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A Beam Line Design Example

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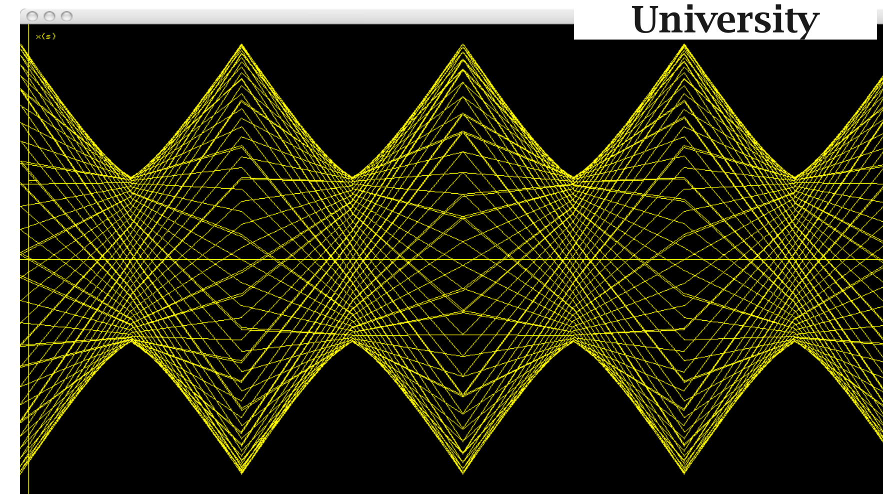
Review

Summary



$$x'' + K(s)x = 0$$ Hill's Equation

trial solution: $x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$

requires:

$$\psi' = 1/\beta \quad \longrightarrow \quad \psi(s) = \int \frac{ds}{\beta(s)}$$

and

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \quad \longrightarrow \quad \beta'' + 4K\beta = const.$$

(for $K' = 0$)

for $K = 0$: $\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$

for $K > 0$: $\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$

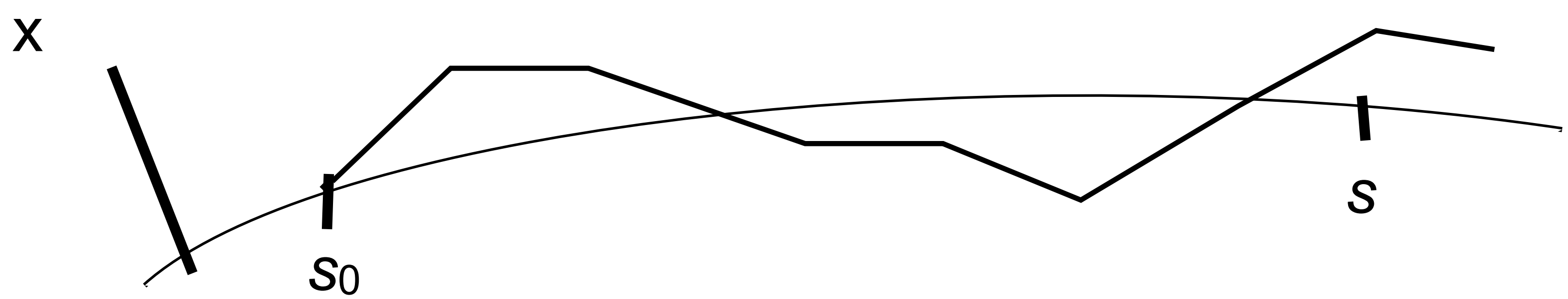
Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle $\Delta x' = \Delta\theta \equiv x_0'$
- Then, downstream, we have

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

or,

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin[\psi(s) - \psi_0]$$



Example:
Suppose $\Delta\theta = 0.4$ mrad, $\beta_0 = 4.0$ m, $\beta(s) = 6.4$ m, and $\Delta\psi = n \times 2\pi + 30°$. Then $x(s) = 1$ mm.

FODO Cell Courant-Snyder Parameters

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- From the matrix: $$M_{periodic} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4 numbers

$$traceM = a + d = 2 - L^2/F^2 = 2\cos\mu \quad \Rightarrow \quad \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1 + L/2F}{1 - L/2F}} \qquad \alpha = \frac{a - d}{2\sin\mu} = \sqrt{\frac{1 + L/2F}{1 - L/2F}}$$

- If go from D quad to D quad, simply replace *F --> -F* in matrix *M* above
  - So, at exit of the D quad:

$$\beta = 2F\sqrt{\frac{1 - L/2F}{1 + L/2F}} \qquad \alpha = -\sqrt{\frac{1 - L/2F}{1 + L/2F}}$$

for completeness,

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

The Stability Criterion

- Through an *eigenvector* approach, one can solve for the eigenvalues of the repetitive matrix *M* and find that for stability, must have

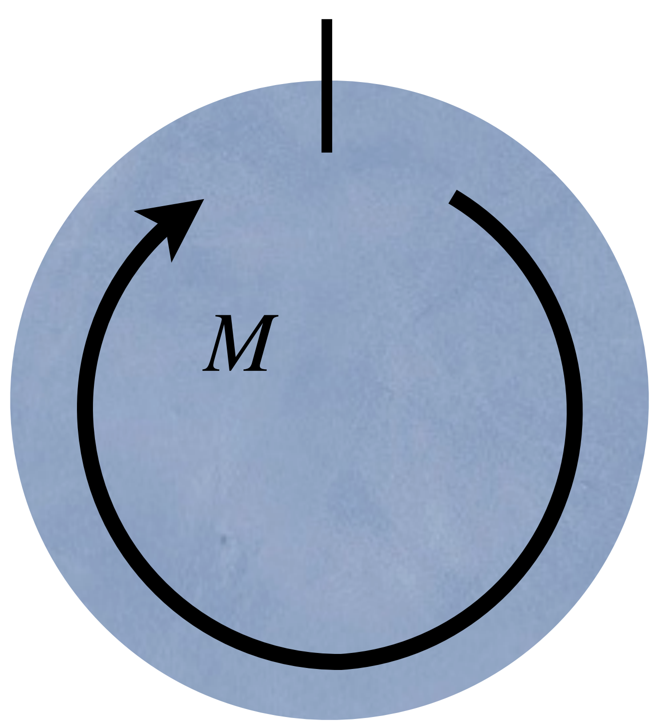
$$|\,tr\ M\,| < 2$$

The Stability Criterion

$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$a + d = trM =$ "trace" of M

$$ad - bc = \det M = 1$$



repeat application of matrix *M* over and over…
the motion will be stable if -2 < tr*M* < 2

The motion $\vec{X} = M^k \vec{X}_0$ is finite and bounded as $k \to \infty$ if $|\,trM\,| < 2$

The Betatron Tune

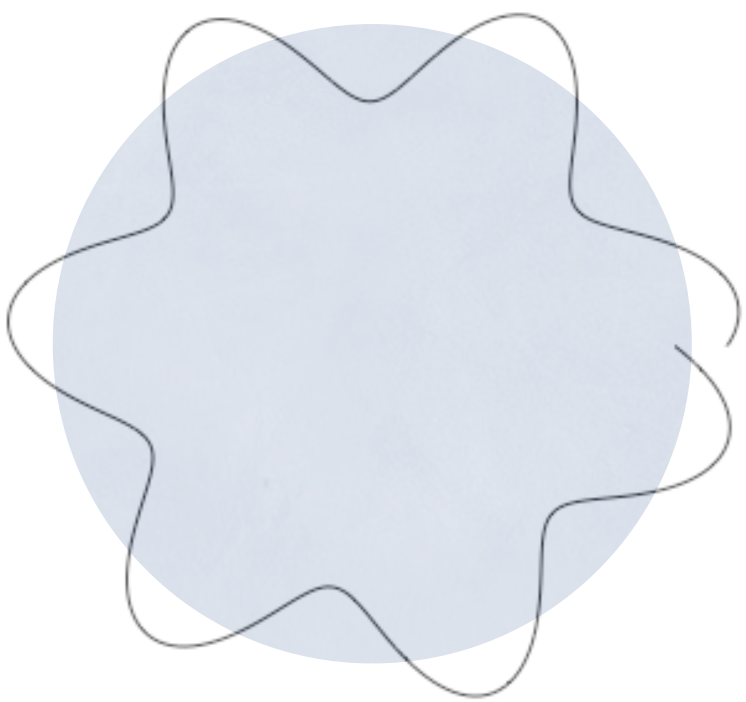
- In a cyclic accelerator (synchrotron), the particles will oscillate transversely (betatron oscillations) with the *betatron frequency*.
- The betatron frequency is determined by the total phase advance once around the ring:

$$\Delta\psi_{total} = \oint \frac{ds}{\beta(s)}$$

$$\nu \equiv \Delta\psi_{total}/2\pi$$

$$\mathrm{tr}M = 2\cos(2\pi\nu)$$

(*M* = 1-turn matrix)



Betatron Tune: # of oscillations per revolution

$$f_{betatron} = \nu f_{rev}$$

Generating Dispersion

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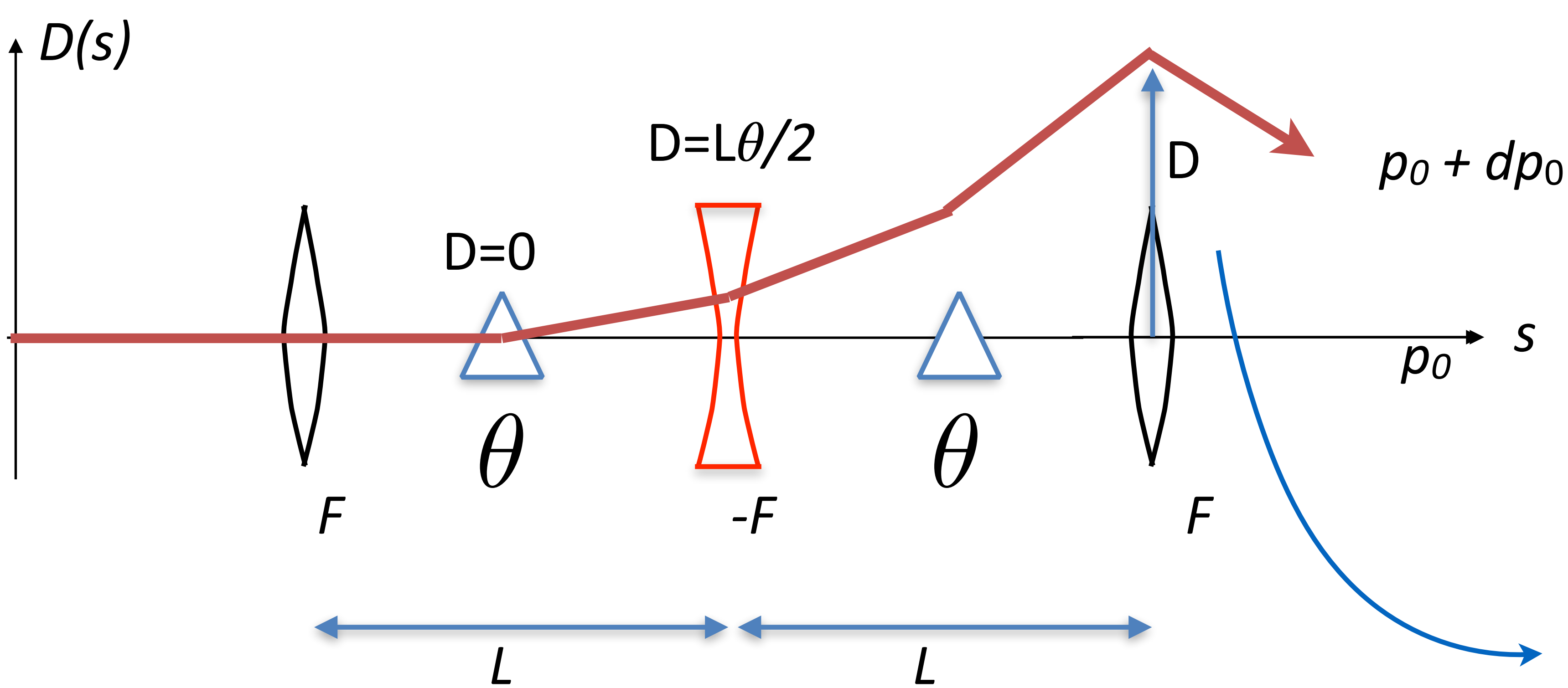
- System of quadrupoles with alternating-sign gradients (F, D, F, …) separated by distance *L*, and with bending magnets present…



$$D = \frac{L\theta}{2}\left(1 + \frac{L}{2f}\right)$$

Ex: D = 3 m, dp/p = 0.3%, then Δx = 9 mm

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New…

Adiabatic Damping from Acceleration

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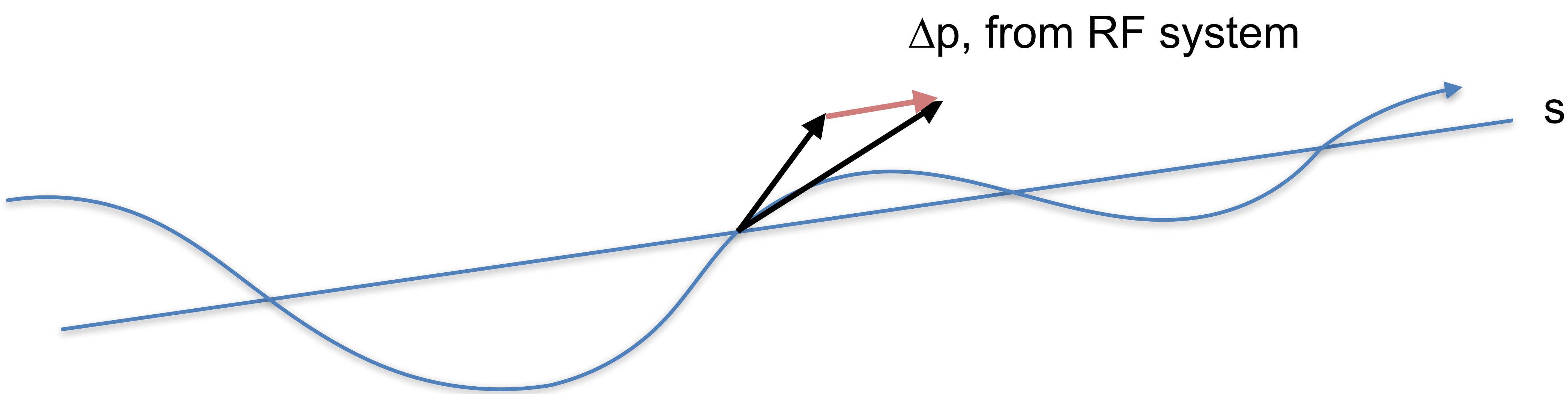
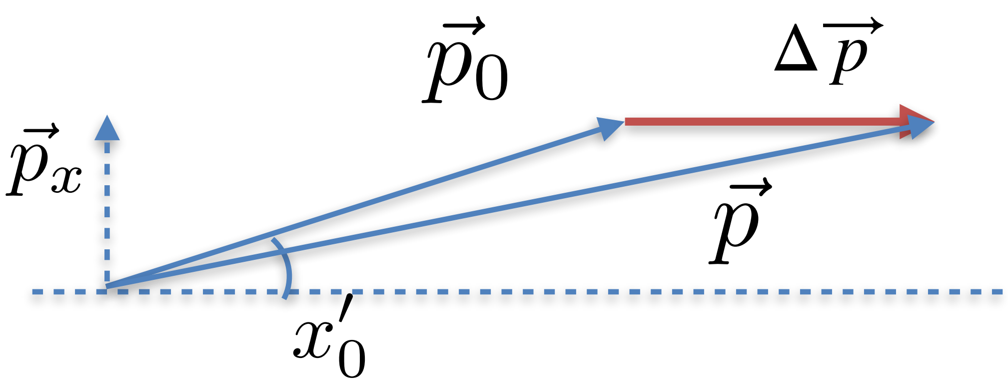
- Transverse oscillations imply transverse momentum. As accelerate, momentum is "delivered" in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.



- The coordinates *x-x'* are not canonical conjugates, but x-p_x are; thus, from classical mechanics, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Adiabatic Damping from Acceleration

details…



Note: assuming that ALL particles receive the same *Δp* from the cavity

$$x' \approx \frac{p_x}{p} \approx \frac{p_x}{p_0 + \Delta p} = \frac{p_x}{p_0}\left(1 + \frac{\Delta p}{p_0}\right)^{-1} \approx x'_0\left(1 - \frac{\Delta p}{p_0}\right)$$

Note: particles at peak of their betatron oscillation will have little/no change in *x'*, while particles with large transverse angles will have their *x'* affected most

relative momentum gain from RF system

$$\Longrightarrow \quad \Delta x' = -x'_0 \, \frac{\Delta p}{p_0}$$

Damping of Oscillations

Now assuming that the incremental change in momentum is small compared to the overall momentum, and that the changes occur gradually on time scales large compared to those of the betatron motion…

$$x = x_0$$
$$x' = x_0'(1-\delta) \qquad \delta \equiv \Delta p / p_0$$

Then if the original emittance is $\epsilon_0 = \pi\sqrt{\langle x_0{}^2\rangle\langle x_0'^2\rangle - \langle x_0 x_0'\rangle^2}$ then, after a single pass through the RF system,

$$\epsilon = \pi\sqrt{\langle x_0{}^2\rangle\langle x_0'^2(1-\delta)^2\rangle - \langle x_0 x_0'\rangle^2(1-\delta)^2} = \epsilon_0\sqrt{(1-2\delta+\delta^2)} \approx \epsilon_0(1-\delta)$$

Thus,

$$\epsilon - \epsilon_0 = \Delta\epsilon = -\epsilon_0\delta$$

or,

$$\frac{\Delta\epsilon}{\epsilon_0} = -\frac{\Delta p}{p_0}$$

So,

$$\epsilon \propto \frac{1}{p} \qquad x_{rms} \propto \frac{1}{\sqrt{p}}$$

Normalized Beam Emittance

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- Hence, as particles are accelerated, the geometrical area (emittance) in *x-x'* phase space is not preserved; however, the product of the emittance and the total momentum **is** preserved. Thus, we define a "normalized" beam emittance, as

$$p = (\gamma m)(\beta c) = (\beta\gamma)\ mc$$

Here: *relativistic* factors!

$$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)_{Lorentz}$$

- **In principle**, the *normalized* beam emittance should be preserved during acceleration. Thus it is a measure of beam quality, and its preservation is a measure of accelerator performance.
- **In practice**, it is not preserved -- non-adiabatic acceleration, especially at the low energy regime; non-linear field perturbations; residual gas scattering; charge stripping; field errors, setting errors; *etc.* -- all contribute at some level to increase the beam emittance. Best attempts are made to keep the emittance as small as possible.

95%, normalized emittance:

$$\epsilon_N = \frac{6\pi\ x_{rms}^2(s)}{\beta(s)}\ (\beta\gamma)_{Lorentz}$$

(for the case of no dispersion)

Linear and Nonlinear Imperfections and Adjustments

Mike Syphers

Operator Lectures

April 2020

Errors and Their Corrections

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- Steering (dipole) Errors
- Focusing (quadrupole) Errors
- Chromatic (momentum) Effects
- Nonlinear Motion and Resonances

- Not only will errors create perturbations in the beam size, etc., but they will also tend to identify operational considerations, such as frequency choices, corrector placement, alignment tolerances, power supply specifications, etc.

Steering (dipole) Errors

- dipole field error:
  - manufacturing; powering; control setting, …

$$B_y = B_0 \quad \longrightarrow \quad B_y = B_0 + \Delta B$$

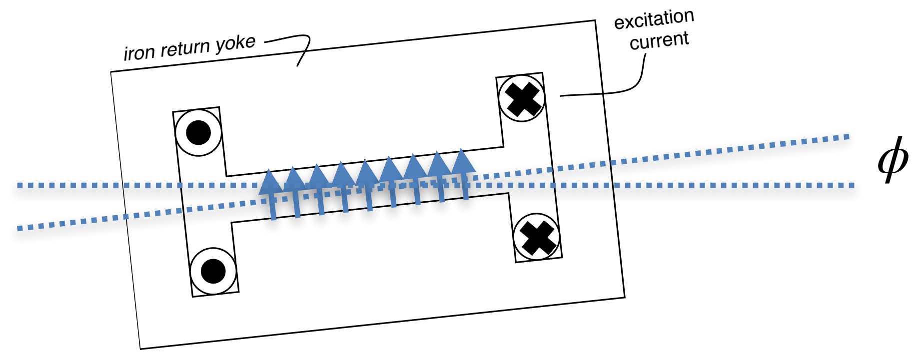
$$\Delta x' = -\frac{\Delta B \ell}{B\rho}$$

- dipole field "roll" (about the longitudinal axis)

$$B_y = B_0, \quad B_x = 0$$

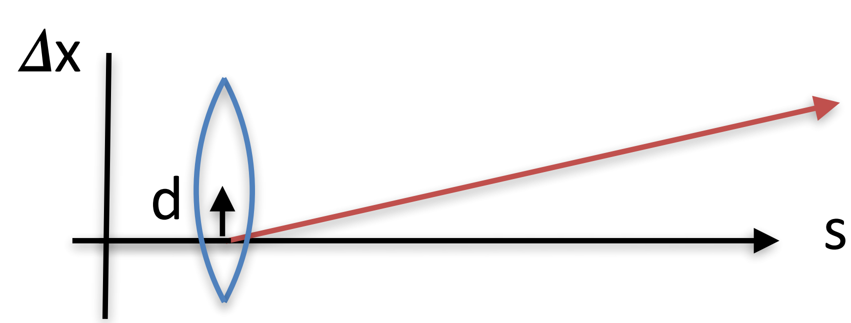
$$\rightarrow \quad B_y = B_0 \cos\phi \approx B_0$$

$$B_x = -B_0 \sin\phi \approx -\phi\, B_0$$



$$\Delta y' = -\phi \frac{B_0 \ell}{B\rho} = -\phi\theta_0$$

- Quadrupole misalignment:



$$\Delta x' = \frac{d}{F}$$

Steering (dipole) Errors

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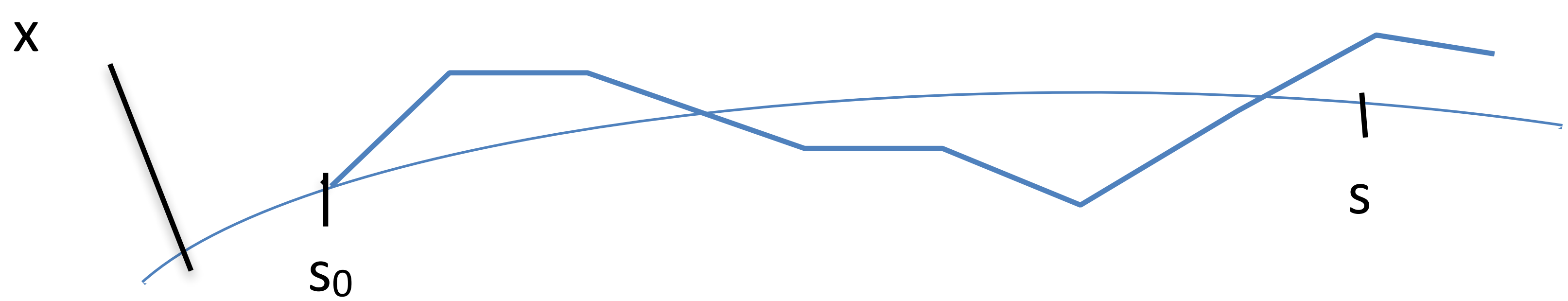
- A field error creates a betatron oscillation…

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$



due to the small error field:

$$\Delta x' = x_0' = \Delta\theta$$

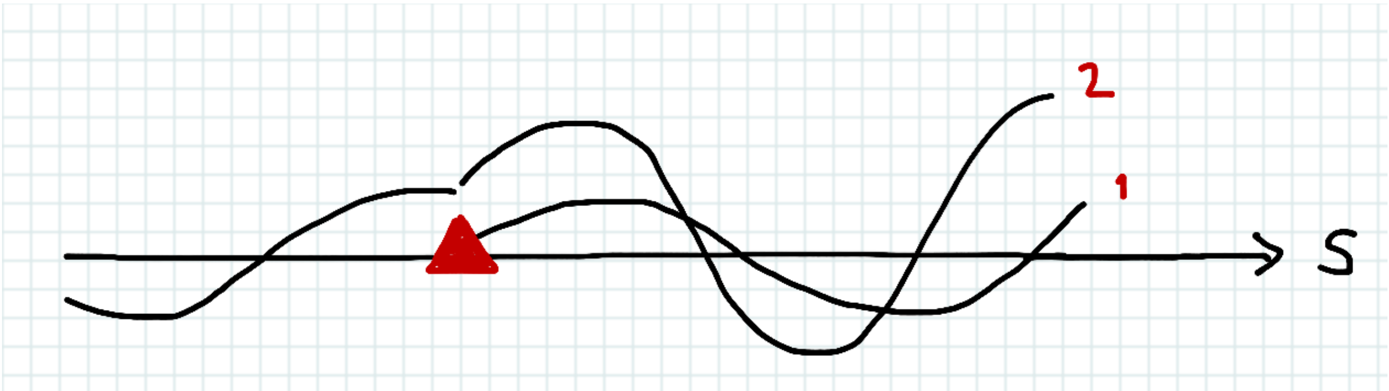
$$\Delta x = x_0 = 0$$

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi$$

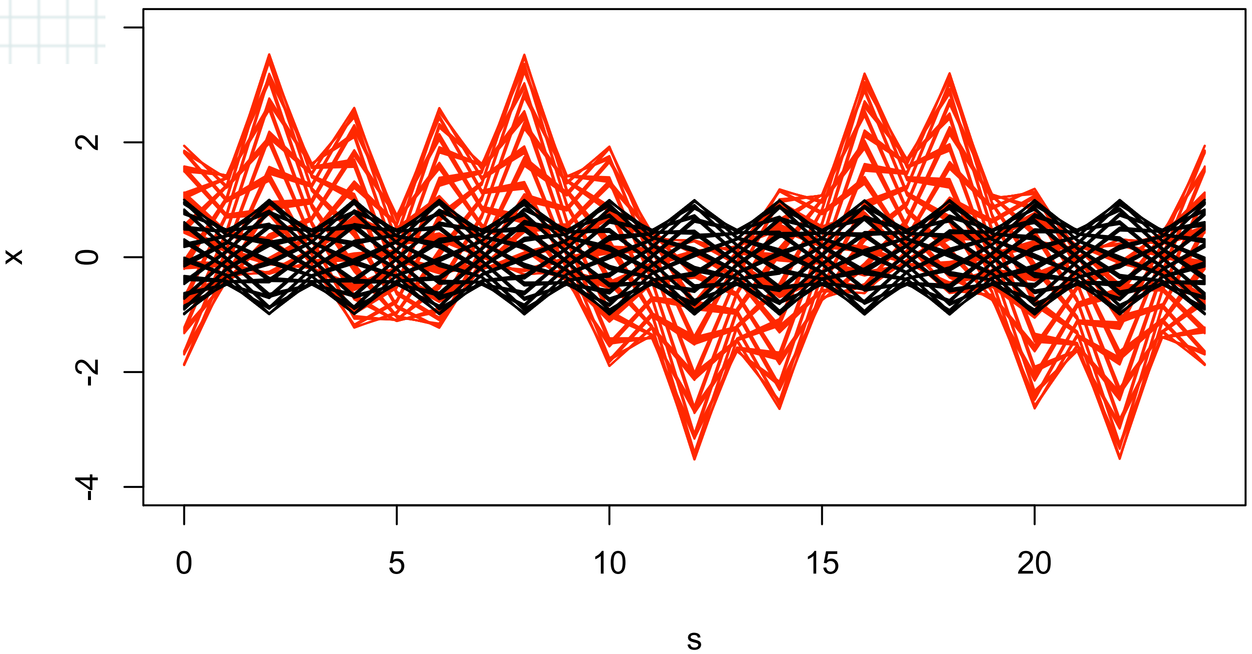
Steering (dipole) Errors

- Closed orbit distortions in a circular accelerator
  - These are not "one-time" kicks; they affect the particle motion every revolution

see ClosedOrbit.R



black = nominal
red = w/ error field



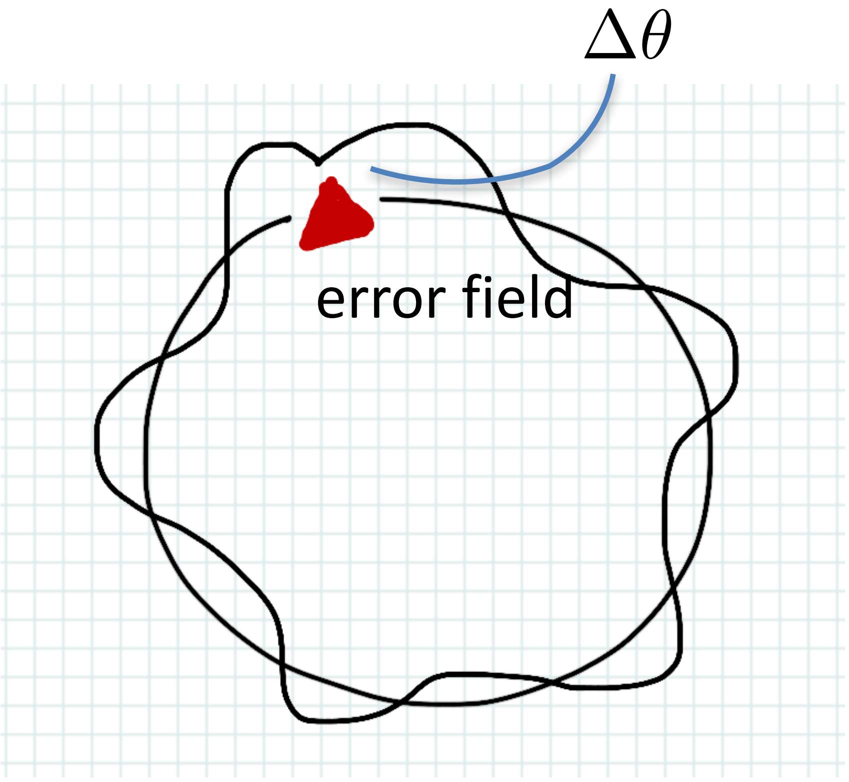
The trajectory of each particle will be altered by the angle $\Delta\theta$ every time it passes through the error field

The Closed Orbit

- Want to find the one trajectory which, upon passing through the error field, will come back upon itself
  - this is the "closed" trajectory, or closed orbit

$$M_0 \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\longrightarrow \qquad \begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M_0)^{-1} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$



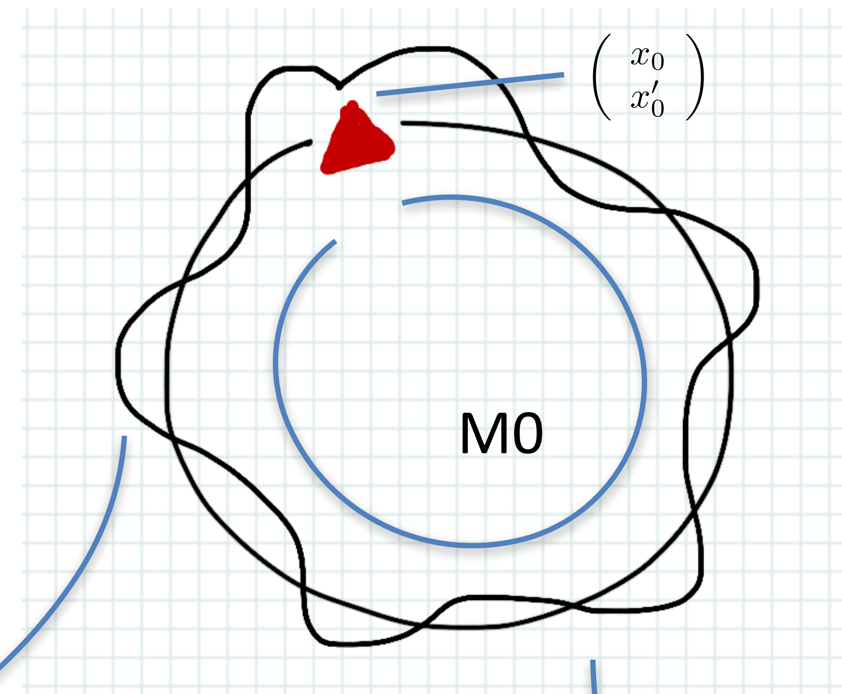
- When find x_0, x'_0, can find x, x' downstream:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

Closed Orbit Distortion from Single Error

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \frac{\Delta\theta}{2\sin\pi\nu}\begin{pmatrix} \beta_0\cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix}\begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$



$$\Delta x(s) = \frac{\Delta\theta\sqrt{\beta_0\beta(s)}}{2\sin\pi\nu}\cos\left[|\psi(s)-\psi_0| - \pi\nu\right]$$

ν oscillations

as ν —> integer, huge distortions

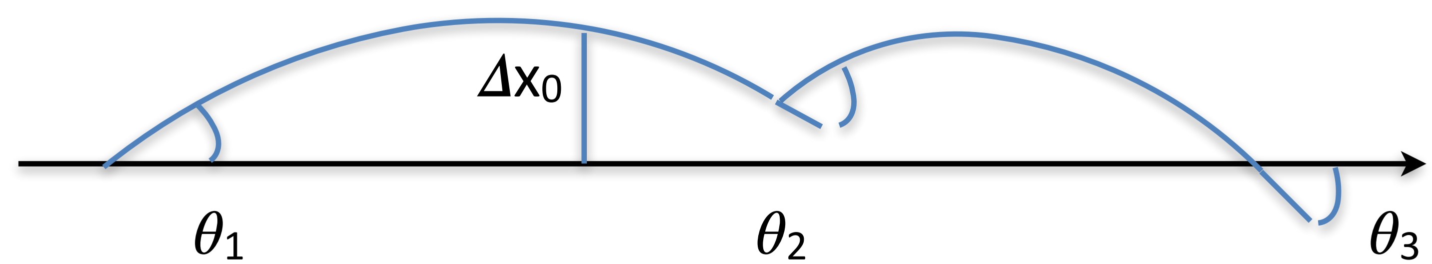
a resonance!

If have a collection of errors about the accelerator, then at any one point:

$$\Delta x(s) = \sum_i \frac{\Delta\theta_i\sqrt{\beta_i\beta(s)}}{2\sin\pi\nu}\cos\left[|\psi(s)-\psi_i| - \pi\nu\right]$$

Trajectory/Orbit Correction

- To make a local adjustment or correction of the position of the beam in a beam line or synchrotron, three correctors are required (in general):



$$\theta_1 = \frac{\Delta x_0}{\sqrt{\beta_0 \beta_1} \sin \psi_{10}}$$

$$\theta_2 = -\theta_1 \sqrt{\frac{\beta_1}{\beta_2}} \frac{\sin \psi_{13}}{\sin \psi_{23}}$$

The trajectory before θ_1 and after θ_3 is left undisturbed

$$\theta_3 = \theta_1 \sqrt{\frac{\beta_1}{\beta_3}} \frac{\sin \psi_{12}}{\sin \psi_{23}}$$

Orbit Corrections

- As an example, in a "FODO" synchrotron, one would place correctors near the location of each quadrupole — at maximum beta locations, and at the source of likely steering errors (misaligned quads)

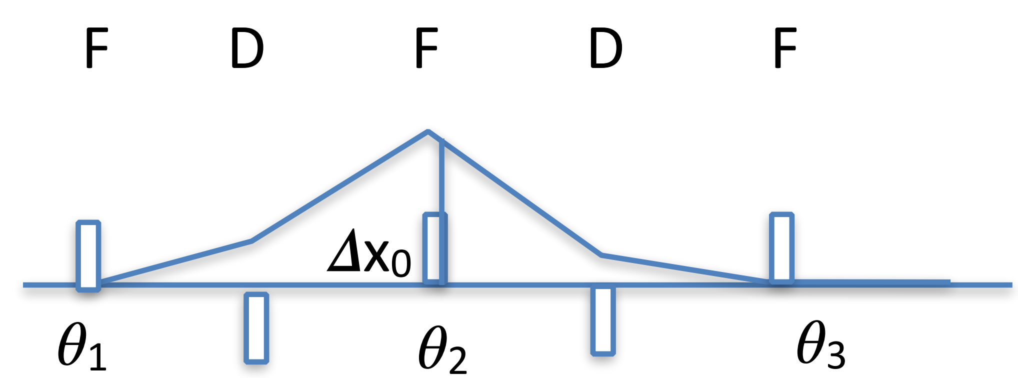
$$\sin(\mu/2) = \frac{L}{2F}$$

$$\psi_{13} = 2\psi_{12} = 2\psi_{23} = 2\mu$$

$$\theta_1 = \frac{\Delta x}{\hat{\beta}\sin\mu}$$

$$\theta_2 = -\theta_1 \frac{\sin 2\mu}{\sin\mu} \quad \Rightarrow \quad \theta_2 = -2\theta_1 \cos\mu$$

$$\theta_3 = \theta_1$$



Focusing (gradient) Errors

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- Sources of gradient focusing errors
  - Quadrupole magnet field error
    - » powering error; control error; manufacturing error
  - Dipole pole tip error (non-parallel poles)
  - etc.

- Impact of gradient errors
  - Look at Hill's Equation: $x'' + K(s)x = 0$
    - » errors in the values of *K* will alter…
      - phase advance (tune, or betatron frequency)
      - amplitude function, β

Focusing (quadrupole) Errors

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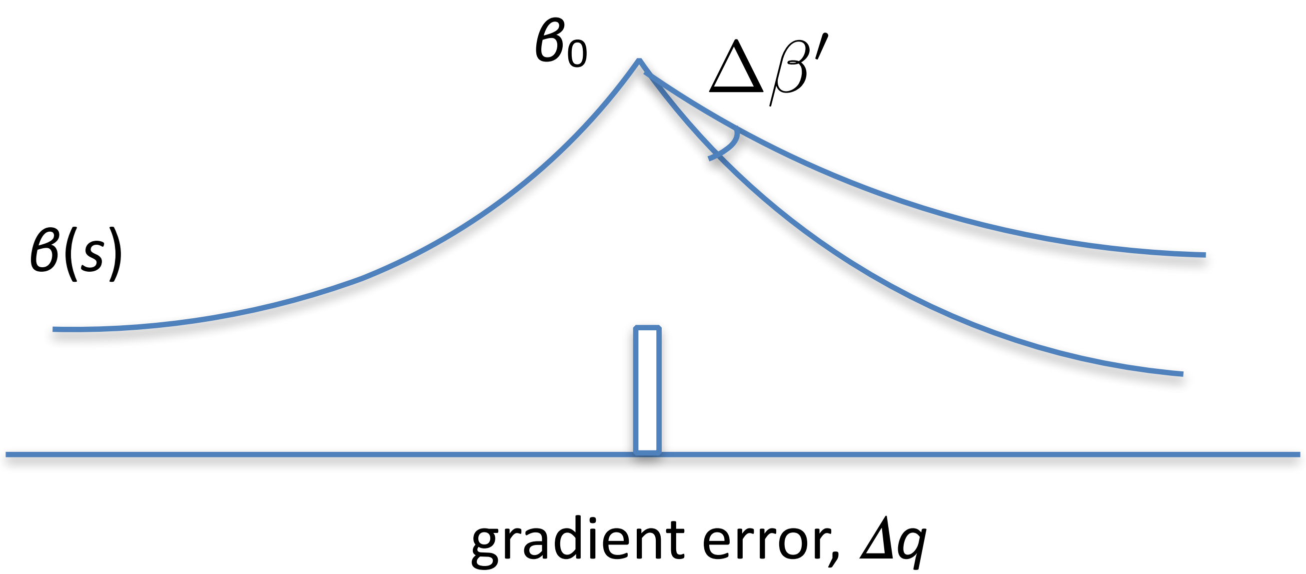
- β, α distortions and "beta-beat"



$$\Delta\alpha = -\frac{1}{2}\Delta\beta'$$

if ideal gradient produces strength $q = B'\ell/(B\rho)$,
then a gradient error will produce $\Delta q = \Delta B'\ell/(B\rho)$
and the slope of β will change according to

$$\Delta\alpha = \beta_0 \Delta q$$

Downstream the distortion will propagate:

$$\frac{\Delta\beta}{\beta}(s) \approx -\Delta q \beta_0 \sin 2\psi_0(s)$$

β Distortion in a Synchrotron

- In a circular accelerator, the closed solution of the amplitude function(s) will be altered by a gradient error. With analysis similar to the situation for a closed orbit distortion, the gradient error will produce a "closed" β-distortion all around the ring according to (for small errors):

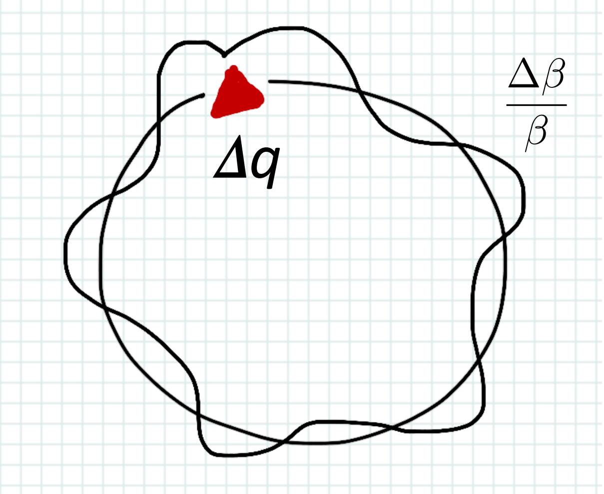
dipole error:

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin\Delta\psi$$

quad error:

$$\frac{\Delta\beta}{\beta}(s) \approx -\Delta q\beta_0 \sin 2\psi_0(s)$$

$$\frac{\Delta\beta}{\beta}(s) \approx -\frac{\Delta q\beta_0}{2\sin 2\pi\nu}\cos(2|\Delta\psi| - 2\pi\nu)$$



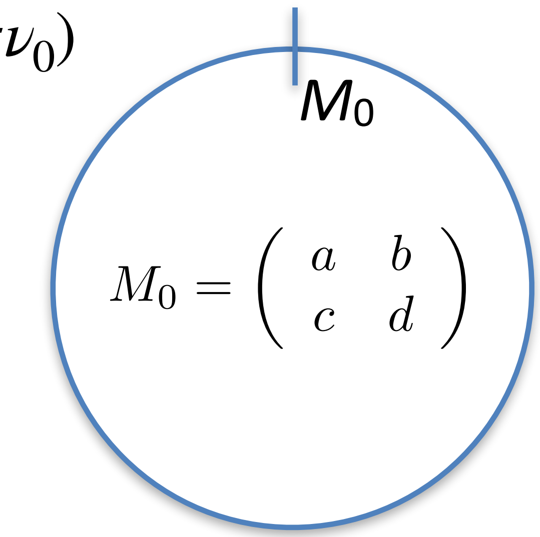
Focusing (quadrupole) Errors

- Phase/tune shift
  - a gradient error will distort the amplitude function, and therefore distort the development of the phase advance downstream. As the β distortion will oscillate about the ideal β function, the phase advance will slightly increase and decrease along the way. This is particularly important in a ring where the betatron tune, ν, might need fine control.
  - To see the change in tune for a synchrotron, we look at the effect on the matrix for one revolution…

The Tune Shift Formula

- M_0 is the one-turn matrix of ideal ring

$$\mathrm{tr}M_0 = 2\cos(2\pi\nu_0)$$



The Tune Shift Formula

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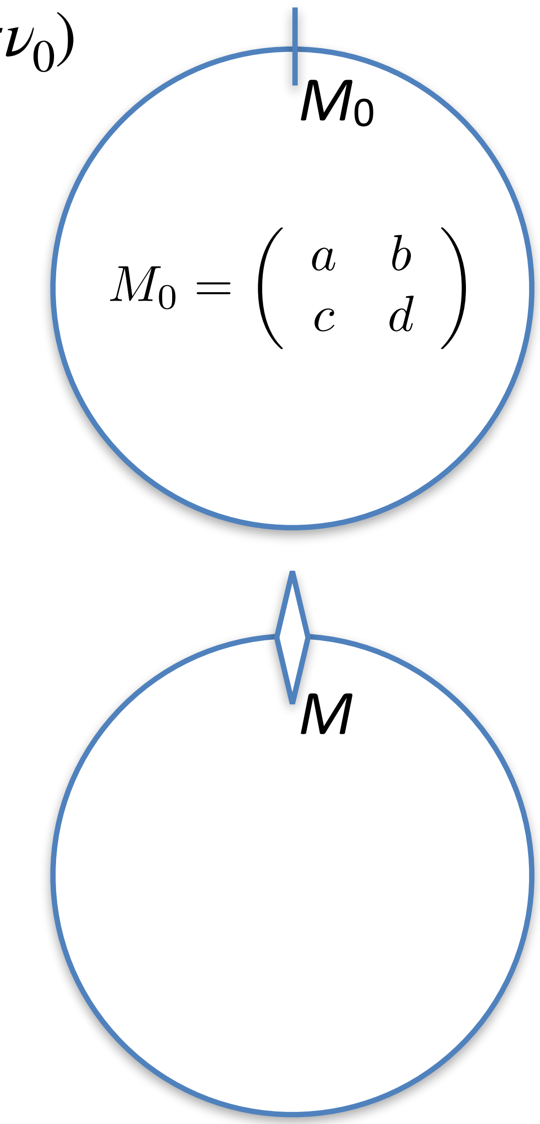
- M_0 is the one-turn matrix of ideal ring
- M is the one-turn matrix of the ideal ring followed by a small gradient error of strength q:

$$\mathrm{tr} M_0 = 2\cos(2\pi\nu_0)$$

$$M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0$$

$$\text{then } M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c - aq & d - bq \end{pmatrix}$$



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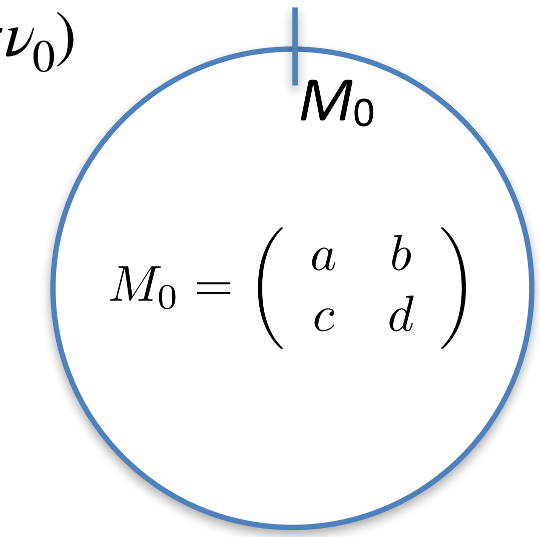

- M is the one-turn matrix of the ideal ring followed by a small gradient error of strength q:

$$M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0$$

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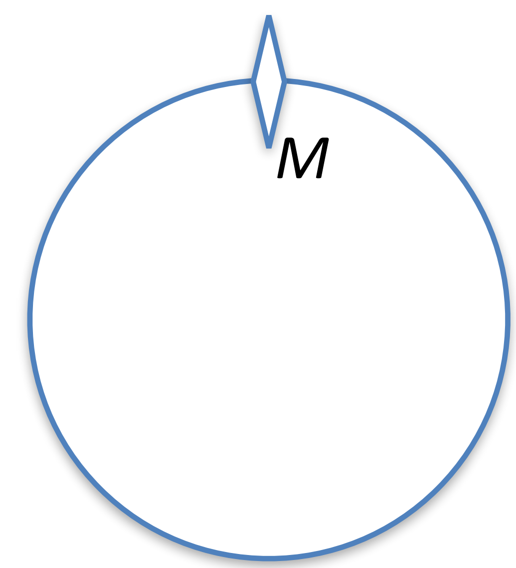

$$\mathrm{trace}M = 2\cos 2\pi\nu = a + d - bq = \mathrm{trace}M_0 - bq = 2\cos 2\pi\nu_0 - (\beta_0 \sin 2\pi\nu_0)q$$

The Tune Shift Formula

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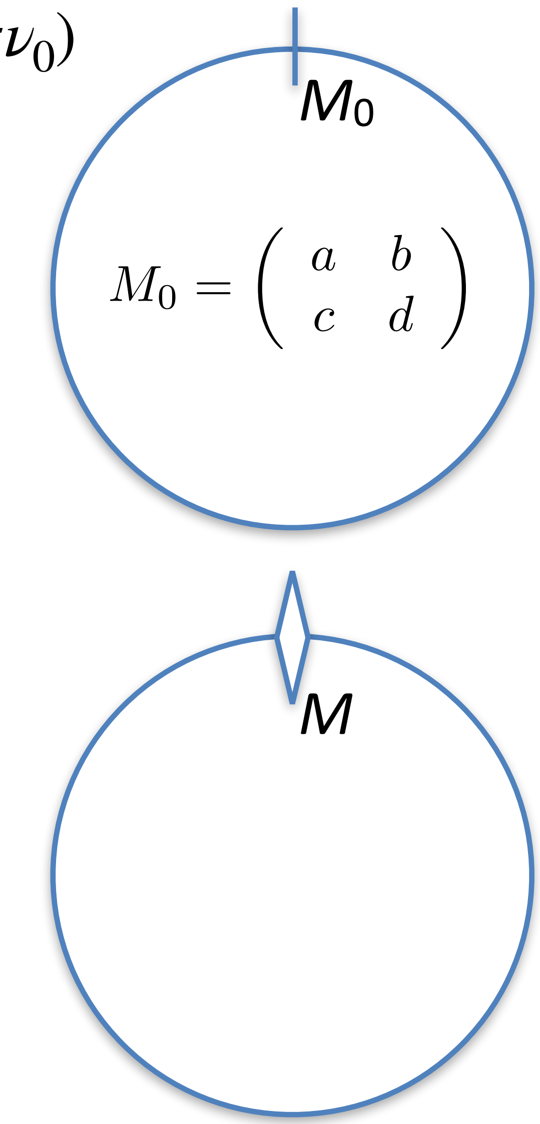
- M_0 is the one-turn matrix of ideal ring
- M is the one-turn matrix of the ideal ring followed by a small gradient error of strength q:

$$\mathrm{tr} M_0 = 2\cos(2\pi\nu_0)$$



$$M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0$$

$$\text{then } M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c - aq & d - bq \end{pmatrix}$$

$$\mathrm{trace} M = 2\cos 2\pi\nu = a + d - bq = \mathrm{trace} M_0 - bq = 2\cos 2\pi\nu_0 - (\beta_0 \sin 2\pi\nu_0) q$$

$$\boxed{\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2} q\beta_0 \sin 2\pi\nu_0}$$

$$\begin{aligned} \cos 2\pi\nu &= \cos 2\pi(\nu_0 + \Delta\nu) \\ &= \cos 2\pi\nu_0 \cos 2\pi\Delta\nu - \sin 2\pi\nu_0 \sin 2\pi\Delta\nu \\ &\approx \cos 2\pi\nu_0 - 2\pi\Delta\nu \sin 2\pi\nu_0 \end{aligned}$$

The Tune Shift Formula

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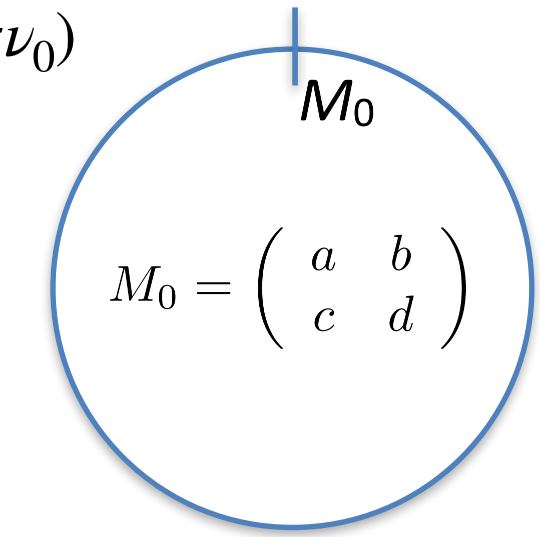
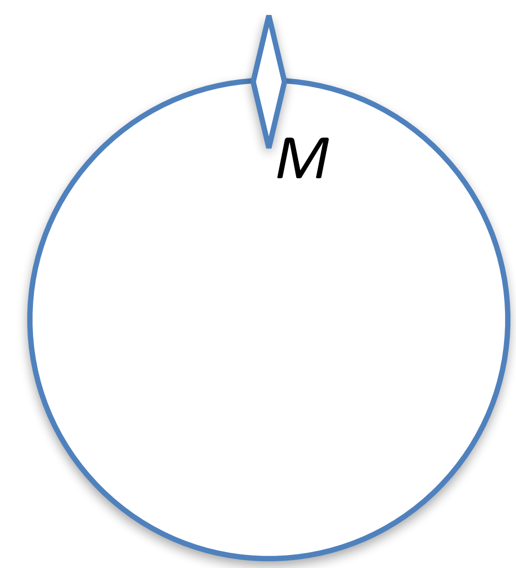
- M_0 is the one-turn matrix of ideal ring

$$\mathrm{tr} M_0 = 2\cos(2\pi\nu_0)$$



- M is the one-turn matrix of the ideal ring followed by a small gradient error of strength q:

$$M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} M_0$$



$$\text{then } M = \begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c - aq & d - bq \end{pmatrix}$$

$$\mathrm{trace} M = 2\cos 2\pi\nu = a + d - bq = \mathrm{trace} M_0 - bq = 2\cos 2\pi\nu_0 - (\beta_0 \sin 2\pi\nu_0) q$$

$$\boxed{\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2} q\beta_0 \sin 2\pi\nu_0}$$

$$\begin{aligned} \cos 2\pi\nu &= \cos 2\pi(\nu_0 + \Delta\nu) \\ &= \cos 2\pi\nu_0 \cos 2\pi\Delta\nu - \sin 2\pi\nu_0 \sin 2\pi\Delta\nu \\ &\approx \cos 2\pi\nu_0 - 2\pi\Delta\nu \sin 2\pi\nu_0 \end{aligned}$$

$$2\pi\Delta\nu \sin 2\pi\nu_0 \approx \frac{1}{2} q\beta_0 \sin 2\pi\nu_0$$

$$\boxed{\Delta\nu \approx \frac{1}{4\pi}\beta_0 q}$$

Tune correction/adjustment

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- In the same way that an error will change the tune of a synchrotron, so can a quadrupole field adjustment be made to implement a desired change in the tune
- Note, however, that a quad change will alter the horizontal tune in one direction, but will alter the vertical tune in the other direction. Also, since the amplitude functions, β_x and β_y, may be different, the actual shifts in the two tunes will also be different in magnitude.
- Thus, to exercise independent control of ν_x and ν_y, there needs to be two quadrupoles (or 2 circuits)

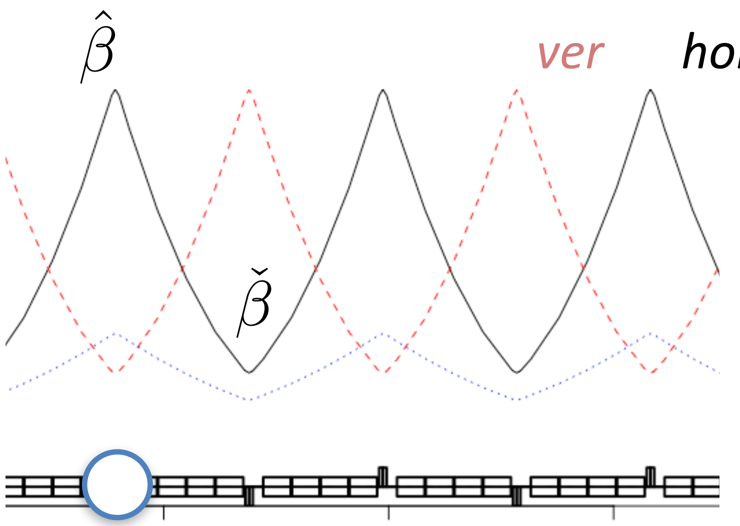
Tune correction/adjustment

- Suppose we have a FODO arrangement, and we put adjustable quadrupoles near every "main" quadrupole (*N* = # quads):

$$\Delta q \equiv \frac{\Delta B'\ell}{B\rho}$$

$$\Delta\nu_x = \frac{1}{4\pi}\hat{\beta}\Delta q_1$$

$$\Delta\nu_y = -\frac{1}{4\pi}\check{\beta}\Delta q_1$$



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$$\Delta q \equiv \frac{\Delta B'\ell}{B\rho}$$

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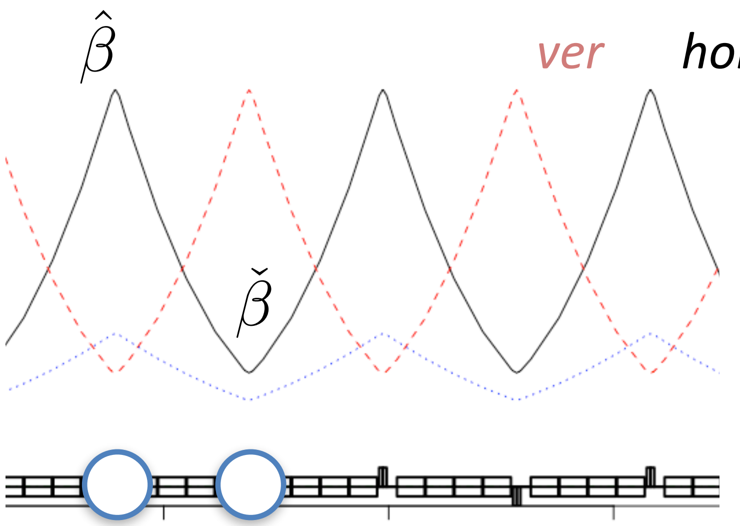
$$\Delta\nu_y = -\frac{1}{4\pi}\left[\check{\beta}\Delta q_1 + \hat{\beta}\Delta q_2\right]$$



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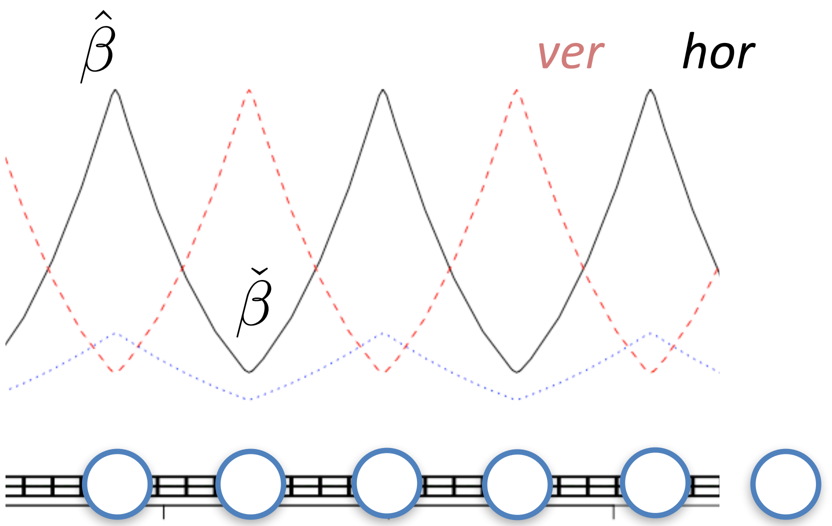

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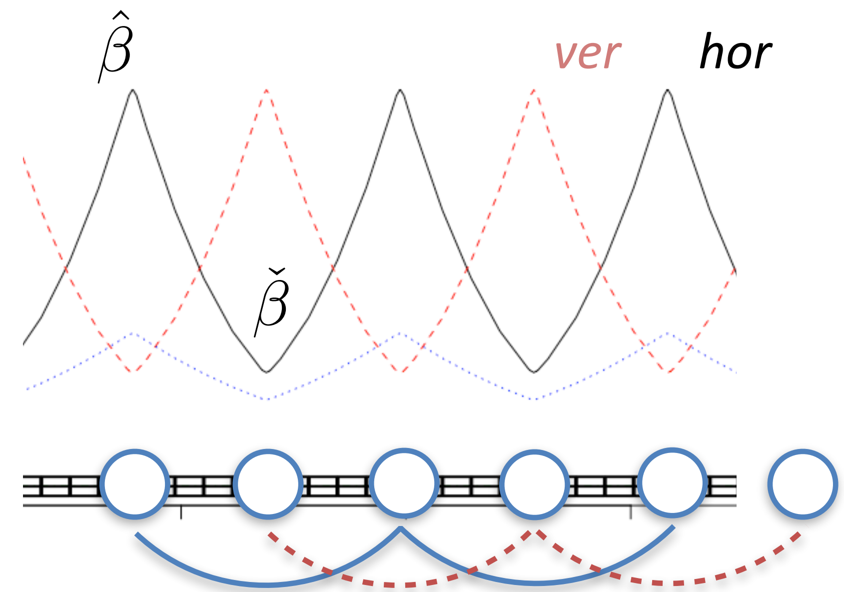

- The quadrupoles can be wired in two separate circuits, and thus the two tunes can be independently adjusted by any (reasonable) amount desired.

Chromatic Effects

- We may think of dispersion (and the Dispersion function) as being the propagation of a steering error, where the error was introduced due to $\Delta p/p$.

- $\Delta p/p$ will similarly introduce gradient "errors"
  - thus, expect the tune to depend upon $\Delta p/p$
  - and, expect the amplitude function $\beta = \beta(\Delta p/p)$

- Some Examples
  - Chromatic Aberration in a final focus
  - Tune spread in a synchrotron due to momentum — chromaticity

Chromaticity of a Circular Accelerator

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- Chromaticity -- change in the betatron tune, ν, with respect to relative momentum deviation ($\Delta p/p$):

$$x'' + K(s)x = x'' + \frac{qB'(s)}{p}x = 0$$

$$\xi \equiv \frac{\Delta \nu}{\Delta p/p}$$

- There will be a different chromaticity value for each degree of freedom:

$$\xi_x = \frac{\Delta \nu_x}{\Delta p/p}$$

$$\xi_y = \frac{\Delta \nu_y}{\Delta p/p}$$

How to estimate the scale of the effect?

The Natural Chromaticity

- While there may be error fields that contribute to chromatic effects (sextupole fields — later), there will be a "natural" chromaticity due to the ideal magnets of the synchrotron lattice
- Starting from $\Delta\nu = \frac{1}{4\pi}\beta\Delta q$ for a single gradient error,

$$\Delta q \equiv \frac{\Delta B'\ell}{B\rho}$$

or,

$$\Delta q \equiv \Delta\left(\frac{eB'\ell}{p}\right)$$

$$= -\left(\frac{eB'\ell}{p}\right)\frac{\Delta p}{p}$$

$$= -\left(\frac{B'\ell}{B\rho}\right)\frac{\Delta p}{p}$$

$$\Delta\nu = \int \frac{1}{4\pi}\beta(s)\left[-\frac{B'(s)}{B\rho}\frac{\Delta p}{p}\right]ds$$

$$\xi = -\frac{1}{4\pi}\int \beta(s)K(s)ds$$

$$\xi \equiv \frac{\Delta\nu}{\Delta p/p}$$

Can show that for a FODO-style lattice, $\xi \approx -\nu$

Chromatic Corrections

- Example: suppose synchrotron has ξ = -10, and the beam has a momentum spread of ±0.1%; then the particle distribution will have a spread in tunes between $\nu_0 \mp 0.01$.
- In order to ensure that all particles have the same tunes (hor/ver), within tolerable levels, need to be able to adjust the overall chromaticity of the ring.
- Desire focusing element with a focusing strength that depends on momentum (linearly, preferably).
- This can be accomplished using *sextupole* fields in regions with horizontal dispersion.

Chromatic Corrections

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- Sextupole Field:

$$B_y = \frac{1}{2}B''(x^2 - y^2)$$

$$B_x = B''xy$$

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} = B''x$$ gradient

So here, if "x" is due to Dispersion: $x = D\frac{\Delta p}{p}$

$$\Delta\nu = \frac{1}{4\pi}\beta/f$$

then, $$\frac{1}{f} = \frac{(\partial B_y/\partial x)\ell}{B\rho} = \frac{B''\ell}{B\rho} \cdot D\,\frac{\Delta p}{p}$$

$$\Delta\xi = \frac{1}{4\pi}\beta\; D\,\frac{B''\ell}{B\rho}$$

$\ell =$ length of the sextupole field

Note: since $\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \propto D \cdot \frac{\Delta p}{p}$, provides focusing in one plane, defocusing in the other plane

Thus, need 2 sextuples (or 2 families of sextuples) for optimal independent corrections/adjustment of ξ_x, ξ_y.

Also Note: introduces (intentionally!) a non-linear field!!

Correction/Adjustment of Chromaticity

- Suppose we have a FODO arrangement, and we put adjustable sextuple magnets near every "main" quadrupole (N = # sextupole magnets):

$$\Delta\xi_x = \frac{N}{4\pi}\left[\hat{\beta}\hat{D}\Delta S_1 + \check{\beta}\check{D}\Delta S_2\right]$$

$$\Delta\xi_y = -\frac{N}{4\pi}\left[\check{\beta}\hat{D}\Delta S_1 + \hat{\beta}\check{D}\Delta S_2\right]$$

$$S \equiv \frac{B''\ell}{B\rho}$$

- The sextupoles can be wired in two separate circuits, and thus the two chromaticities can be independently adjusted by any (reasonable) amount desired.

The Introduction of a Non-Linear Element

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- For the first time in our discussion, have introduced a “non-linear” transverse magnetic field for explicit use in the accelerator system — sextupoles for chromatic and/or chromaticity correction
- This opens the door to new and interesting phenomena, just as in the nonlinear longitudinal motion:
  - phase space distortions
  - tune variation with amplitude
  - dynamic aperture
  - …

Sextupole Tracking Code Demonstration

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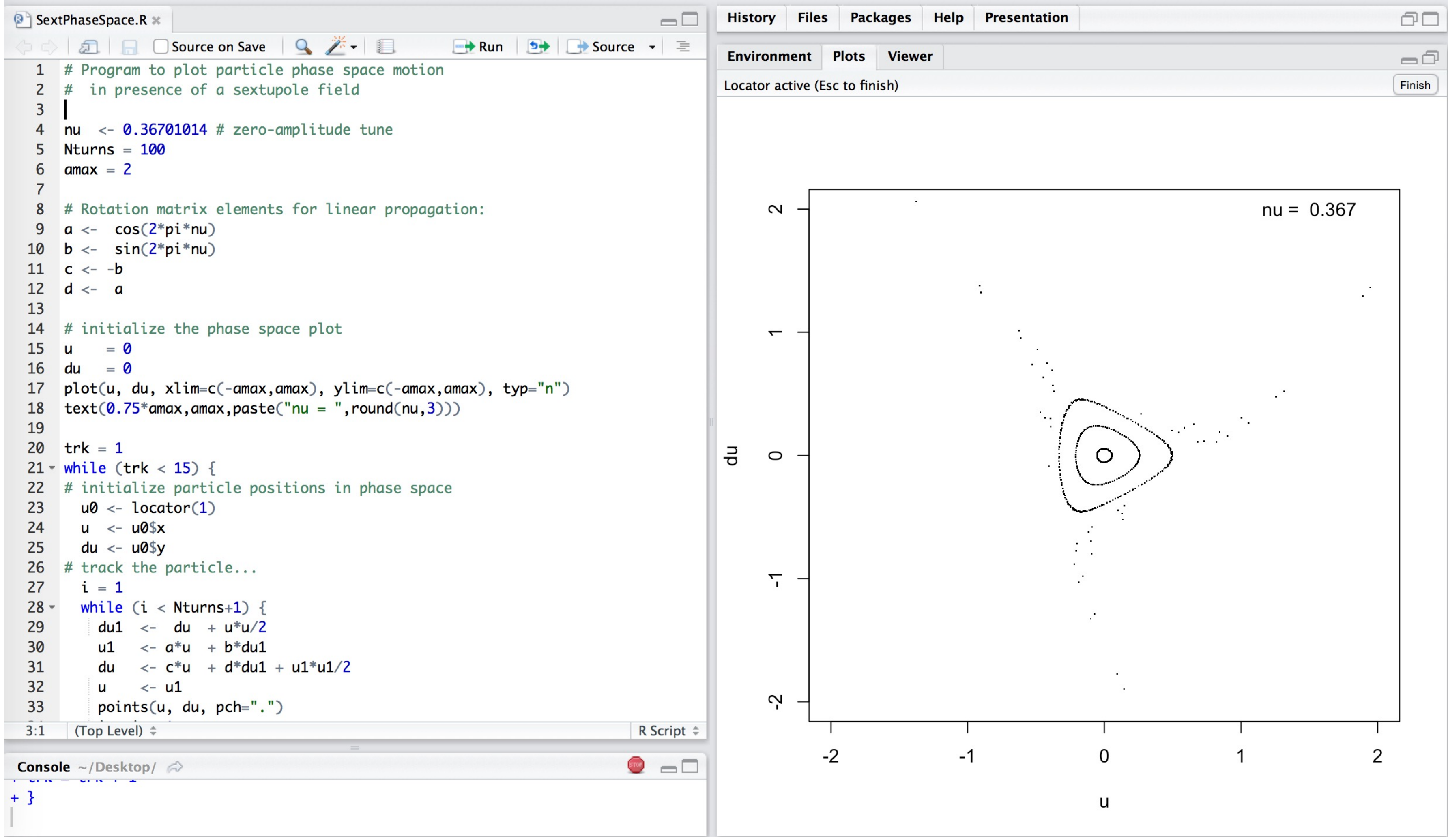
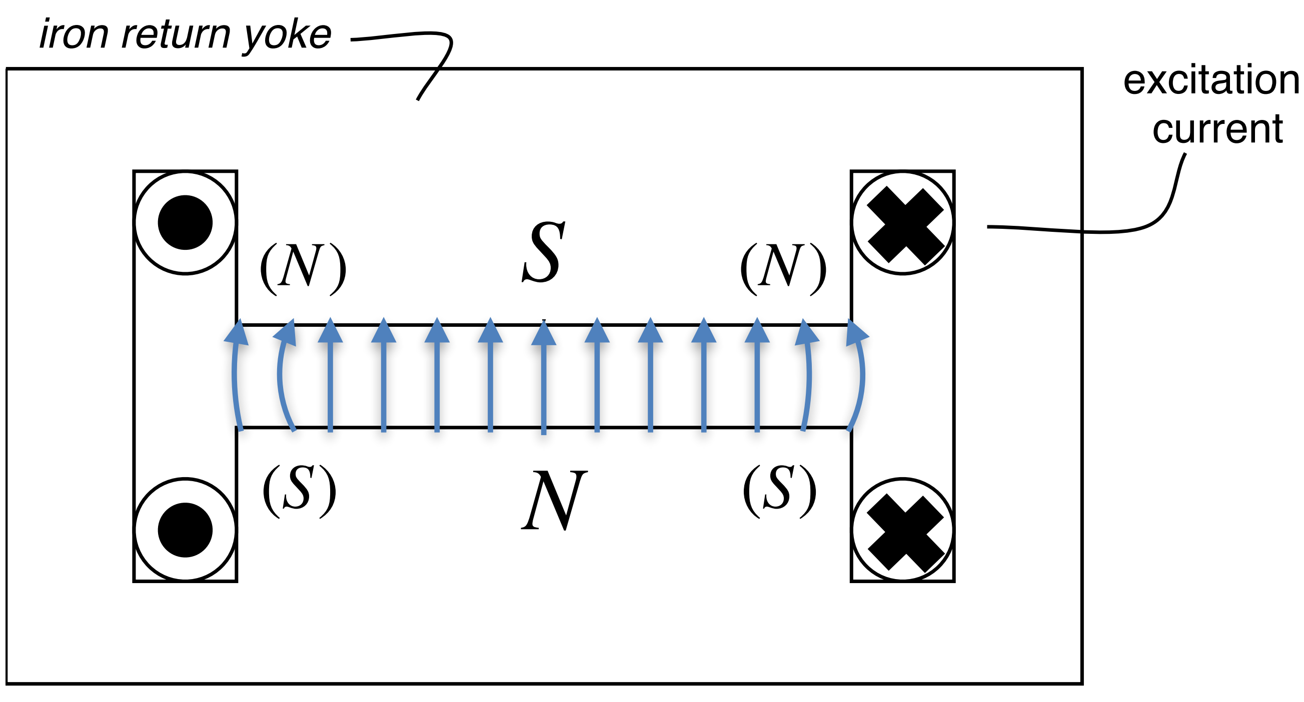
- while (i < Nturns+1) {
- du1 <- du + u*u/2
- u1 <- a*u + b*du1
- du <- c*u + d*du1 + u1*u1/2
- u <- u1
- points(u, du, pch=".")
- i = i + 1
- }



- Let's run a code…

Sources of Transverse Nonlinearities

- Real accelerator magnets
  - Finite width of the field region in a dipole magnet produces a 6-pole (sextupole) term -- $B_y(y = 0) \sim x^2$
  - Real magnets also have:
    - » Systematic construction errors
    - » Random construction errors
    - » Eddy currents in vacuum
    - » chambers as fields ramp up



- So, real life will introduce sources of linear AND nonlinear field perturbations which can affect the region of stable phase space ....

Resonances and Tune Space

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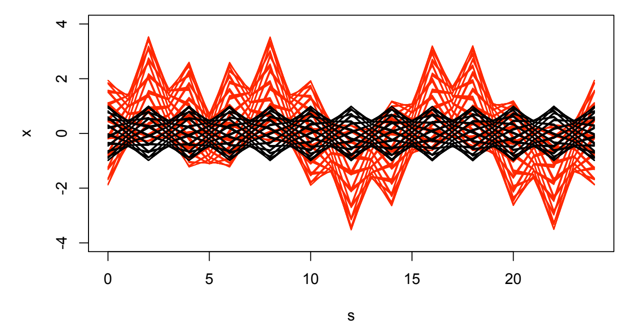
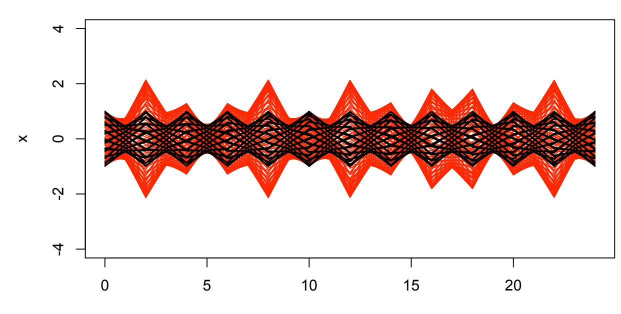
- Error fields are encountered repeatedly each revolution -- thus, can be resonant with the transverse oscillation frequency

- Let the "tune" ν = no. of oscillations per revolution
  - repeated encounter with a steering (dipole) error produces an orbit distortion:
    - » thus, avoid integer tunes



$$\Delta x \sim \frac{1}{\sin \pi\nu}$$

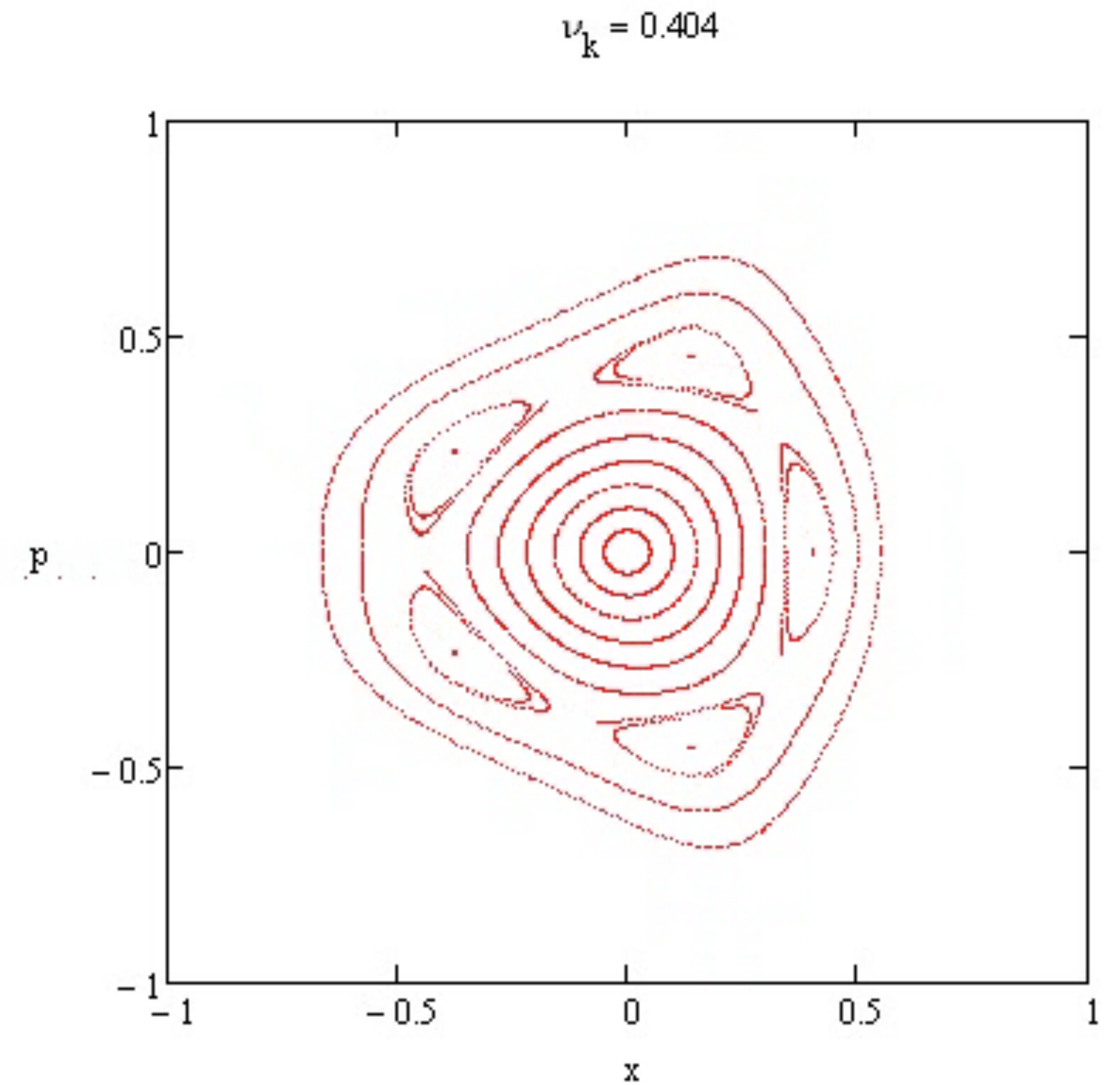
  - repeated encounter with a focusing error produces distortion of amplitude function, β:
    - » thus, avoid half-integer tunes



$$\Delta\beta/\beta \sim \frac{1}{\sin 2\pi\nu}$$

Nonlinear Resonances

- Phase space w/ sextupole field present ($\sim x^2$)
  - topology is tune dependent:
  - frequency depends upon amplitude
  - "dynamic aperture"



With sextupole field present, must avoid tunes:

integer, integer/2, integer/3, ...

"normalized" phase space; ideal trajectories are circular

An Application

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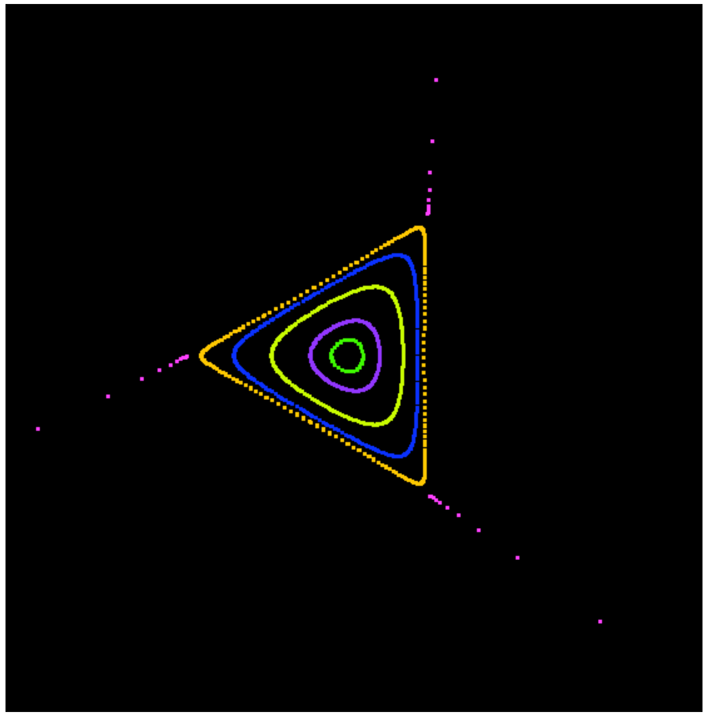
- Put the transverse nonlinear fields to work for us
- Can pulse an electromagnet to send the particles out of the accelerator all at once; but Particle Physics experiments often desire smooth flow of particles from the accelerator toward their detectors
- Resonant Extraction
  - developed in 1960's, particles can be put "on resonance" in a controlled manner and slowly extracted
  - third-integer: carefully approach $\nu = k/3$
    - » driven by sextupole fields
  - half-integer: carefully approach $\nu = k/2$
    - » driven by quadrupole and octupole (8-pole) fields

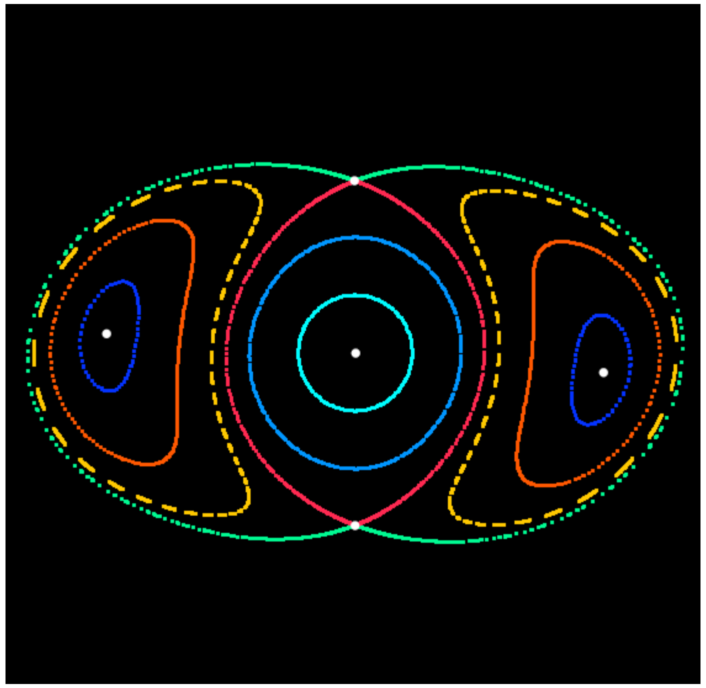
Phase Space used for Extraction

- Linear restoring forces with Sextupole perturbation, running near a tune of k/3



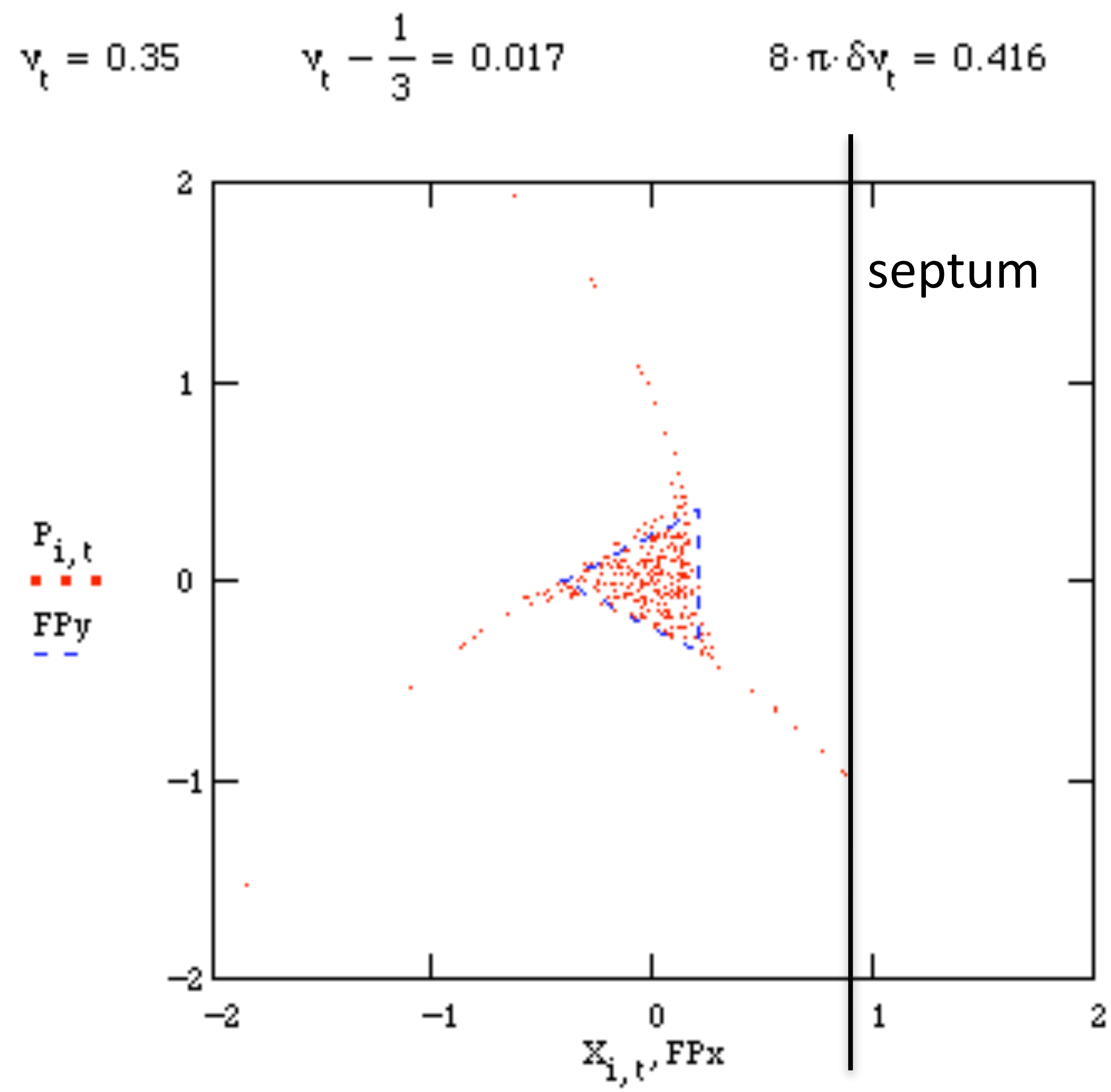
k = “integer”

- Linear restoring forces with Octupole (8-pole) and quadrupole perturbations, running near a tune of k/2



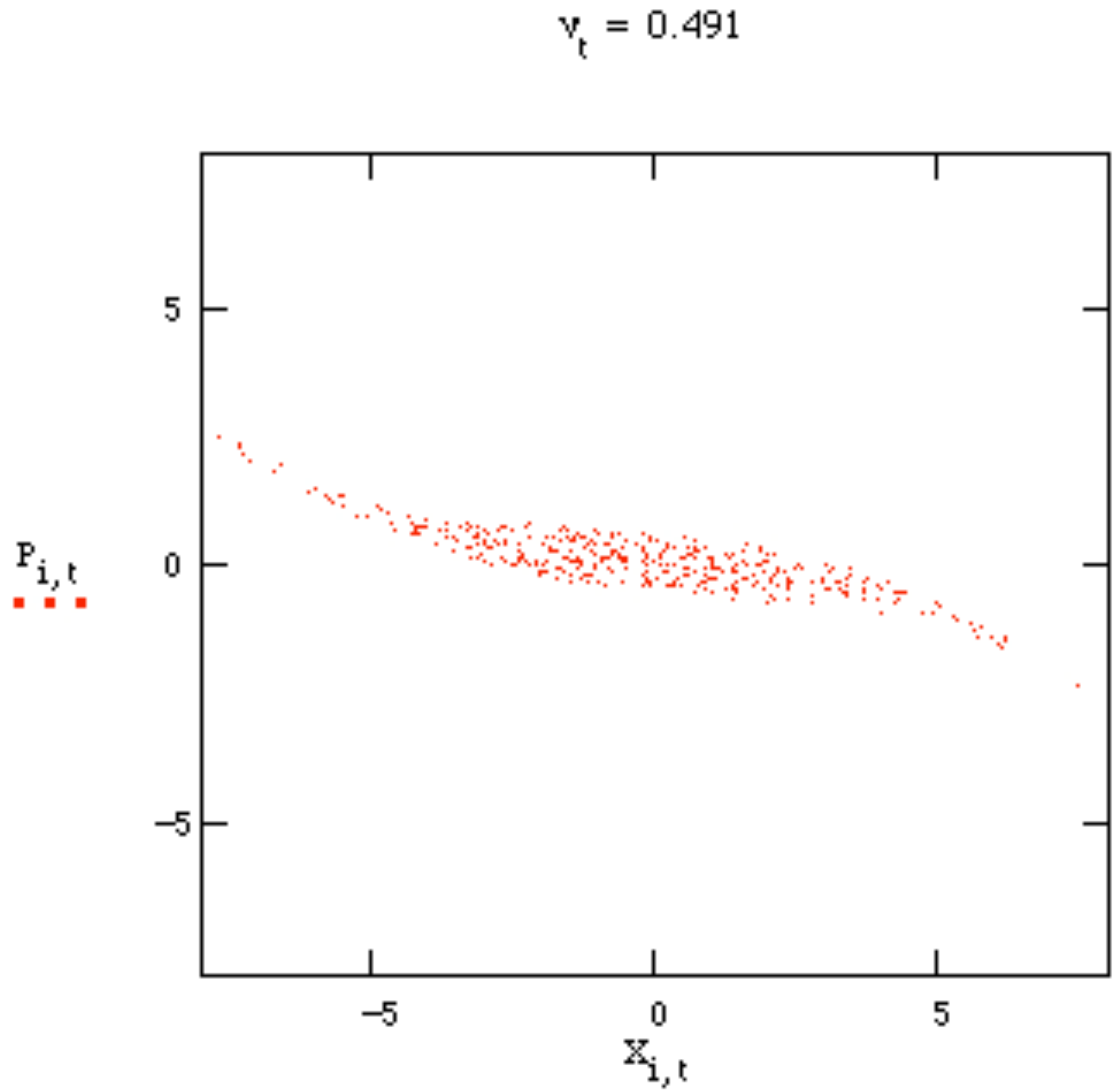
Third-integer Extraction

- Example: particles oscillate in phase space in presence of a single sextupole
- Slowly adjust the tune toward a value of k/3
  - (here, k=1)
- Tune is exactly 1/3 at the separatrix
- The lines that appear are derived from a first-order perturbation calculation
- Particles stream away from the "unstable fixed points", stepping across a "septum" which leads out of the accelerator



Half-integer Extraction

- Similar to last movie, but "ideal" accelerator has extra quadrupole and octupole (8-pole) fields
- Slowly adjust the tune toward a value of k/2
  - (here, k=1)
- Here, lowest-order separatrices defined by two intersecting circles
- Eventually, when very close to half-integer tune, entire phase space becomes unstable (|trM|>2)



Coupling Resonances

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- Always "error fields" in the real accelerator

- "Skew" fields can couple the motion between the two transverse degrees of freedom

  - thus, can also generate coupling resonances
    - » (sum/difference resonances)

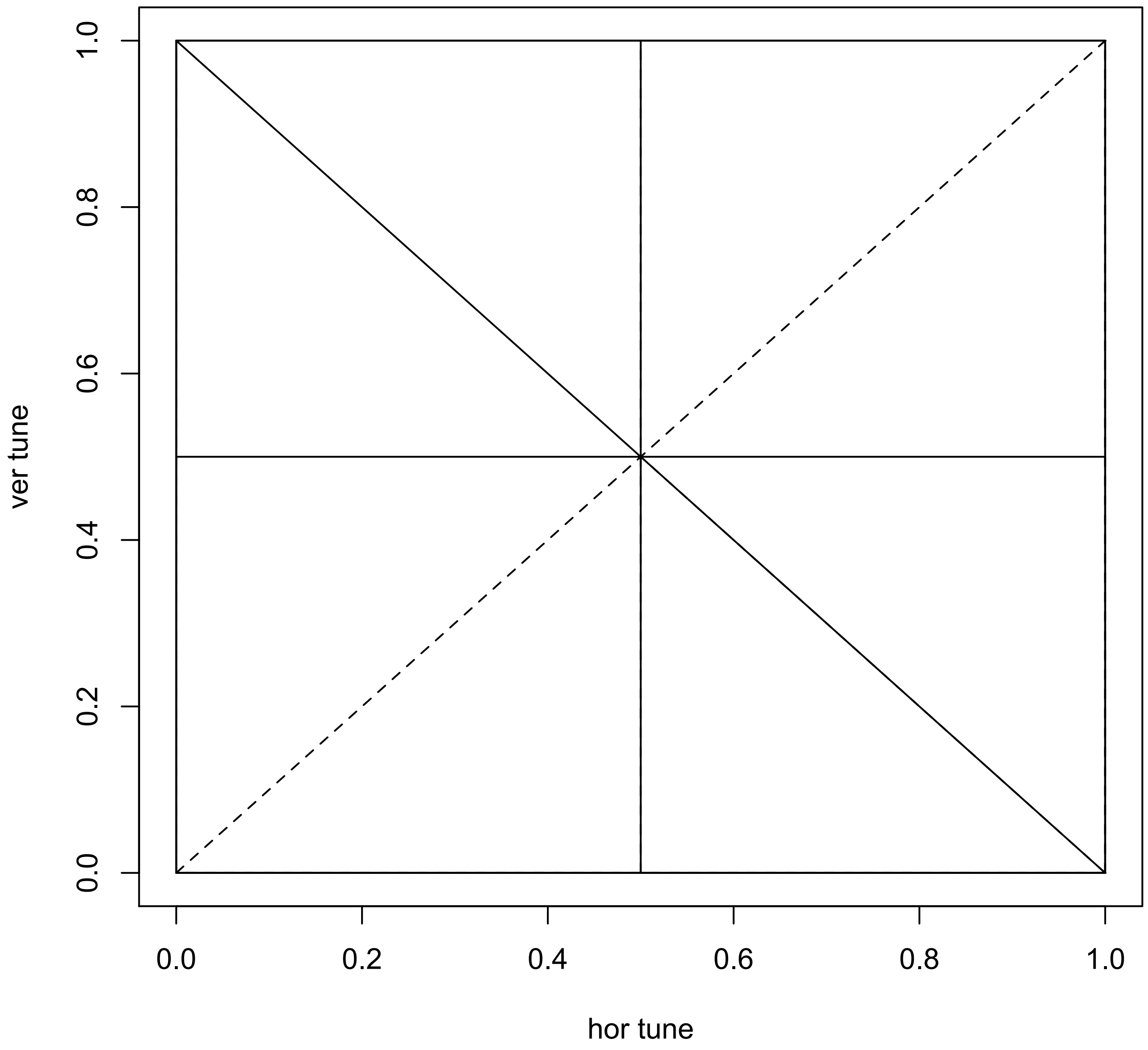
  - in general, should avoid: $$m\,\nu_x \pm n\,\nu_y = k$$

avoid ALL rational tunes???

Tune Diagram

lines of $m\,\nu_x \pm n\,\nu_y = k$

Through order
k =2



Tune Diagram

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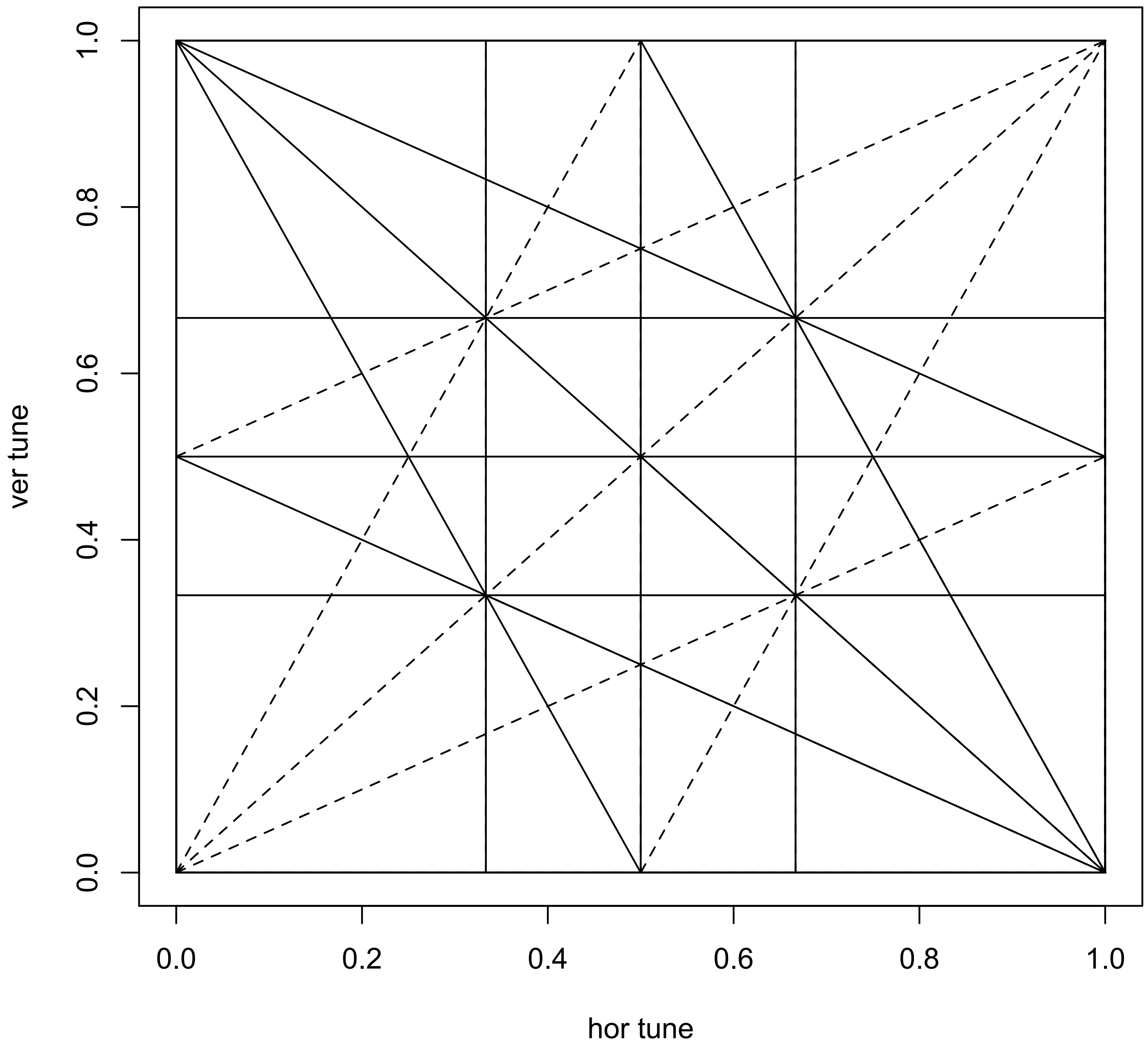
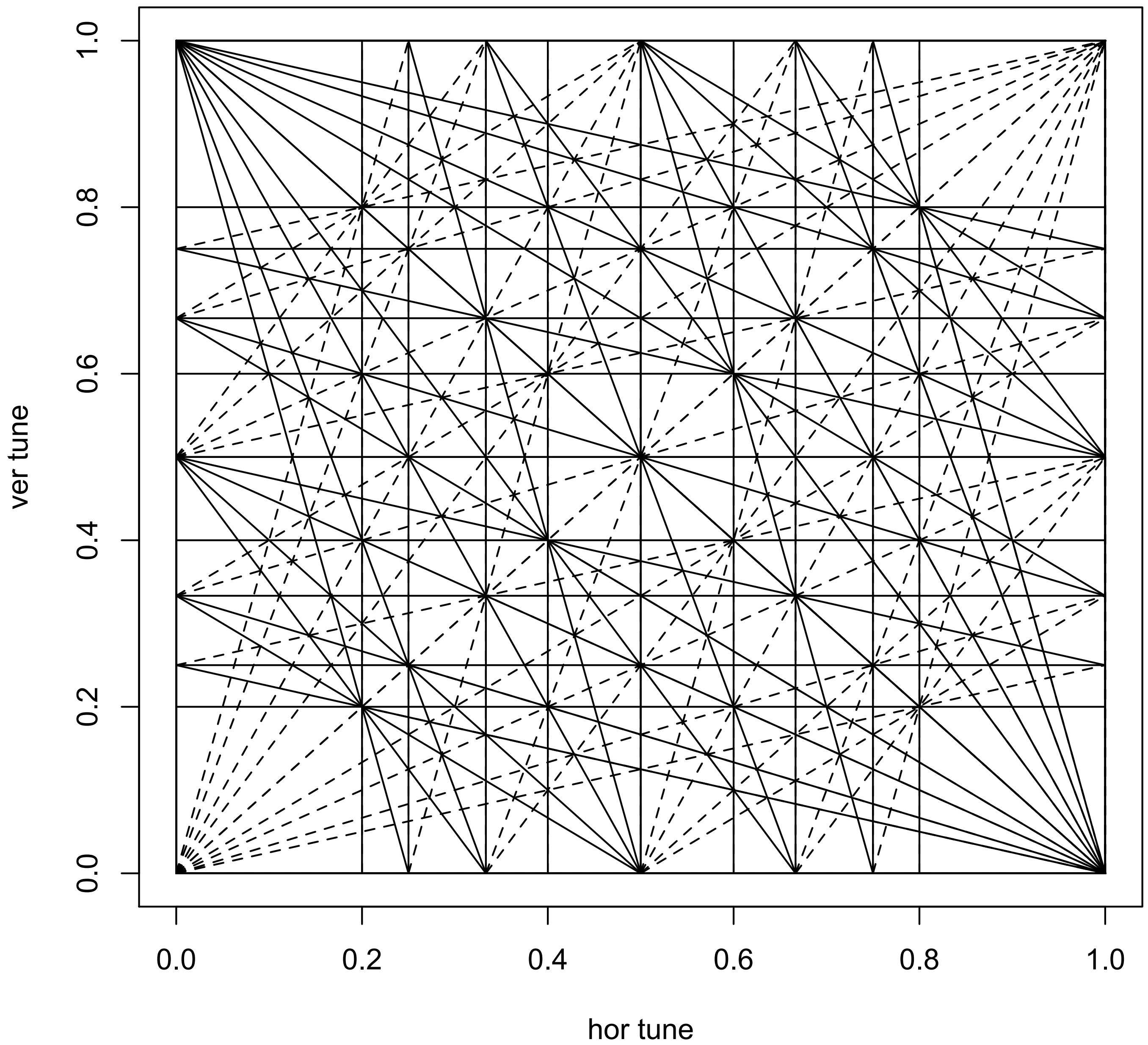
lines of $m\,\nu_x \pm n\,\nu_y = k$

Through order
k =3



Tune Diagram

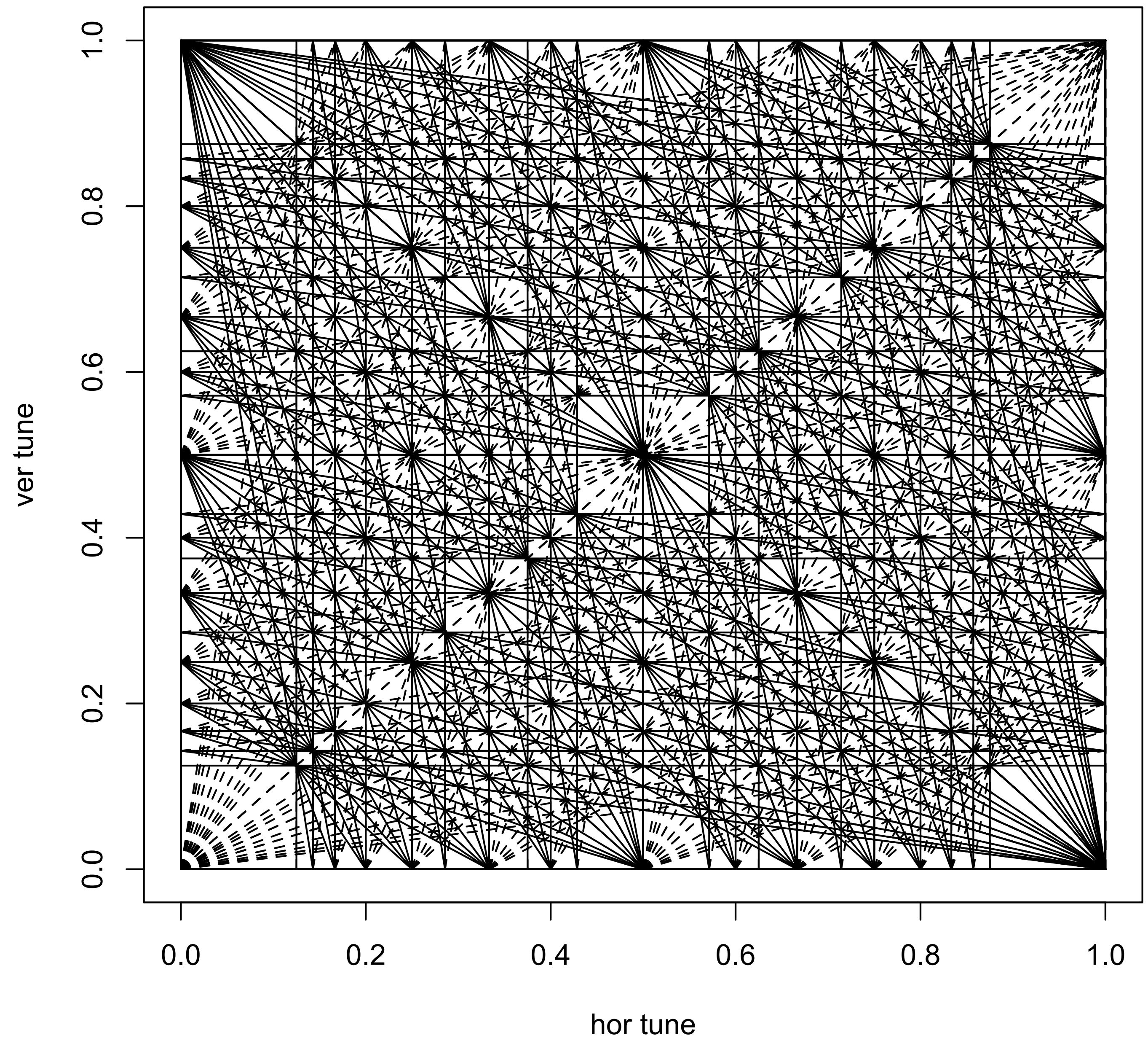
lines of $m\,\nu_x \pm n\,\nu_y = k$



Through order
k =5

Tune Diagram

lines of $m\,\nu_x \pm n\,\nu_y = k$



Through order
k =8

Tune Diagram

lines of $m\,\nu_x \pm n\,\nu_y = k$