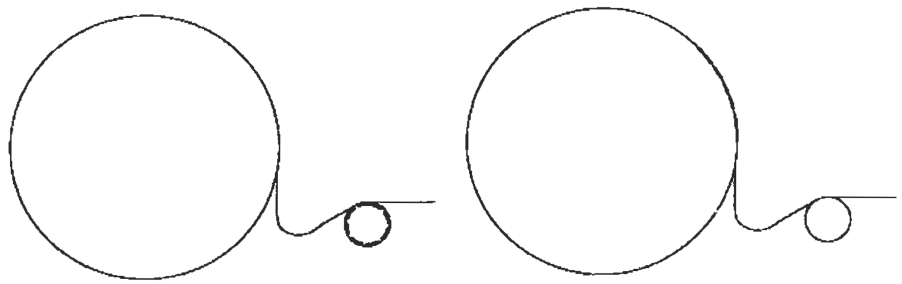
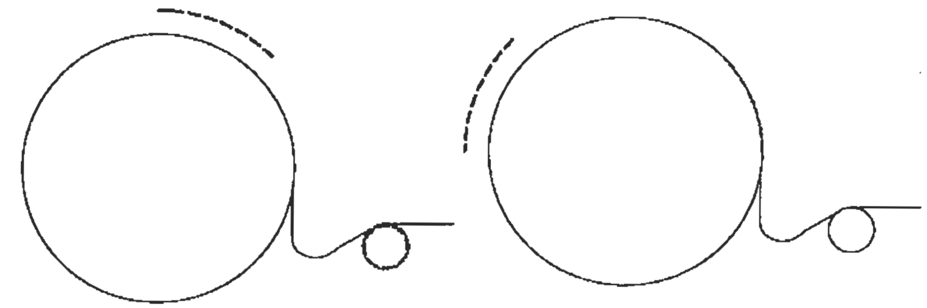
- Essentially "cogging" during injection
  - inject ~ 1/2-circumference-worth of beam
    - » accelerate slightly —> moves orbit outward
      - (use RF system "A", say)
  - inject 2nd batch into the ring, behind the first batch
    - » decelerate slightly —> moves orbit inward
      - (using RF system "B", say)
  - Δp between these 2 orbits implies they will "slip" in time until they line up:
  - $$\frac{\Delta t}{t} = \left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p}$$
  - re-capture with a higher voltage RF in order to match the bucket shape to the beam emittance

Slip Stacking cartoon (1)



- First Booster Batch accelerated in Booster
- First Booster Batch injected onto MI central orbit with RF system A

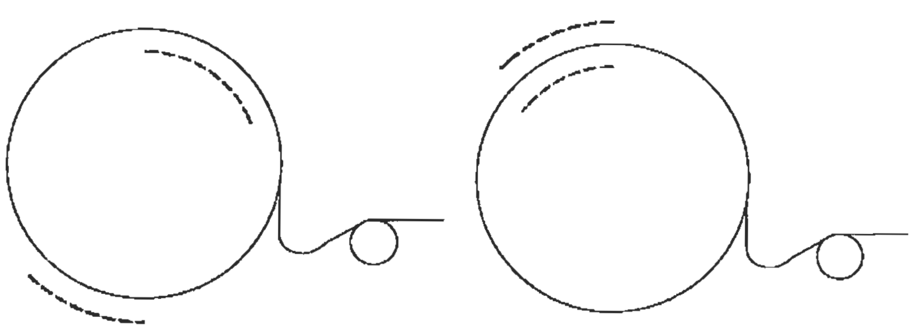
Slip Stacking Cartoon (2)



- First Booster Batch slightly accelerated in MI with RF System A
- Second Booster Batch accelerated in Booster
- Second Booster Batch injected onto MI central orbit with RF system B

Protons on Target, I. Kourbanis

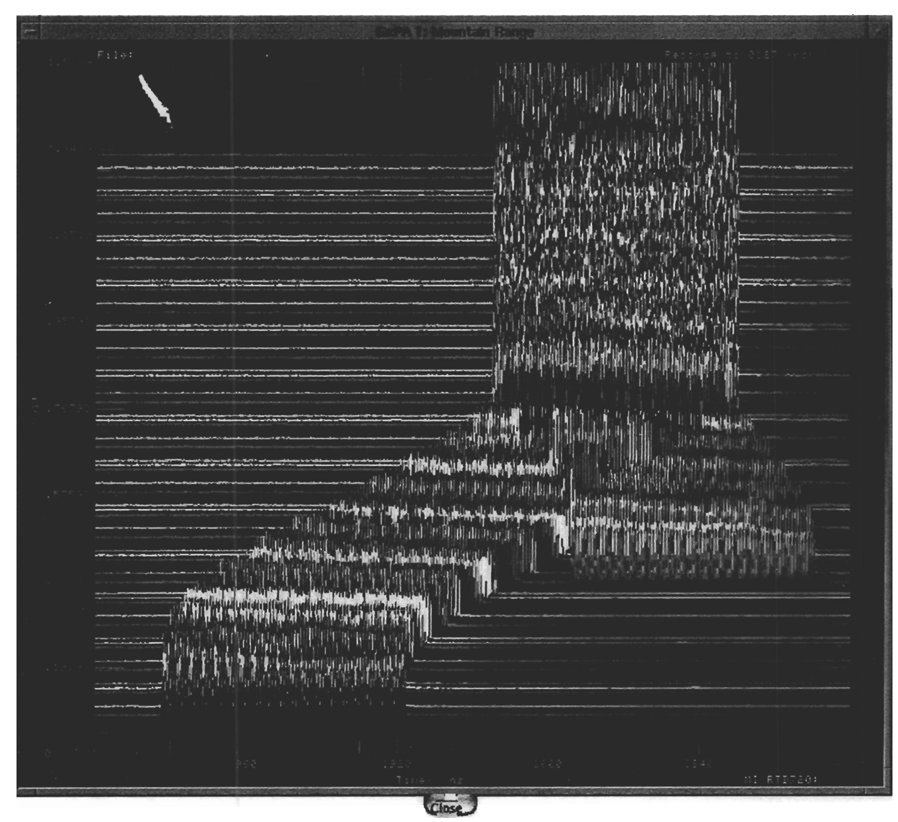
Slip Stacking Cartoon (3)



- Second Booster Batch slightly decelerated in MI with RF System B
- Wait till batches line up and snap on RF system C while turning of RF systems A & B

Protons on Target, I. Kourbanis

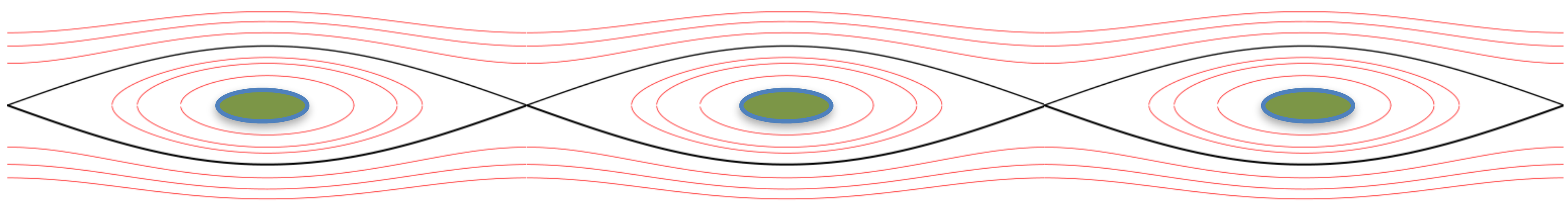
Protons on Target, I. Kourbanis



data

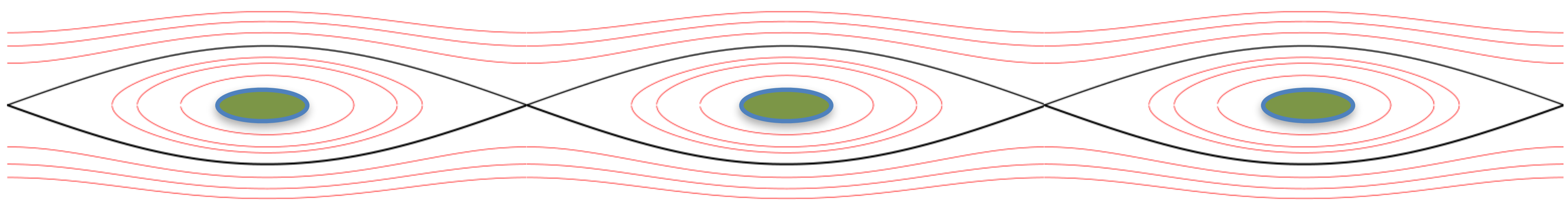
Bunch Rotation

start:

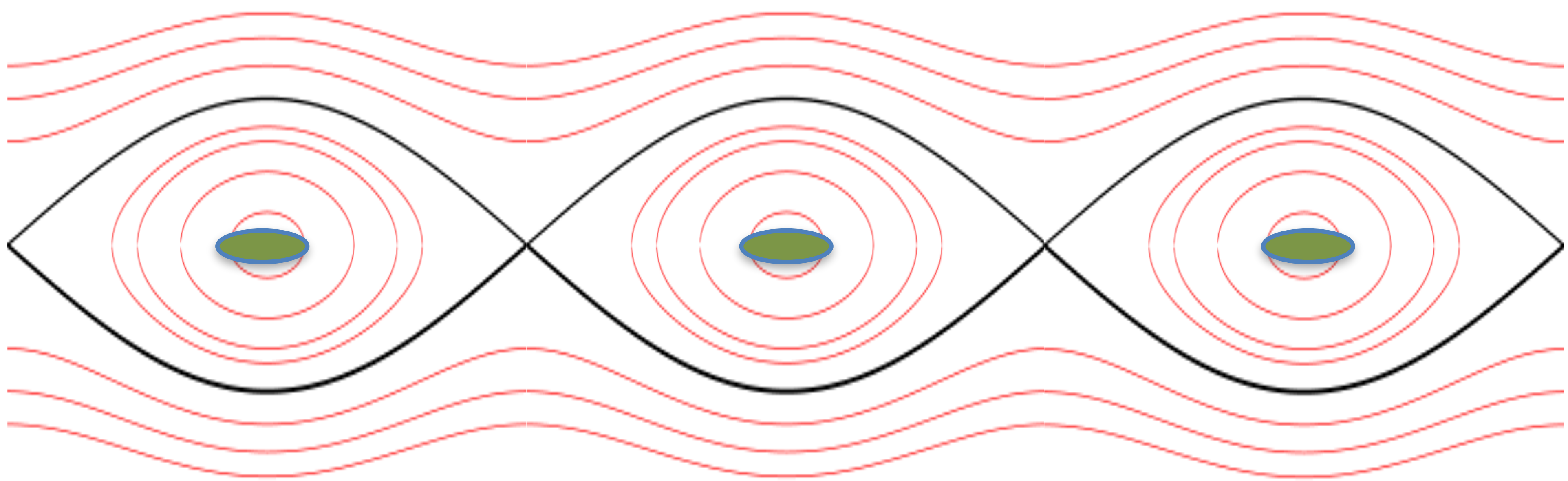


Bunch Rotation

start:



instantly raise RF voltage… bunches will begin to rotate in phase space:



Bunch Rotation

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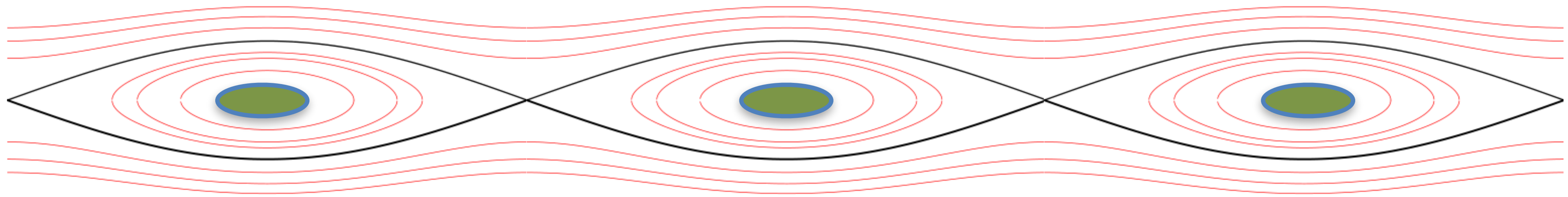
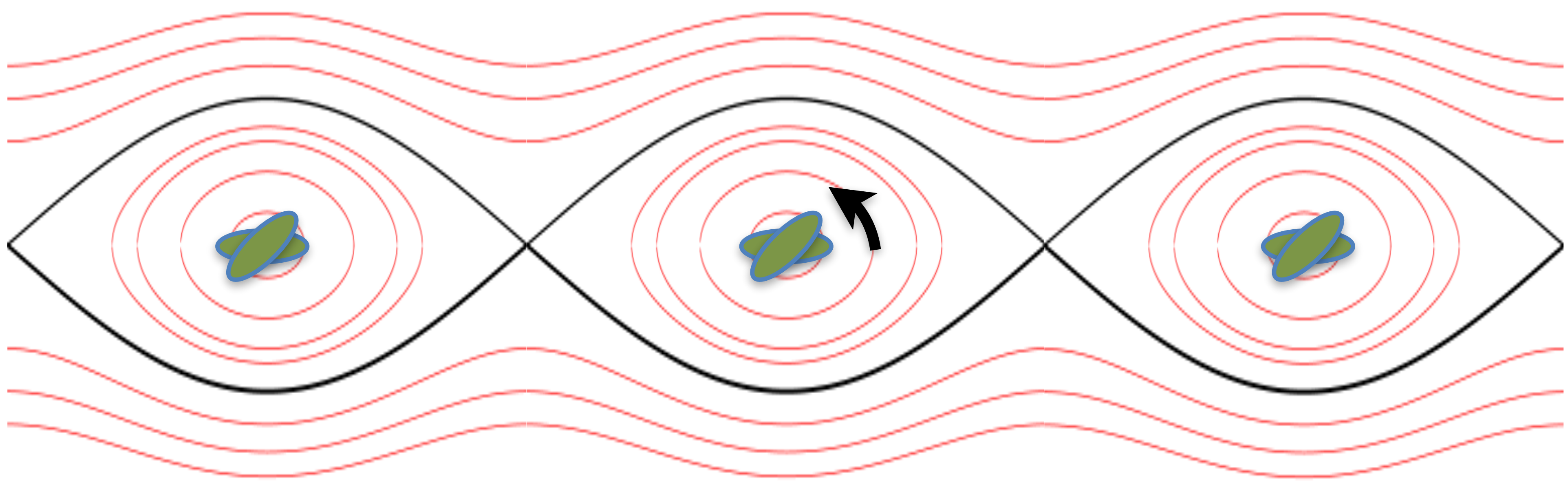
start:



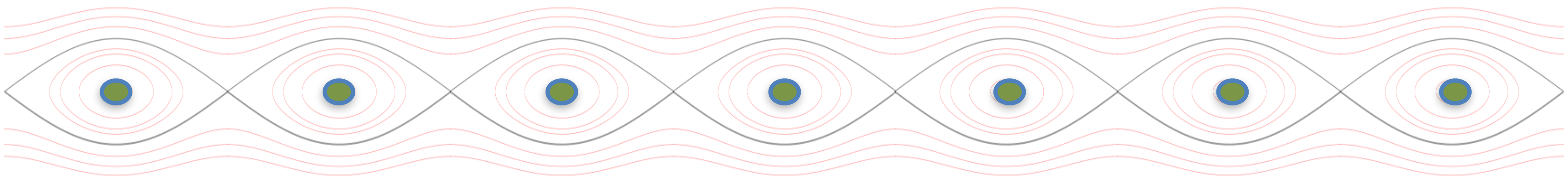
instantly raise RF voltage…   bunches will begin to rotate in phase space:



when rotated by 90° can rapidly switch to a higher-harmonic RF system in order to maintain the shorter bunch length; *or*, for example, extract the beam and send to a target!

Bunch Coalescing

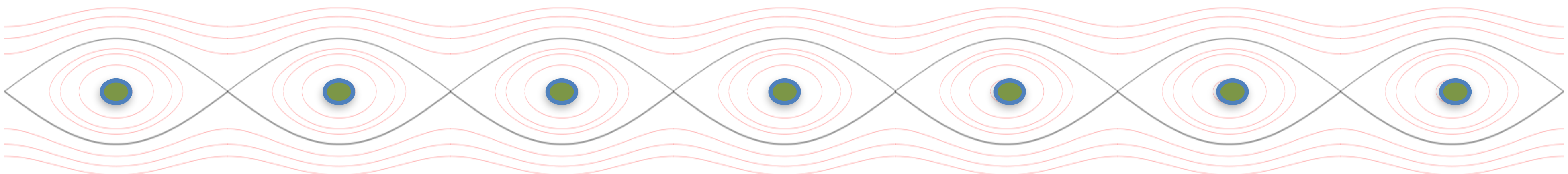
similar to bunch rotation, but also involves a change in RF frequency (harmonic)



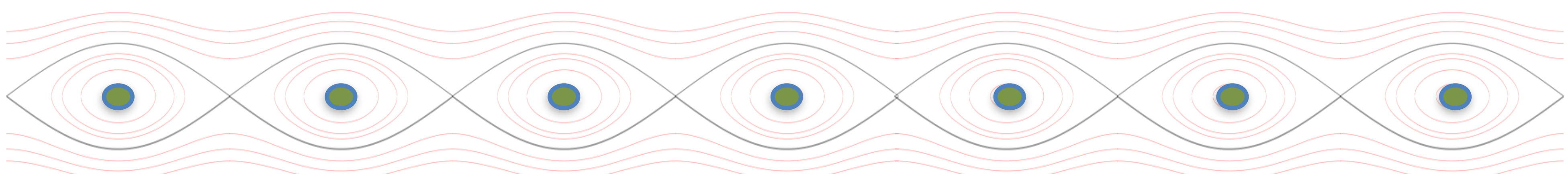
Bunch Coalescing

Northern Illinois University

similar to bunch rotation, but also involves a change in RF frequency (harmonic)

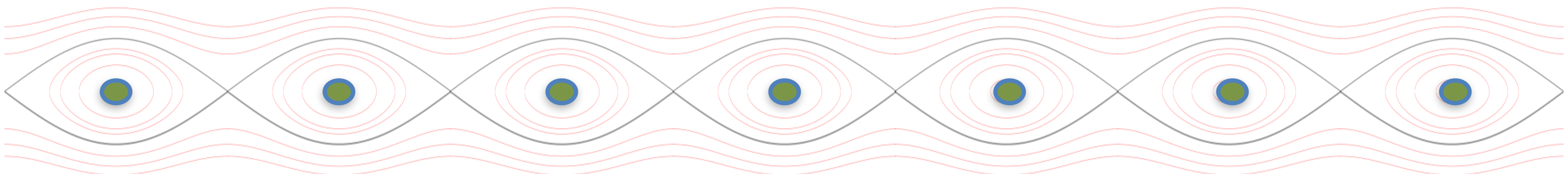


switch off high frequency, low voltage system,
switch on low frequency, high voltage system…

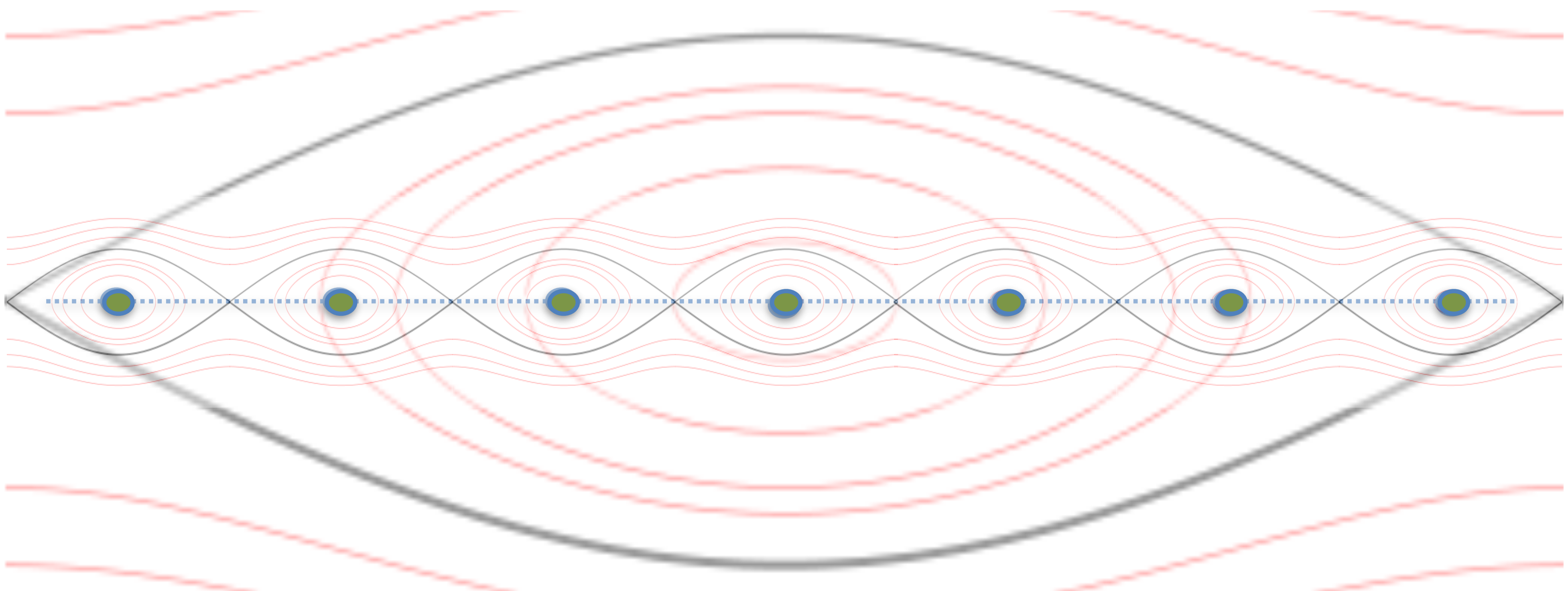


Bunch Coalescing

similar to bunch rotation, but also involves a change in RF frequency (harmonic)

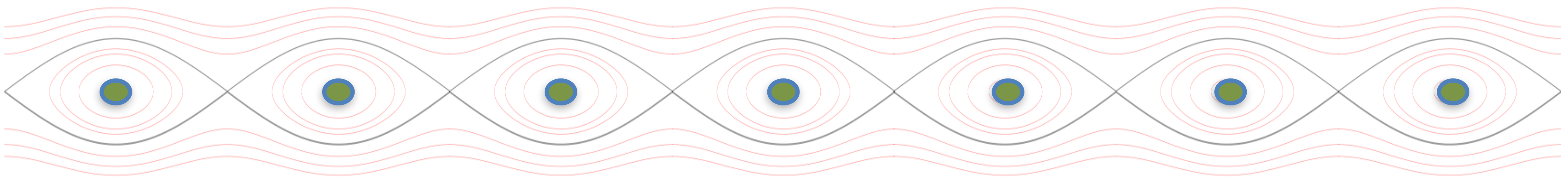


switch off high frequency, low voltage system,
switch on low frequency, high voltage system…

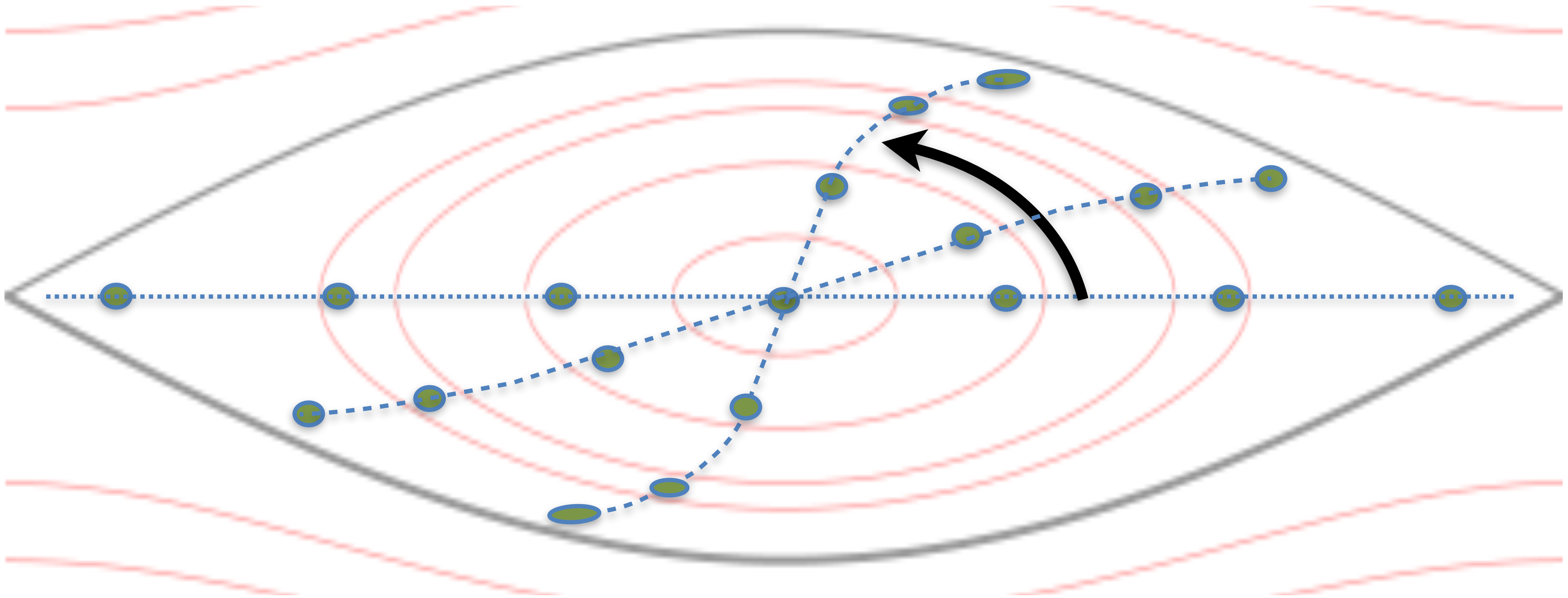


Bunch Coalescing

similar to bunch rotation, but also involves a change in RF frequency (harmonic)



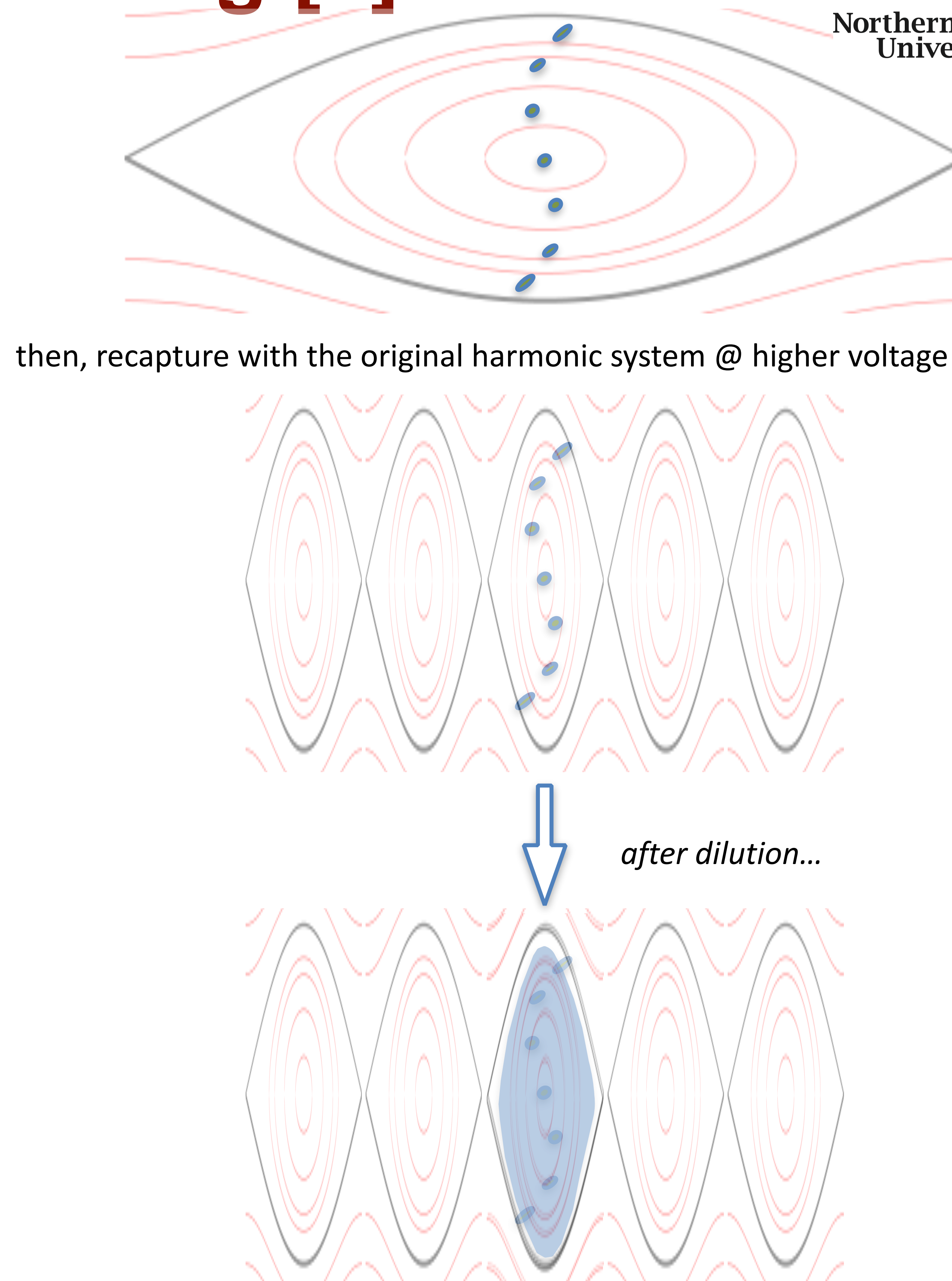
switch off high frequency, low voltage system,
switch on low frequency, high voltage system…



Bunch Coalescing [2]

- Can use coalescing techniques to take bunched beam from one accelerator, make intense bunches, and inject into downstream accelerators to increase bunch intensities

  - downside: increased longitudinal emittance



Ex: Muon g-2 Bunch Formation
(approximately)

21 bunches from Booster: