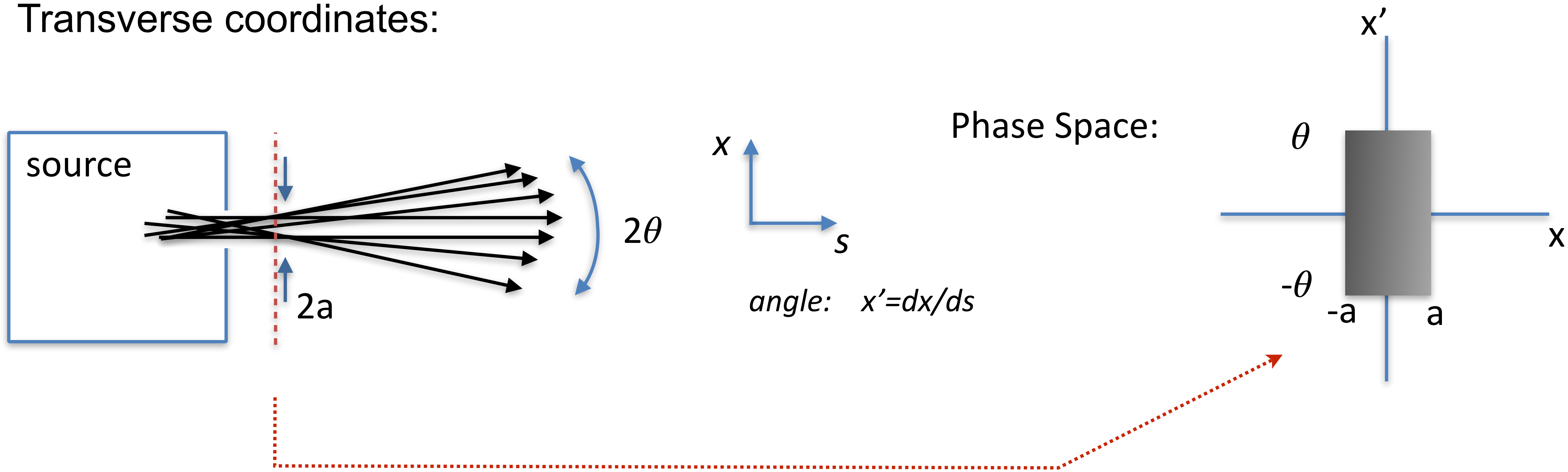
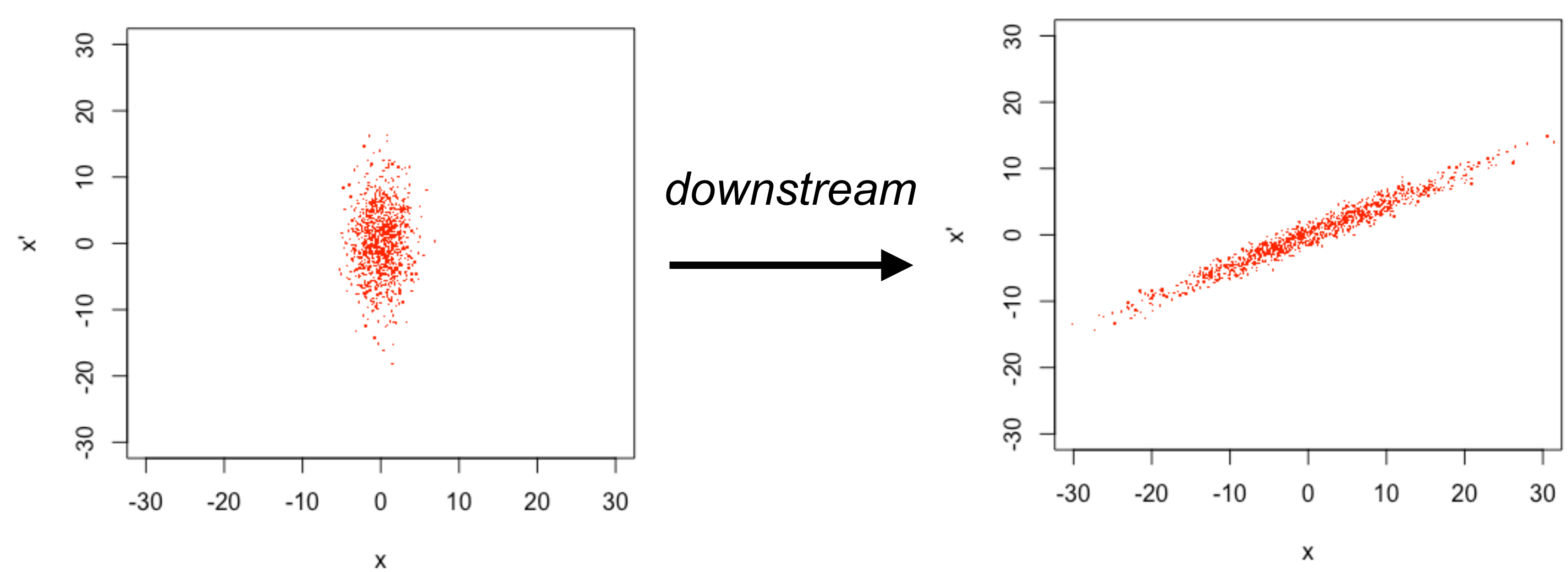
Particle Beams and Phase Space

Transverse coordinates:



Shape, orientation of distribution in "phase space" will change as particles progress downstream, but effective "area" of distribution will remain constant (*Liouville*); correlations will naturally develop

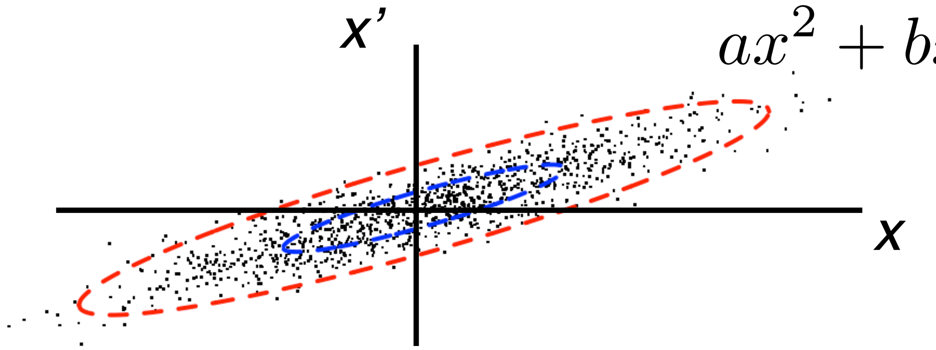


Emittance in Terms of Moments

see: <u>http://nicadd.niu.edu/~syphers/tutorials/ellipseNotes.html</u>

- Considering the general equation of an ellipse, the area enclosed by the ellipse is related to its coefficients by:

$$ax^2 + bxx' + cx'^2 = 1$$



area of ellipse:

$$\mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

- Can define quantities scaled by an area, ϵ, of our elliptical distribution:

$$\alpha \equiv -\frac{\langle xx' \rangle}{\epsilon/\pi} \qquad \beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \qquad \gamma \equiv \frac{\langle x'^2 \rangle}{\epsilon/\pi}$$

$$\boxed{\epsilon = \pi\sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}}$$

the "rms emittance"

α, β, γ collectively are called the ***Courant-Snyder* parameters**, or *Twiss* parameters

So, equation of the **blue** curve above:

$$\boxed{\gamma x^2 + 2\alpha xx' + \beta x'^2 = \epsilon/\pi}$$

The ellipse (red curve above) that contains ~95% has area ~6ϵ

(*for Gaussian distribution*)

TRANSPORT of Beam Moments

- Transport of particle state vector downstream from position 0

$$\vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix} \qquad \vec{X} = M\vec{X}_0$$

- Create a "covariance matrix" of the resulting vector…

$$\vec{X}\vec{X}^T = \begin{pmatrix} x^2 & xx' \\ x'x & x'^2 \end{pmatrix} = M\vec{X}_0(M\vec{X}_0)^T = M\vec{X}_0\vec{X}_0^T M^T$$

- … then, by averaging over all the particles in the distribution,

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

we get: $$\Sigma = M\Sigma_0 M^T$$

Let's Think About the Numbers & Units…

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$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

- If $<x^2>$ ~ mm^2, and $<x'^2>$ ~ mrad^2,
  - then the emittance can have units of **mm-mrad** (*also* = μm)
- Courant-Snyder parameters

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon}$$

mm^2/(mm-mrad) ~ mm/mrad = m

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

(mm-mrad)/(mm-mrad) = dimensionless

$$\gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

mrad^2/(mm-mrad) ~ mrad/mm = 1/m

The "π" comes from our definition of emittance as an area in phase space; emittance is often expressed in units of "π mm-mrad"

Summary

- Given an initial particle distribution in phase space at the input to a beam transport system, can describe that distribution (sometimes not all that well, but we try…) using Courant-Snyder parameters:

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon} \qquad \gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

- The C-S parameters can then be computed downstream, using

$$\Sigma = M\Sigma_0 M^T$$

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Courant-Snyder Approach to Linear Transverse Dynamics

Mike Syphers

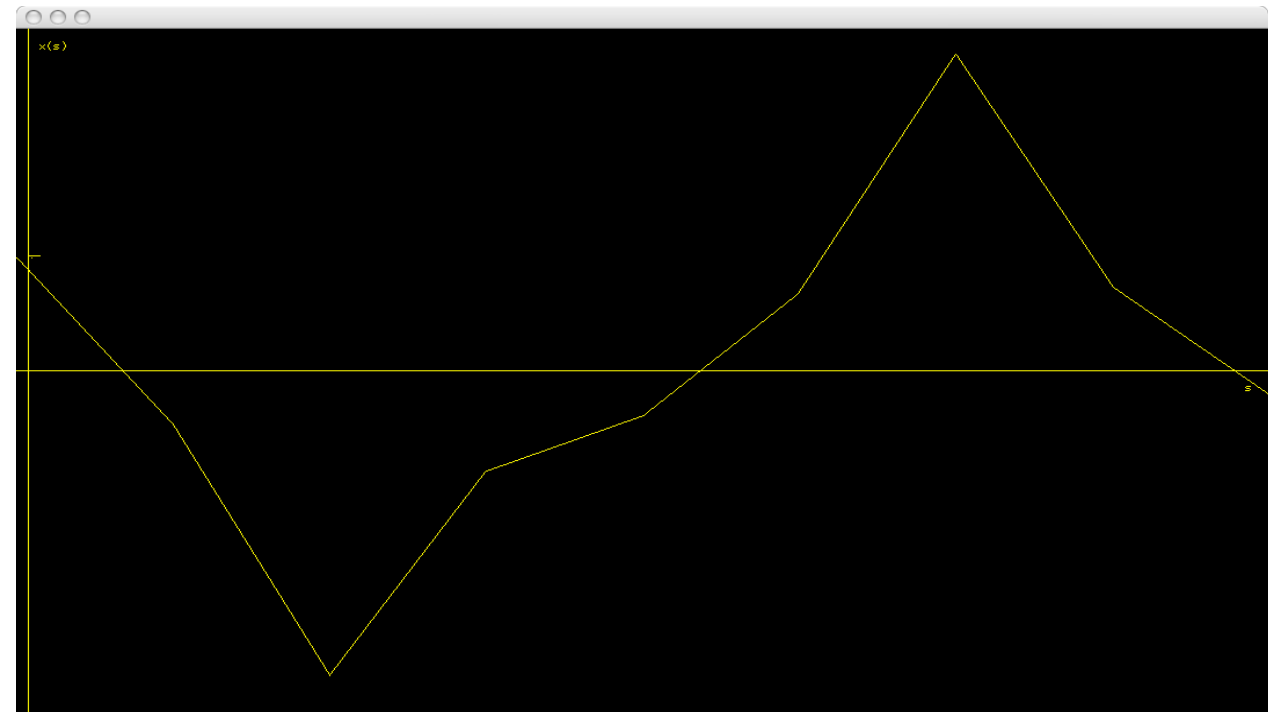
Operator Lectures

April 2020

The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line



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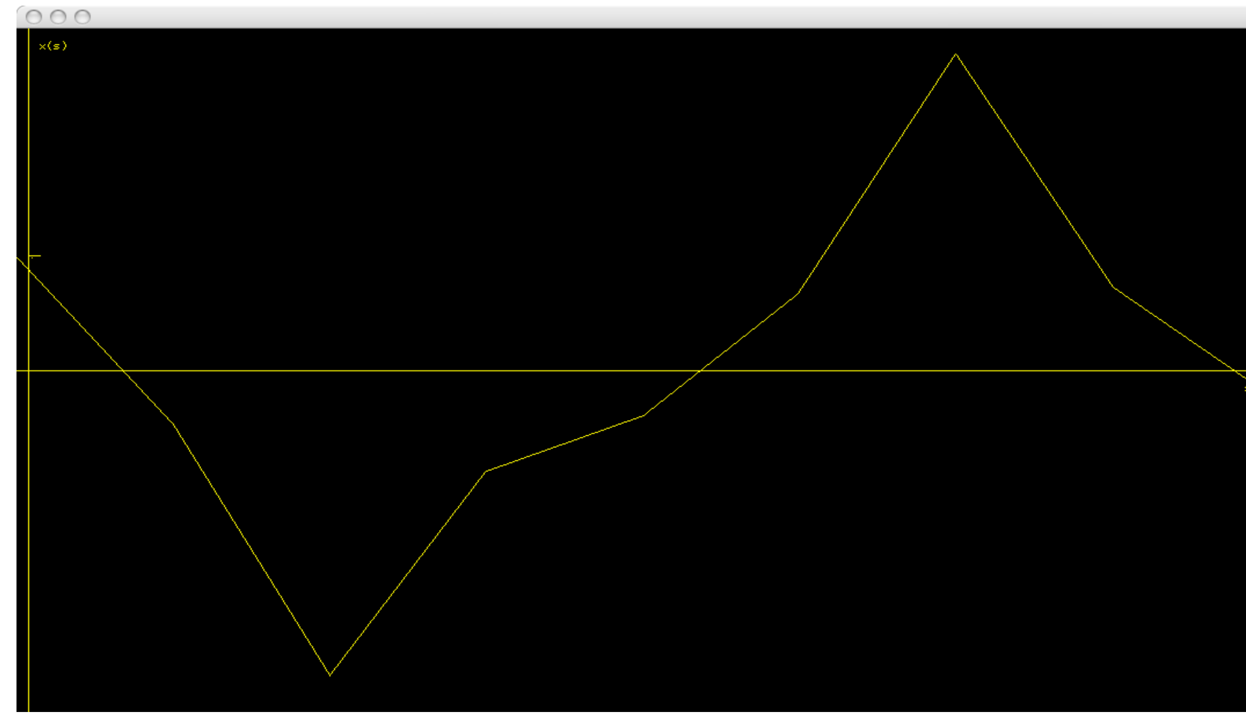

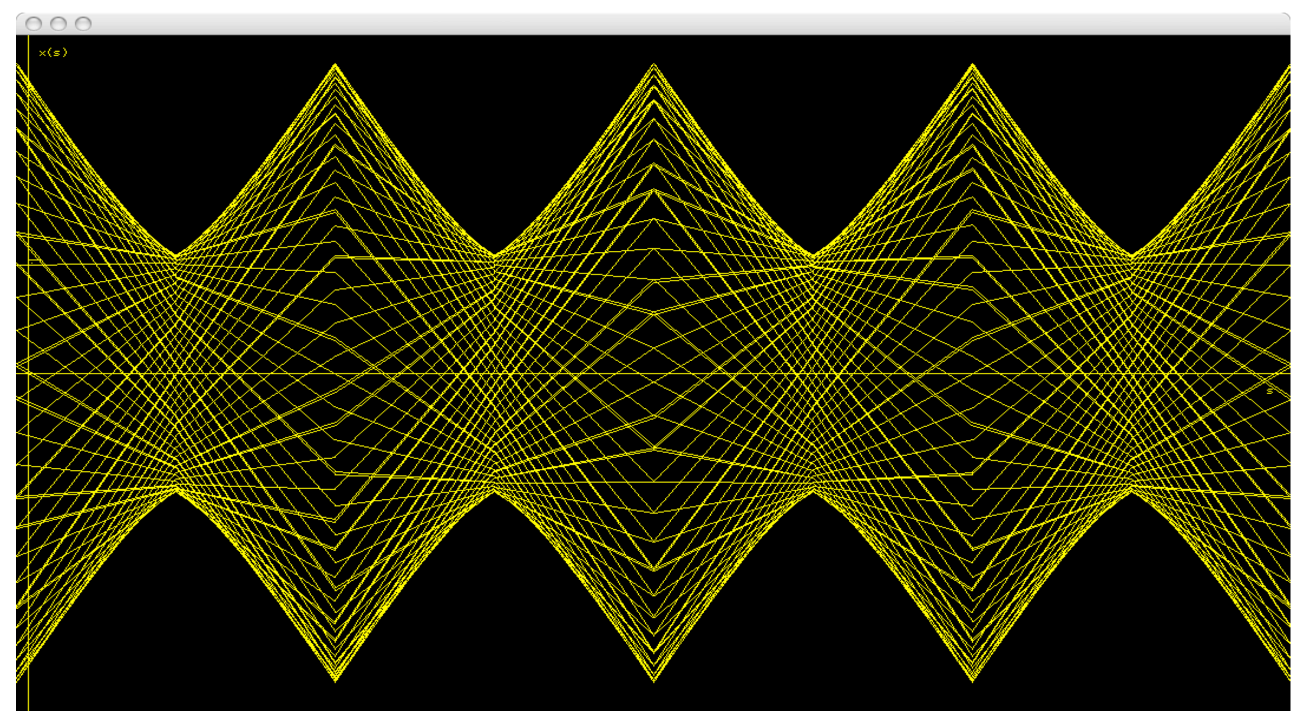

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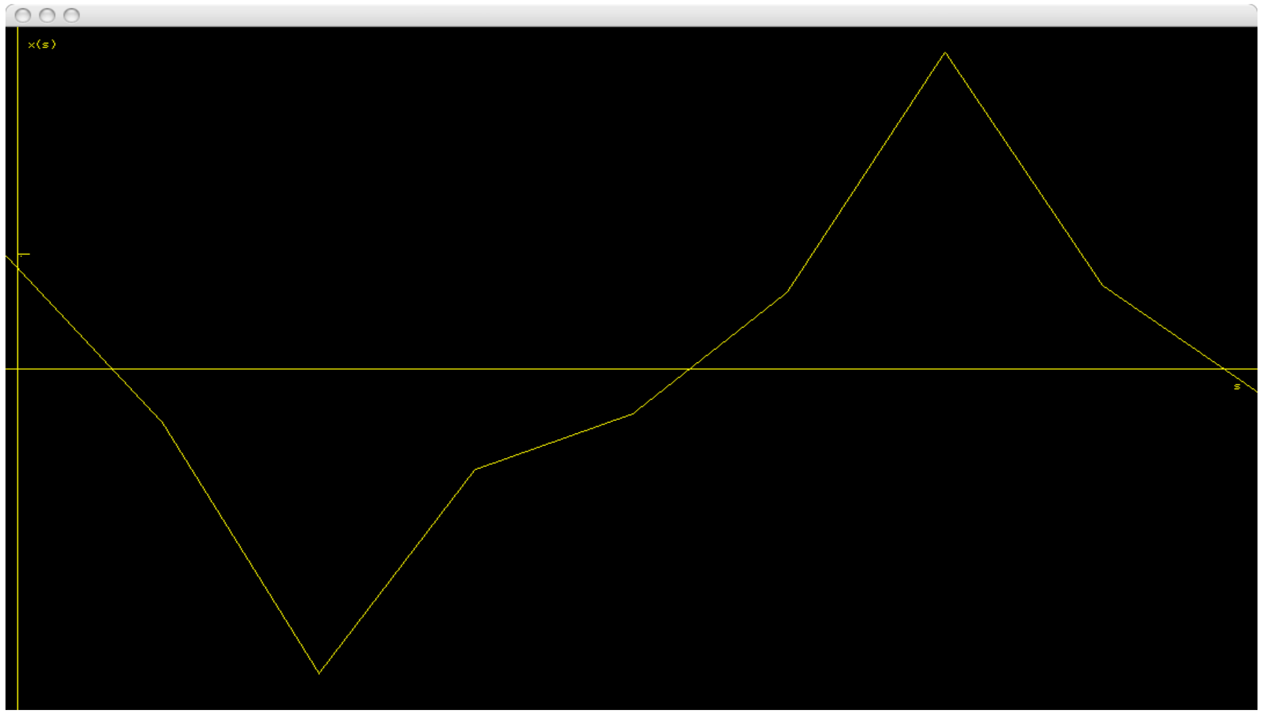
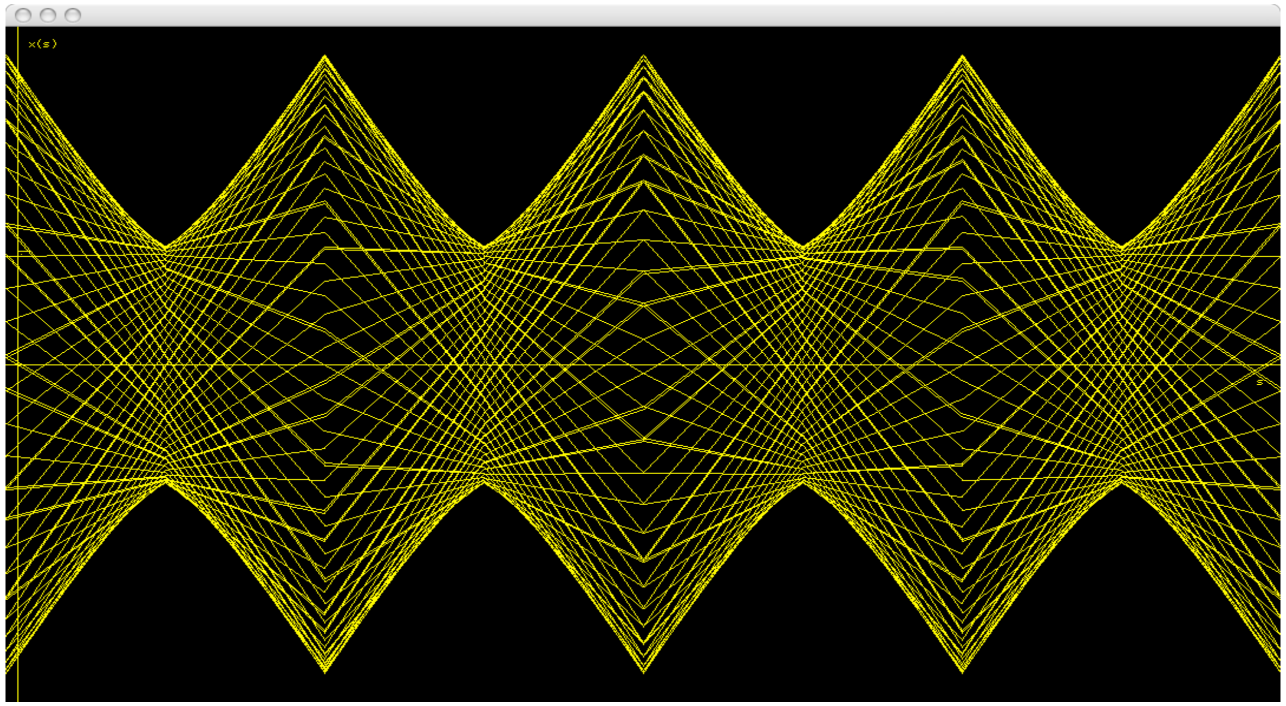




Can we describe the maximum amplitude of particle excursions in analytical form?

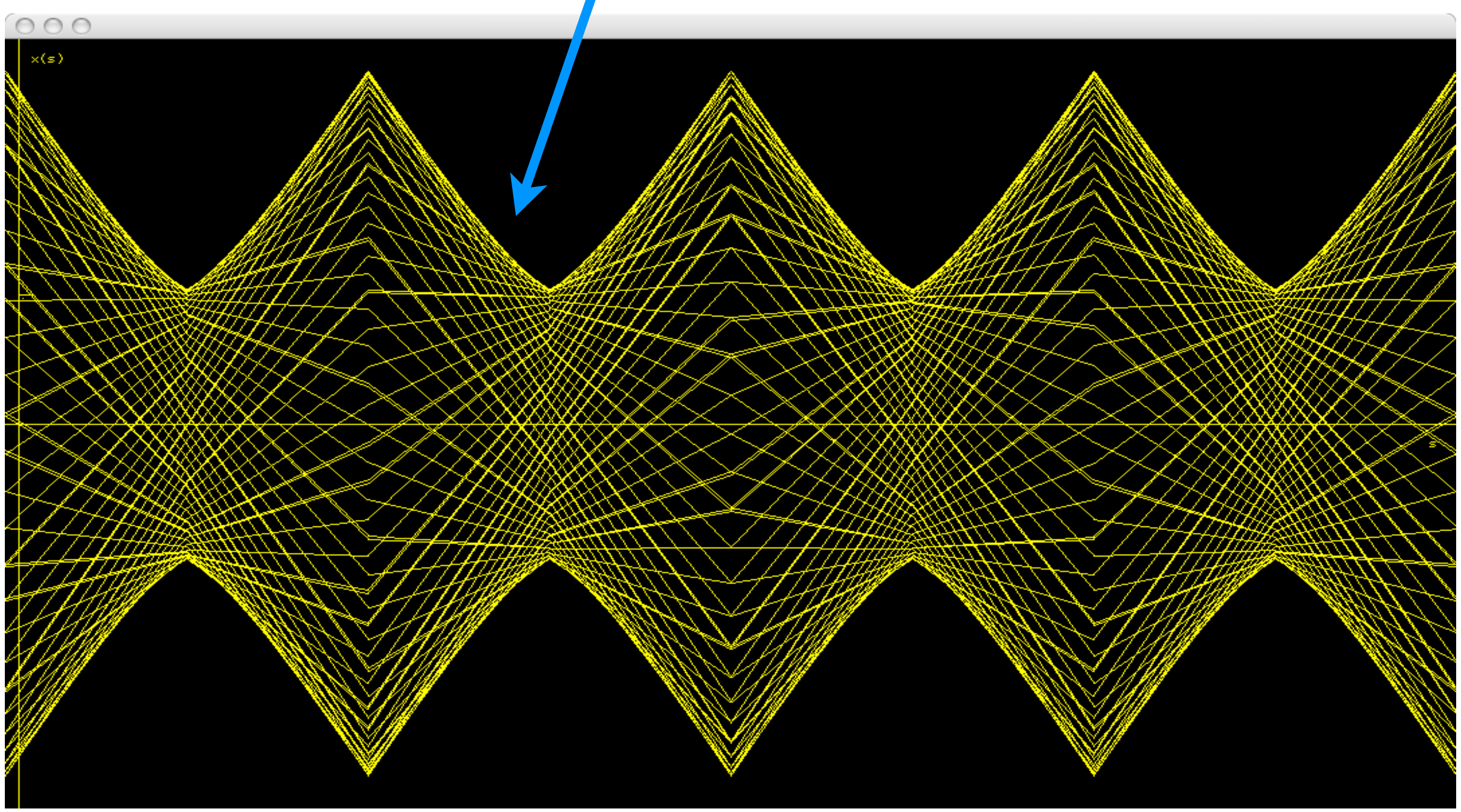
of course! *coming up next …*

Pushing the “Envelope”

- Wish to look for a functional form of the outer envelope of particle motion, and the rate at which the phase of the oscillatory motion develops within that envelope

- This will enable us to decouple the motion of individual particle from intrinsic properties of the accelerator design

Envelope described by an “amplitude function”



Hill's Equation — Analytical Solution

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- We saw that the equation of transverse motion is Hill's Equation:

$$x'' + K(s)x = 0$$

- Note: "similar" to simple harmonic oscillator equation, but "spring constant" is not *constant* -- depends upon longitudinal position, *s*.
- So, assume solution is sinusoidal, with a phase which advances as a function of location *s*; also assume amplitude is modulated by a function which also depends upon *s*:

$$\boxed{x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]}$$

  - Then, plug into Hill's Equation ...

Analytical Solution (cont'd)

$$\begin{aligned} x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\ x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\ x'' &= \ldots \end{aligned}$$

Plugging into Hill's Equation, and collecting terms...

$$\begin{aligned} x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\ &\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0 \end{aligned}$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, must have contents of each [ ] = 0 simultaneously for *sum* = 0.

Analytical Solution (cont'd)

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- Thus, we must have ...

thus, we need

$$\boxed{\psi'' + \frac{\beta'}{\beta}\psi' = 0}$$

and

$$\boxed{-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0}$$

Analytical Solution (cont'd)

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

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and

$$\boxed{-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0}$$

$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Note: the phase advance is an observable quantity. So, while could choose different value of `const`, then β would just scale accordingly; thus, valid to choose `const` = 1.

Analytical Solution (cont'd)

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s)+\delta]$$

- Thus, we must have ...

thus, we need

$$\boxed{\psi'' + \frac{\beta'}{\beta}\psi' = 0}$$

and

$$\boxed{-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0}$$

$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

$$2\beta\beta'' - (\beta')^2 - 4\beta^2(\psi')^2 + 4K\beta^2 = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

Differential equation that the amplitude function must obey

Note: the phase advance is an observable quantity. So, while could choose different value of `const`, then β would just scale accordingly; thus, valid to choose `const` = 1.

Some Comments

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$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But the amplitude function is also a local wavelength of the motion.
- This seems strange at first, but ...
  - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
  - Thus, the spacing and/or strengths (i.e., *K(s)*) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes must be tied together.

Equation of Motion of Amplitude Function

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From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

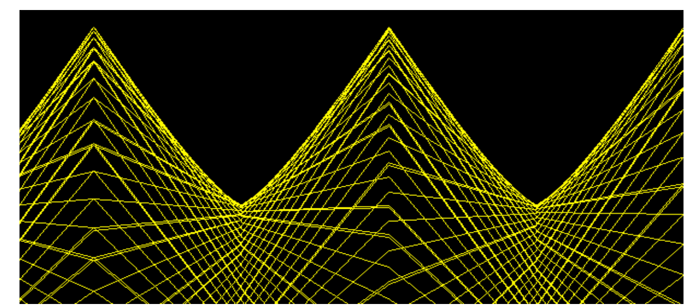
$$\boxed{\beta'' + 4K\beta = const.}$$

is the general equation of motion for the amplitude function, β.

(in regions where K is either zero or constant)

Piecewise Solutions

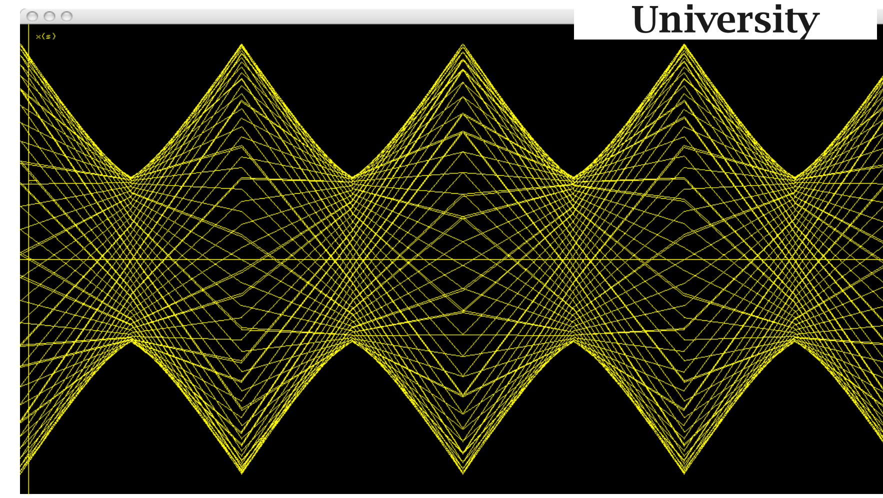
- $K = 0$: $$\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$ Parabola!
  - since $\beta > 0$, then from original diff. eq. …
    $$2\beta\beta'' - (\beta')^2 = 4 \qquad \beta'' = \frac{4 + (\beta')^2}{2\beta} > 0$$
  - Therefore, parabola is ***always*** concave up



- ($K < 0$) $K > 0$: (hyperbolic) sinusoidal + *constant*

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$$

Summary



$$x'' + K(s)x = 0$$ Hill's Equation

trial solution: $x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$

requires:

$$\psi' = 1/\beta \quad \longrightarrow \quad \psi(s) = \int \frac{ds}{\beta(s)}$$

and

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \quad \longrightarrow \quad \beta'' + 4K\beta = const.$$

(for *K'* = 0)

for *K* = 0 : $\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$

for *K* > 0 : $\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1 - \cos(2\sqrt{K}s)]$

Courant-Snyder Parameters, & Connection to Matrix Approach

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- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Previously have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,

$$\boxed{\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}}$$

- Collectively, β, α, γ are called the *Courant-Snyder Parameters* (sometimes called "Twiss parameters" or "lattice parameters")

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \qquad \textit{becomes} \qquad K\beta = \gamma + \alpha'$$

Solutions using Courant-Snyder Parameters

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- Define two new variables: $$\boxed{\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}}$$
  - will see that these are the *same* α and γ from our *ellipse* description!

- Our previous results become
  - drift space: –>

$$\beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

  - gradient field:

$$\beta(s) = \beta_0 + \frac{\beta_0'}{2\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\beta_0''}{4K}[1-\cos(2\sqrt{K}s)]$$

–>

$$\beta(s) = \frac{\beta_0}{2}[1+\cos(2\sqrt{K}s)] - \frac{\alpha_0}{\sqrt{K}}\sin(2\sqrt{K}s) + \frac{\gamma_0}{2K}[1-\cos(2\sqrt{K}s)]$$

The Transport Matrix

- We can always write: $x(s) = a\sqrt{\beta}\sin\Delta\psi + b\sqrt{\beta}\cos\Delta\psi$

- So, solve for *a* and *b* in terms of initial conditions and write in matrix form
  - we get:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

So, can write any of our transport matrices in terms of values of C-S parameters at the two end points, and the phase advance between them.

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

Tracking β, α, γ … and Phase Advance

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- Saw earlier that if given values of the Courant-Snyder parameters at one location in the beam line, and if know the matrix of the linear motion between that location and another location downstream, then can compute the values at the second location *via*:

$$\boxed{K = M\ K_0\ M^T} \qquad \text{where} \quad K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- Also, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan\Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase *and* the Courant-Snyder parameters along a beam line from one point to another

Simple Examples

- Propagation through a Drift:

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = \tan^{-1}\left(\frac{L}{\beta_1 - L\alpha_1}\right)$$

$$\beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha = \alpha_0 - \gamma_0 L$$

$$\gamma = \gamma_0$$

- Propagation through a Thin Lens:

$$M = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = 0$$

$$\beta = \beta_0$$

$$\alpha = \alpha_0 + \beta_0/F$$

$$\gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2$$

$$\boxed{K = M\ K_0\ M^T}$$

- Given *α, β* at one point, can calculate *α, β* at all downstream points

Another Summary

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- So, with knowledge of the layout of (linear) magnetic (and electrostatic) fields from which matrices describing the horizontal and vertical motion can be derived, and with an initial set of Courant-Snyder parameters describing the beam distribution, can *transport* the Courant-Snyder parameters along the beam line
  - Hence, can design a first-order focusing system without having to track particles. Within such a system the beam size will be determined by the value of the emittance used.
- These same C-S parameters describing the beam ellipse in phase space are found to be the same parameters found in the analytical solution to Hill's Equation if we identify

$$\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta} \qquad \Delta\psi = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}$$

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s)+\delta]$$

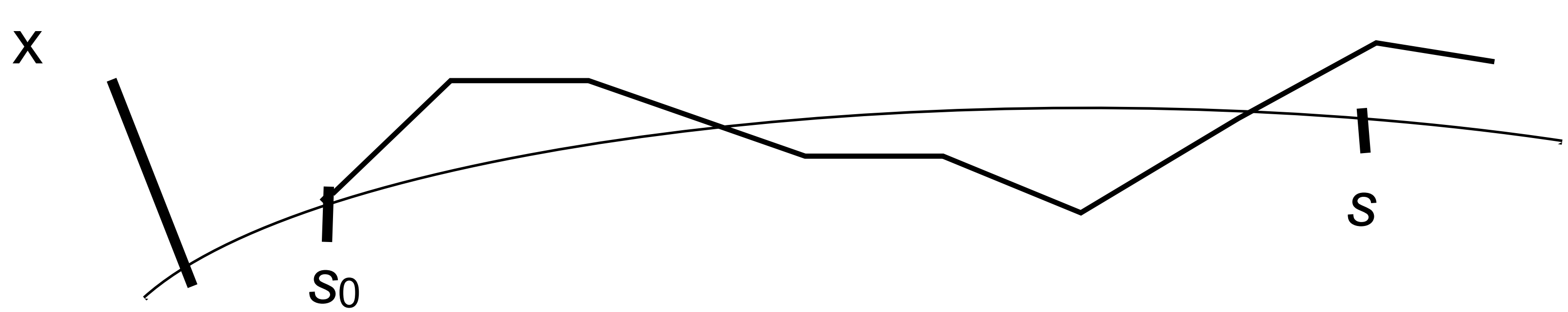
Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle $\Delta x' = \Delta\theta \equiv x_0'$
- Then, downstream, we have

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} 0 \\ \Delta\theta \end{pmatrix}$$

or,

$$\boxed{x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin[\psi(s) - \psi_0]}$$



Example:
Suppose $\Delta\theta = 0.4$ mrad, $\beta_0 = 4.0$ m, $\beta(s) = 6.4$ m, and $\Delta\psi = n \times 2\pi + 30°$. Then $x(s) = 1$ mm.

Courant-Snyder Invariant

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- In general,

$$
\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
&= A\sqrt{\beta}\cos\psi - \alpha x \\
\beta x' + \alpha x &= A\sqrt{\beta}\cos\psi
\end{aligned}
$$

Courant-Snyder Invariant

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&= A\sqrt{\beta}\cos\psi - \alpha x \\
\boxed{\beta x' + \alpha x} &\boxed{= A\sqrt{\beta}\cos\psi}
\end{aligned}
$$

$$
\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}
$$

Courant-Snyder Invariant

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- In general,

$$\begin{aligned}
x &= A\sqrt{\beta}\sin\psi \\
x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\
\beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\
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\boxed{\beta x' + \alpha x = A\sqrt{\beta}\cos\psi}
\end{aligned}$$

$$\begin{aligned}
x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\
A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\
&= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta}
\end{aligned}$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

Courant-Snyder Invariant

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- In general,

$$\begin{aligned} x &= A\sqrt{\beta}\sin\psi \\ x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\ \beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\ &= A\sqrt{\beta}\cos\psi - \alpha x \\ \boxed{\beta x' + \alpha x = A\sqrt{\beta}\cos\psi} \end{aligned}$$

$$\begin{aligned} x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\ A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\ &= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta} \end{aligned}$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

From our general analytical approach today, we arrive at the result that for a single particle, the combination above will remain constant —

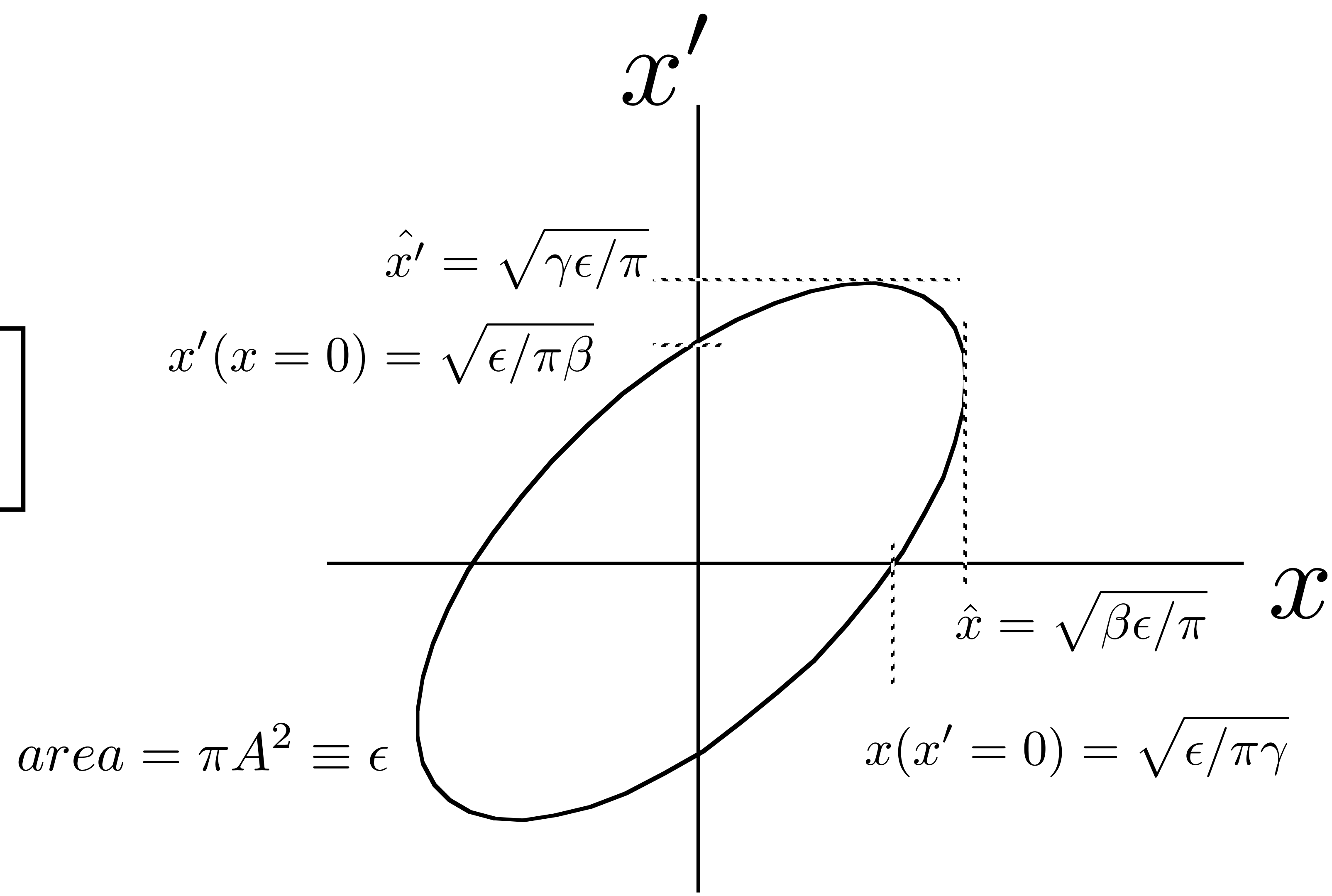
it's the same ellipse from before, and the "beta function" is the same parameter that we showed last time. The seemingly arbitrary parameters α and γ are related to the β ***function*** by: $\alpha = -\frac{1}{2}\beta', \quad \gamma = (1+\alpha^2)/\beta$

Properties of the Phase Space Ellipse

- The initial conditions of a freely-oscillating particle in the beam optics system determine its C-S invariant and hence the particle's individual phase space ellipse

while the ellipse changes shape along the beam line, its area remains constant

$$\boxed{area = \pi A^2}$$



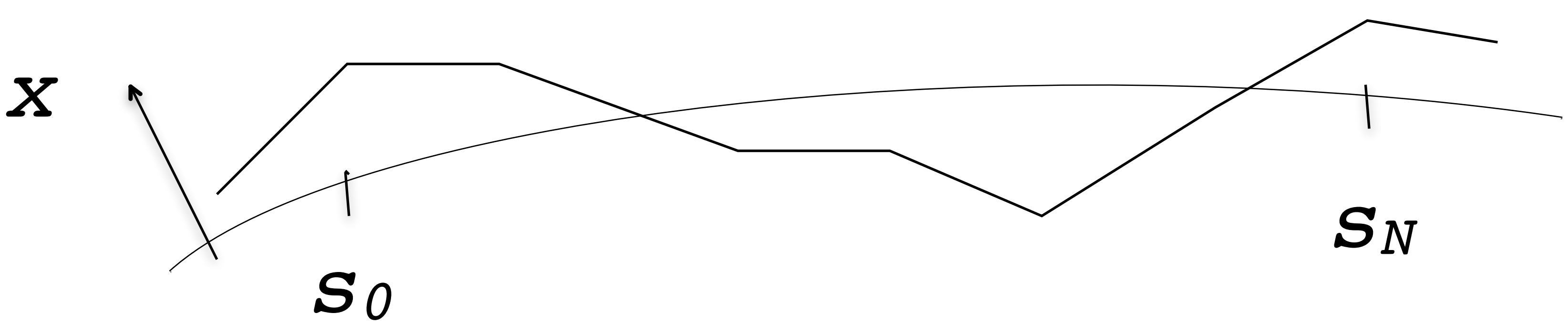
Emittance = area within a phase space trajectory

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2$$

Long-Term Stability of Arbitrary Focusing System

- Generate a single-turn matrix of the linear motion, made from matrices of individual elements (Note: each with unit determinant)
- Look at matrix describing motion for one passage through a repetitive period:

$$M = M_N M_{N-1} \cdots M_2 M_1$$



- Now suppose **repeat** this operation k times. We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \rightarrow \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

The Stability Criterion

- Through an *eigenvector* approach, one can solve for the eigenvalues of the repetitive matrix *M* and find that for stability, must have

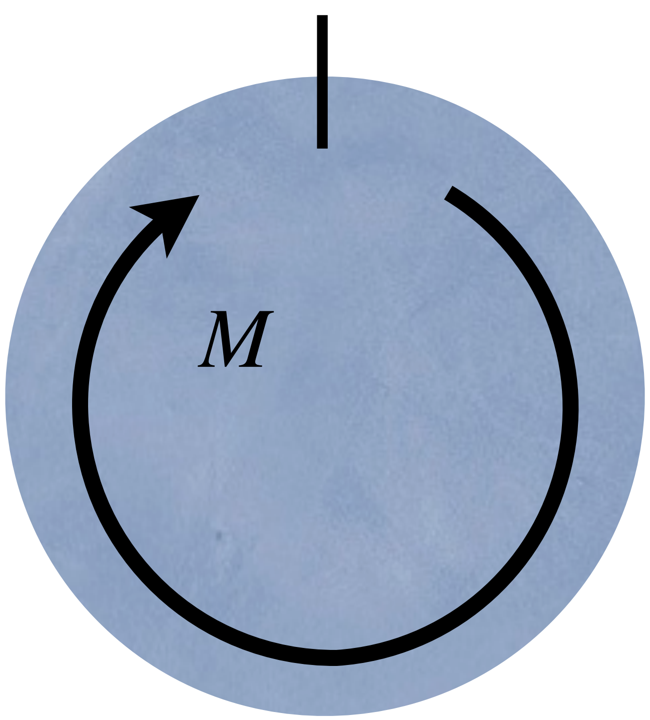
$$|tr\ M| < 2$$

The Stability Criterion

$$\text{if}\ \ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$a + d = trM =$ "trace" of M

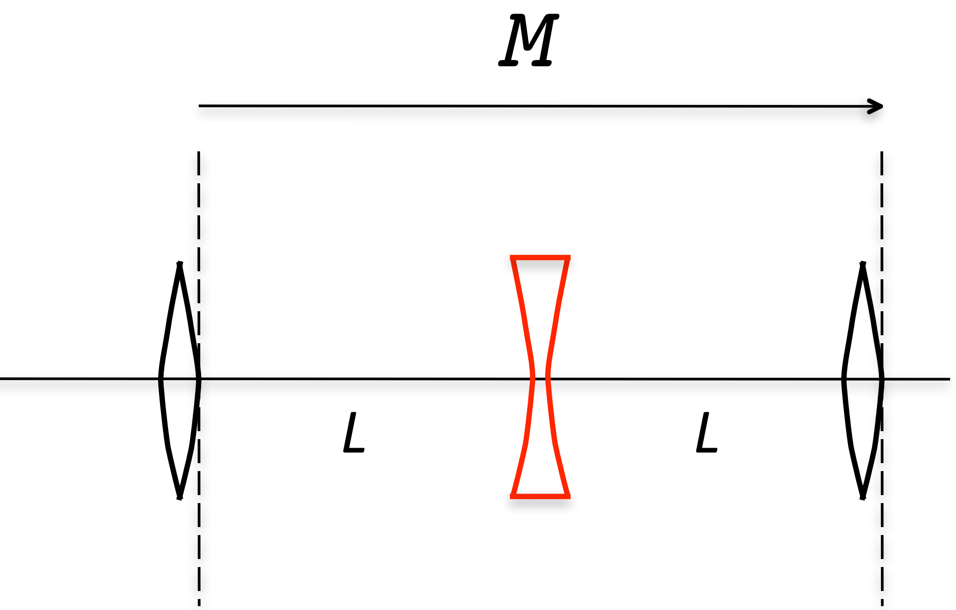
$$ad - bc = \det M = 1$$



repeat application of matrix *M* over and over…
the motion will be stable if -2 < tr*M* < 2

The motion $\overrightarrow{X} = M^k \overrightarrow{X}_0$ is finite and bounded as $k \to \infty$ if $|trM| < 2$

Example: FODO system



and repeat…

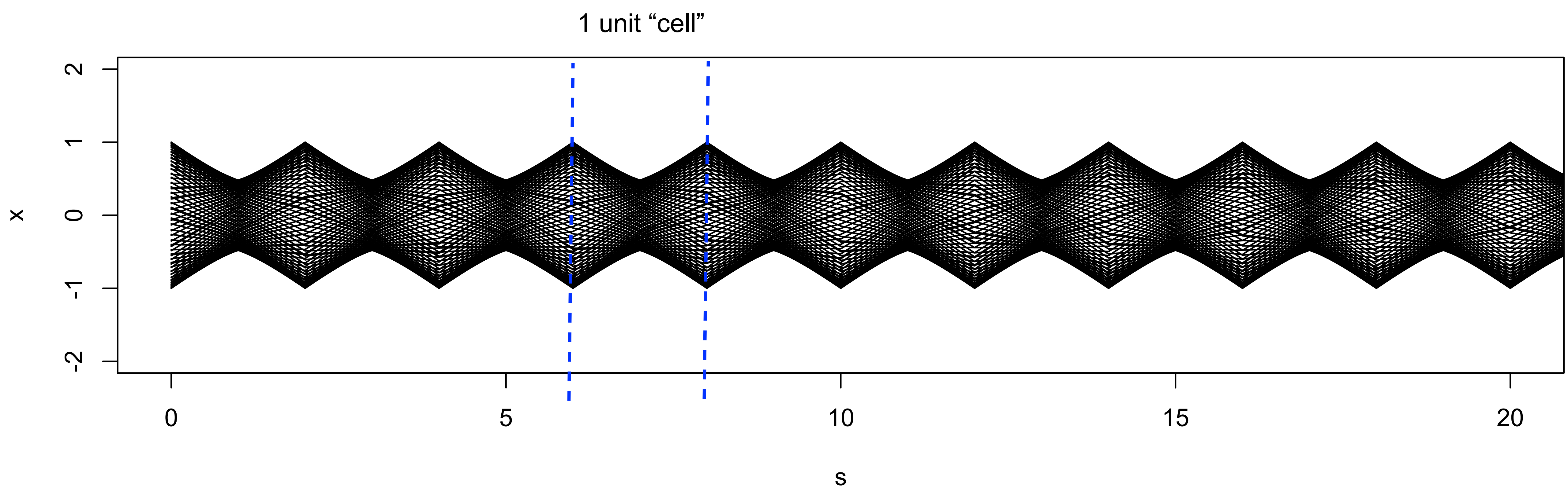
$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1-L/F \end{pmatrix}\begin{pmatrix} 1 & L \\ 1/F & 1+L/F \end{pmatrix} \\
&= \begin{pmatrix} 1+L/F & 2L+L^2/F \\ -L/F^2 & 1-L/F-L^2/F^2 \end{pmatrix}
\end{aligned}
$$

So, $tr\, M = 2 - L^2/F^2$ and thus, for stability, $-2 < 2 - L^2/F^2 < 2$

$$-4 < -L^2/F^2 < 0$$

$$F > L/2$$

Particle Trajectories in a Periodic Lattice



$$\frac{dx'}{ds} = \frac{d^2x}{ds^2} = -\frac{eB'(s)}{p}x$$

$$x'' + K(s)x = 0$$

(Hill's Equation)

$$K(s) = \frac{e}{p}\frac{\partial B_y}{\partial x}(s)$$

$$\boxed{x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]}$$

The Periodic Amplitude Function

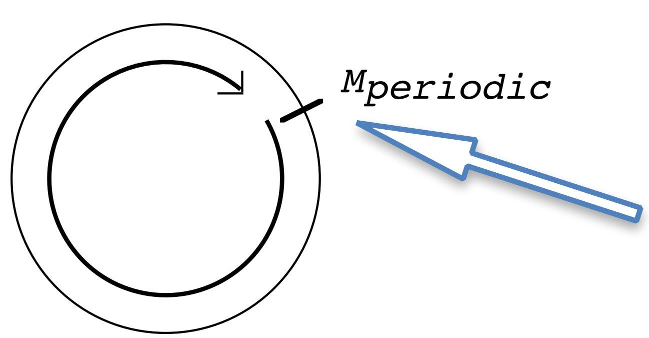
- Previously, …
  - Transport matrix, in terms of amplitude function at end points, and phase advance between:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2} (\cos\Delta\psi + \alpha_0 \sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2} (\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

$$M_{periodic} = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$

sometimes "μ" is used to denote the **periodic** phase advance



Natural choice in a circular accelerator, when values of β, α above correspond to one particular point in the ring

Choice of Initial Conditions

Northern Illinois University

- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a "ring," then natural to choose the **periodic** solutions for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- In a system that takes a distribution from a source or off of a target, wish to "match" to desired initial conditions at the input to the downstream beam line system by using an arrangement of tunable focusing elements

Computation of Courant-Snyder Parameters

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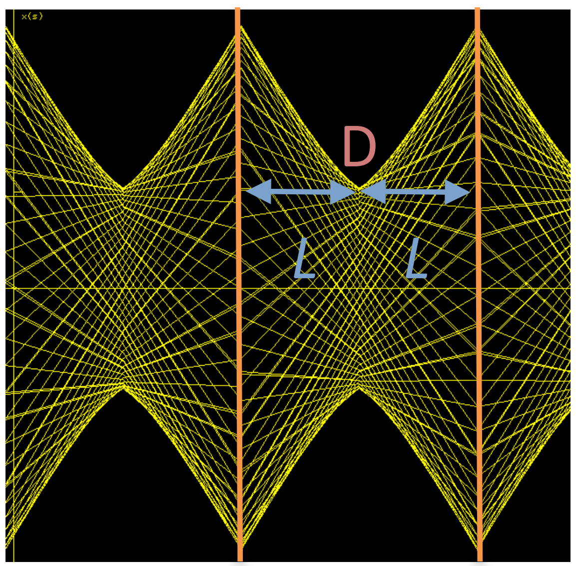
- As an example, consider again the FODO system



F

-F

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$

- Let’s, use above matrix of the periodic section to compute functions at exit of the F quad..

FODO Cell Courant-Snyder Parameters

Northern Illinois University

- From the matrix: $$M_{periodic} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

$$M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

4 numbers

$$trace M = a + d = 2 - L^2/F^2 = 2\cos\mu \quad \Rightarrow \quad \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1 + L/2F}{1 - L/2F}} \qquad \alpha = \frac{a - d}{2\sin\mu} = \sqrt{\frac{1 + L/2F}{1 - L/2F}}$$

- If go from D quad to D quad, simply replace *F* --> -*F* in matrix *M* above
  - So, at exit of the D quad:

$$\beta = 2F\sqrt{\frac{1 - L/2F}{1 + L/2F}} \qquad \alpha = -\sqrt{\frac{1 - L/2F}{1 + L/2F}}$$

for completeness,

$$\gamma = \frac{1 + \alpha^2}{\beta}$$

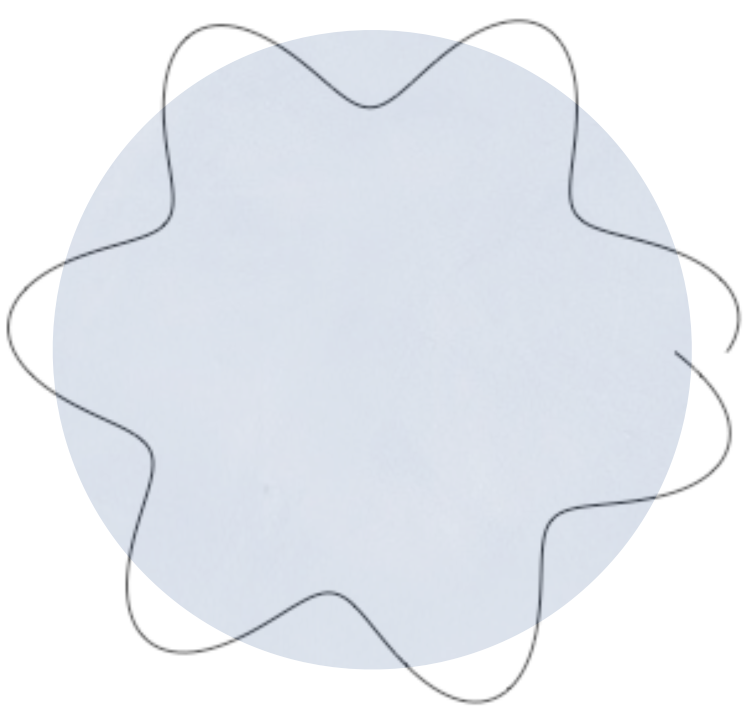
The Betatron Tune

- In a cyclic accelerator (synchrotron), the particles will oscillate transversely (betatron oscillations) with the *betatron frequency*.
- The betatron frequency is determined by the total phase advance once around the ring:

$$\Delta\psi_{total} = \oint \frac{ds}{\beta(s)}$$

$$\nu \equiv \Delta\psi_{total}/2\pi$$

$$\mathrm{tr}M = 2\cos(2\pi\nu)$$



Betatron Tune: # of oscillations per revolution

$$f_{betatron} = \nu f_{rev}$$

Ex: Tune of a FODO synchrotron

- Suppose a ring is made up of *N* FODO cells
- Each cell has phase advance given by the lens spacing and lens focal length:

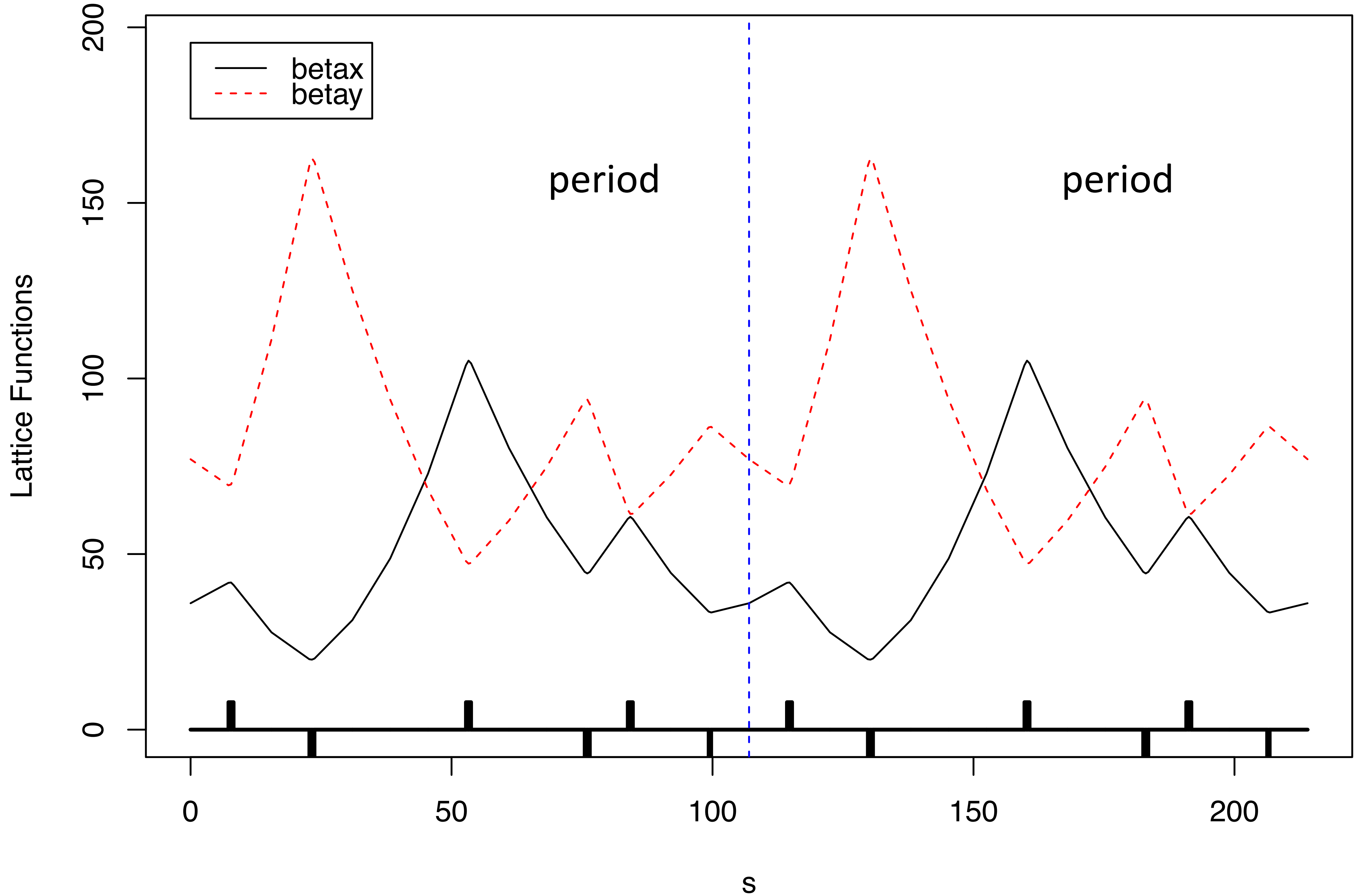
$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

- So, the tune of this simple synchrotron would be:

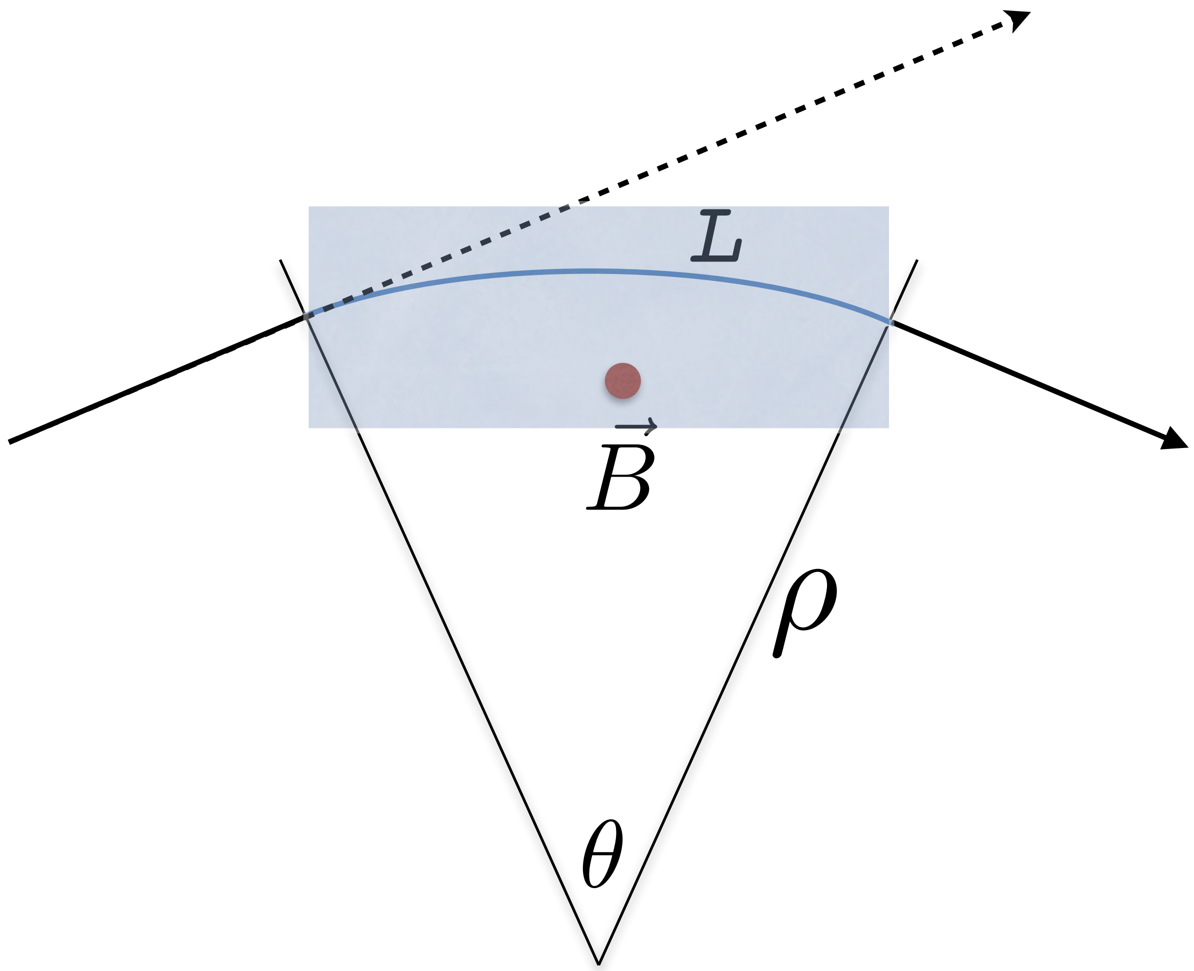
$$\nu = N\mu/2\pi \approx N\frac{L}{2\pi F} = \frac{2LN}{4\pi F} = \frac{C}{2\pi}\frac{1}{2F} = \frac{R}{2F}$$

- Ex: Main Injector at Fermilab: *R* ~ 500 m; *F* ~ 13 m
  - so, ν ~ 20
  - thus, if initiate a betatron oscillation in this synchrotron it will oscillate ~20 times per revolution around the ring

Arbitrary Distribution of Quadrupoles



Bending through Dipole Field



$$\theta = \frac{L}{\rho} = \frac{B \cdot L}{(B\rho)}$$

$$= \frac{q \cdot B \cdot L}{p}$$

Dispersion

$$B\rho = \frac{p}{q} \qquad \theta = \frac{qB \cdot \ell}{p}$$

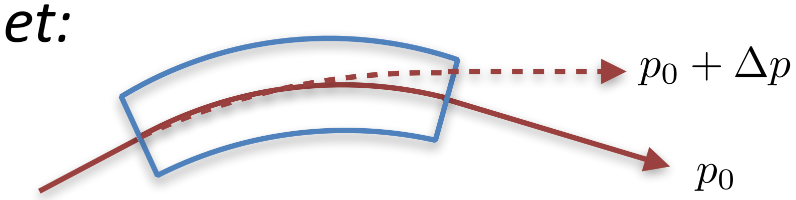
The bend angle (and/or focusing strength) depends upon momentum

Similar to index of refraction depending upon frequency

dipole steering "error" due to a different momentum —> "dispersion"

focusing "error" due to a different momentum —> "chromatic aberration"

dipole magnet:



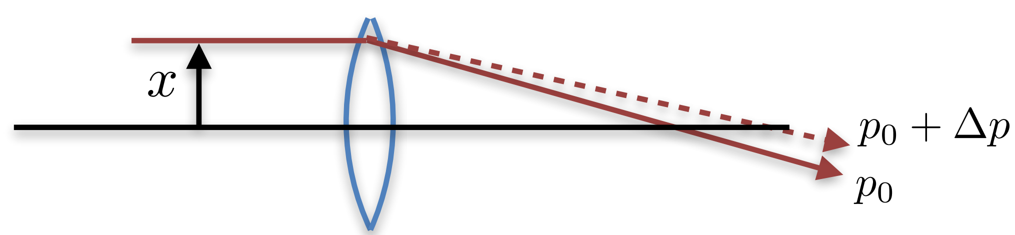
$$\frac{\Delta\theta}{\theta_0} = -\frac{\Delta p}{p}$$

[i.e., in "opposite" direction of bend]

at exit, to lowest order, $\Delta x' = \theta_0 \frac{\Delta p}{p}$

$$\Delta x \approx \frac{1}{2}\ell\theta_0 \frac{\Delta p}{p}$$

likewise, for quadrupole:



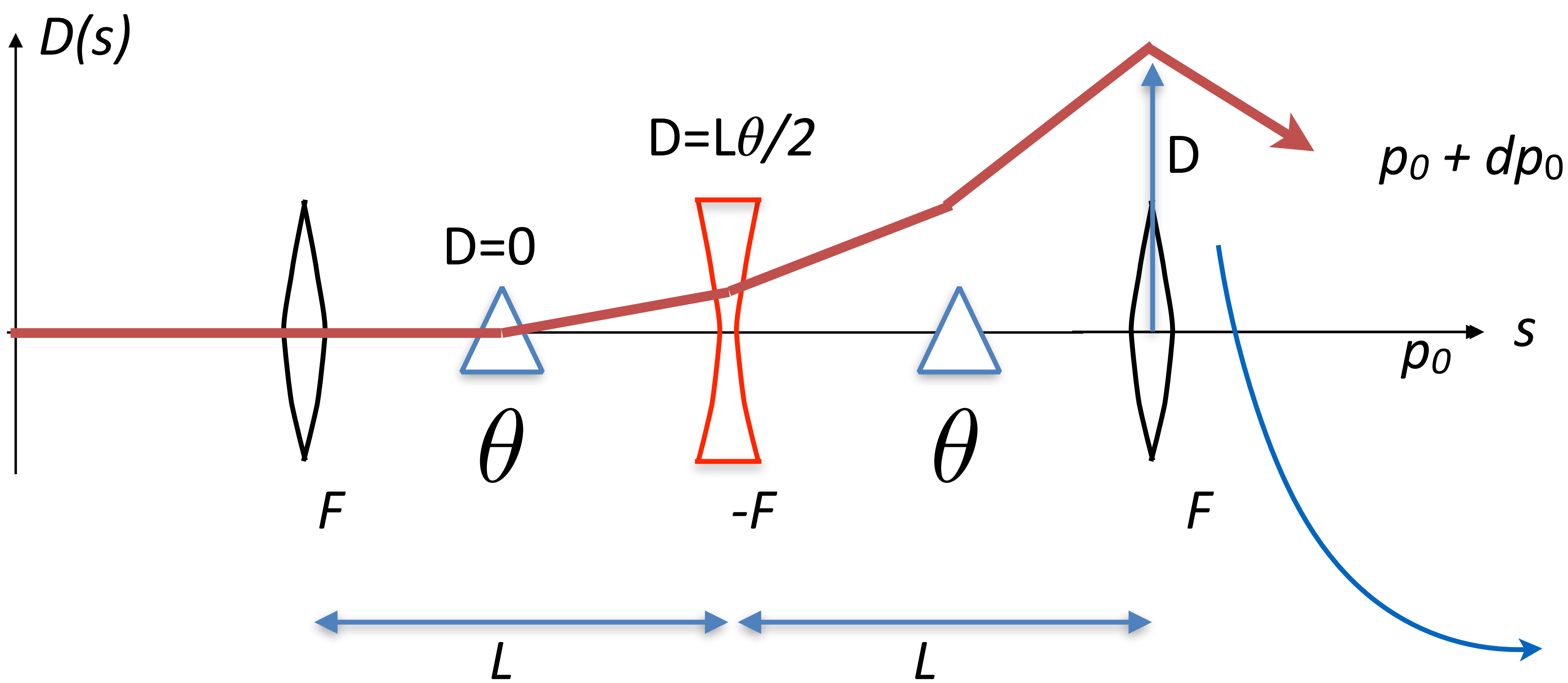
$$f = f_0(1 + \frac{\Delta p}{p})$$

Trajectory differences due to momentum differences referred to as "dispersion"

and, $$\boxed{D(s, \Delta p/p) \approx D(s) \equiv \frac{\Delta x(s)}{\Delta p/p}}$$ *"dispersion function"*

Generating Dispersion

- System of quadrupoles with alternating-sign gradients (F, D, F, …) separated by distance *L*, and with bending magnets present…



$$D = \frac{L\theta}{2}\left(1 + \frac{L}{2f}\right)$$

Ex: D = 3 m, dp/p = 0.3%, then Δx = 9 mm

Beam Size Including Dispersion

- Total excursion due to “off momentum” plus betatron oscillation:

$$x = x_\beta + D\ \delta \qquad\qquad \delta \equiv \Delta p/p$$

$$x^2 = x_\beta^2 + 2x_\beta D\delta + D^2\delta^2$$

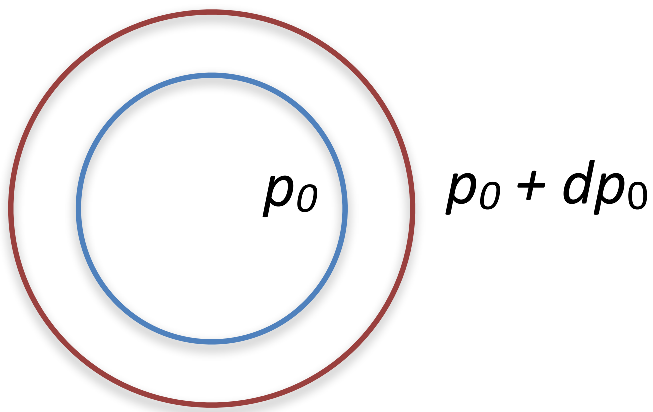
- Assuming no correlation between x_β and particle’s momentum:

$$\langle x^2 \rangle = \langle x_\beta^2 \rangle + D^2 \langle \delta^2 \rangle$$

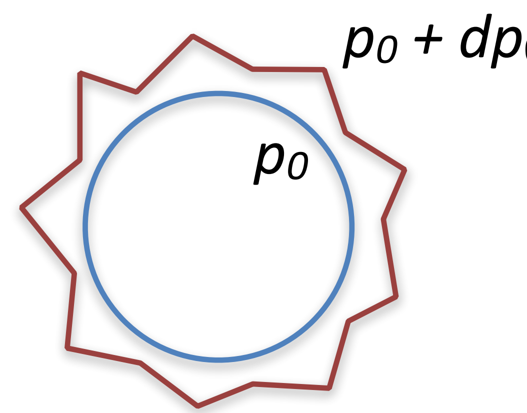
$$\langle x^2 \rangle = \epsilon\beta/\pi + D^2 \langle \delta^2 \rangle$$

Periodic Dispersion Function

uniform bend field:



add gradients…

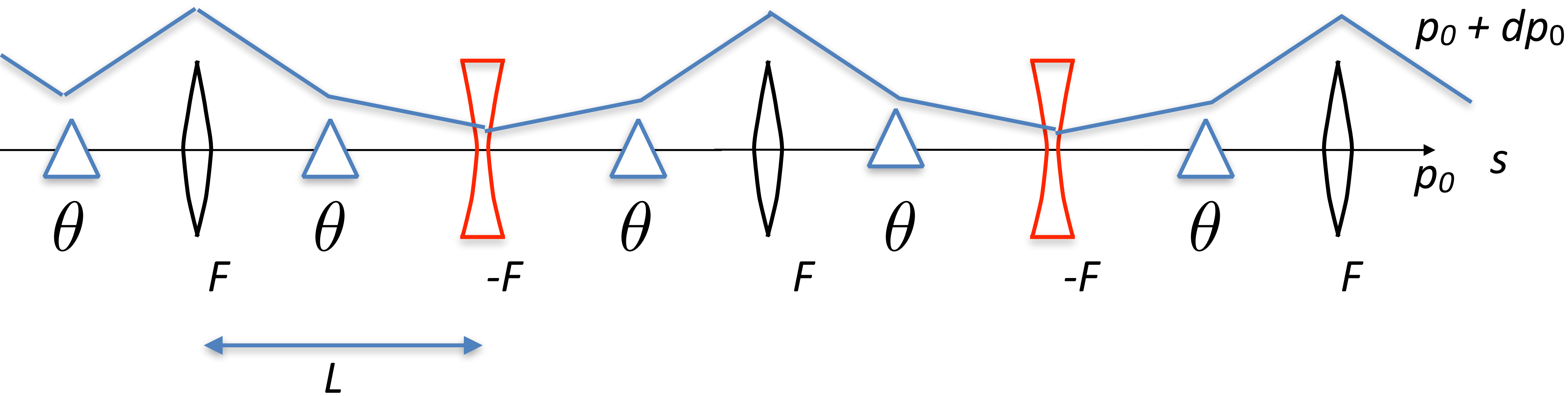


the trajectory "closed" orbit for momentum p+Δp

$$D(s,p) = \frac{\Delta x(s,p)}{\Delta p/p}$$

the orbit of an off-momentum particle which closes on itself is described by the *periodic* dispersion function

Ex: FODO Cells with Bending Magnets



$$\sin\frac{\mu}{2} = \frac{L}{2F}$$

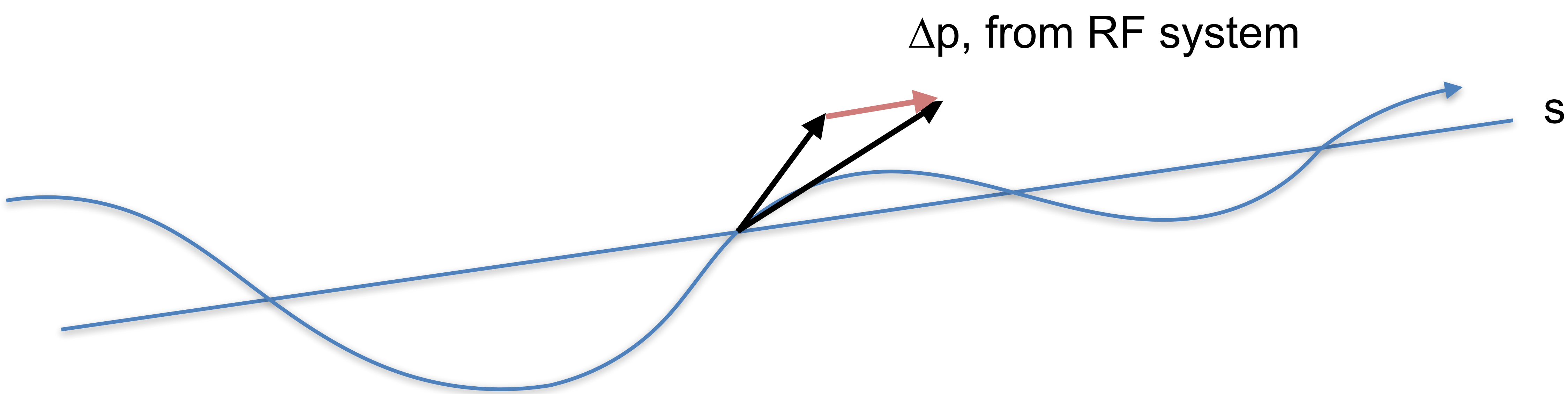
$$D_{max,min} = \frac{L\theta}{2\sin^2(\mu/2)}\left(1 \pm \frac{1}{2}\sin(\mu/2)\right)$$

Values of dispersion function are typically ~ few meters

Note: in a weak-focusing synchrotron, would have $D = R_0$!

Adiabatic Damping from Acceleration

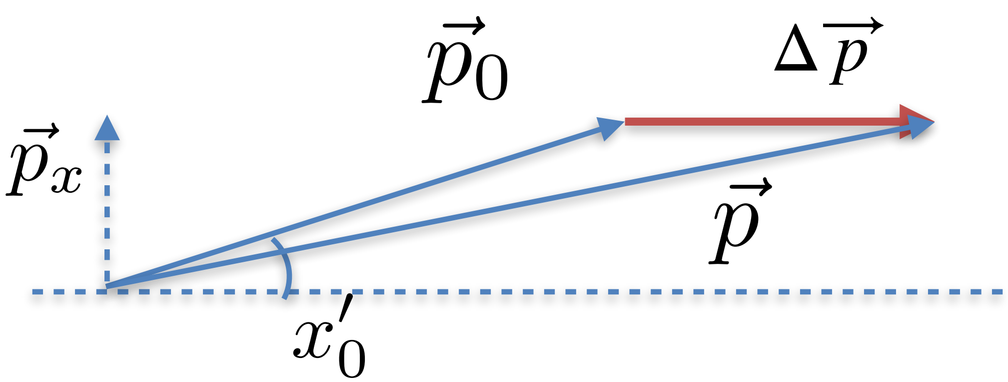
- Transverse oscillations imply transverse momentum. As accelerate, momentum is "delivered" in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.



- The coordinates *x-x'* are not canonical conjugates, but x-p_x are; thus, from classical mechanics, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Adiabatic Damping from Acceleration

details…



Note: assuming that ALL particles receive the same *Δp* from the cavity

$$x' = \frac{p_x}{p} = \frac{p_x}{\sqrt{p_0^2 + \Delta p^2 - 2\Delta p\, p_0 \cos\phi}} = \frac{p_x}{p_0}\left(1 - \frac{\Delta p}{p_0} + \ldots\right) \approx x'_0\left(1 - \frac{\Delta p}{p_0}\right)$$

Note: particles at peak of their betatron oscillation will have little/no change in *x'*, while particles with large transverse angles will have their *x'* affected most

relative momentum gain from RF system

$$\Longrightarrow \quad \Delta x' = -x'_0 \frac{\Delta p}{p_0}$$

Damping of Oscillations

Now assuming that the incremental change in momentum is small compared to the overall momentum, and that the changes occur gradually on time scales large compared to those of the betatron motion…

$$x = x_0$$
$$x' = x_0'(1-\delta) \qquad\qquad \delta \equiv \Delta p / p_0$$

Then if the original emittance is $\epsilon_0 = \pi\sqrt{\langle x_0{}^2\rangle\langle x_0'^2\rangle - \langle x_0 x_0'\rangle^2}$ then, after passing through the RF system,

$$\epsilon = \pi\sqrt{\langle x_0{}^2\rangle\langle x_0'^2(1-\delta)^2\rangle - \langle x_0 x_0'\rangle^2(1-\delta)^2} = \epsilon_0\sqrt{(1-2\delta+\delta^2)} \approx \epsilon_0(1-\delta)$$

Thus,

$$\epsilon - \epsilon_0 = \Delta\epsilon = -\epsilon_0\delta$$

or,

$$\frac{\Delta\epsilon}{\epsilon} = -\frac{\Delta p}{p}$$

So,

$$\boxed{\epsilon \propto \frac{1}{p} \qquad x_{rms} \propto \frac{1}{\sqrt{p}}}$$

Normalized Beam Emittance

- Hence, as particles are accelerated, the area in *x-x'* phase space is not preserved; however, the area in x-p_x **is** preserved. Thus, we define a "normalized" beam emittance, as

$$p = (\gamma m)(\beta c) = (\beta\gamma)\, mc$$

Here: *relativistic* factors!

$$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)_{Lorentz}$$

- In principle, the *normalized* beam emittance should be preserved during acceleration, and hence along the chain of accelerators from source to target. Thus it is a measure of beam quality, and its preservation is a measure of accelerator performance.
- In practice, it is not preserved -- non-adiabatic acceleration, especially at the low energy regime; non-linear field perturbations; residual gas scattering; charge stripping; field errors and setting errors; *etc.* -- all contribute at some level to increase the beam emittance. Best attempts are made to keep the emittance as small as possible.