Particle Transport, Beams, and Emittance

Mike Syphers

Northern Illinois University & Fermilab

Operator Lectures

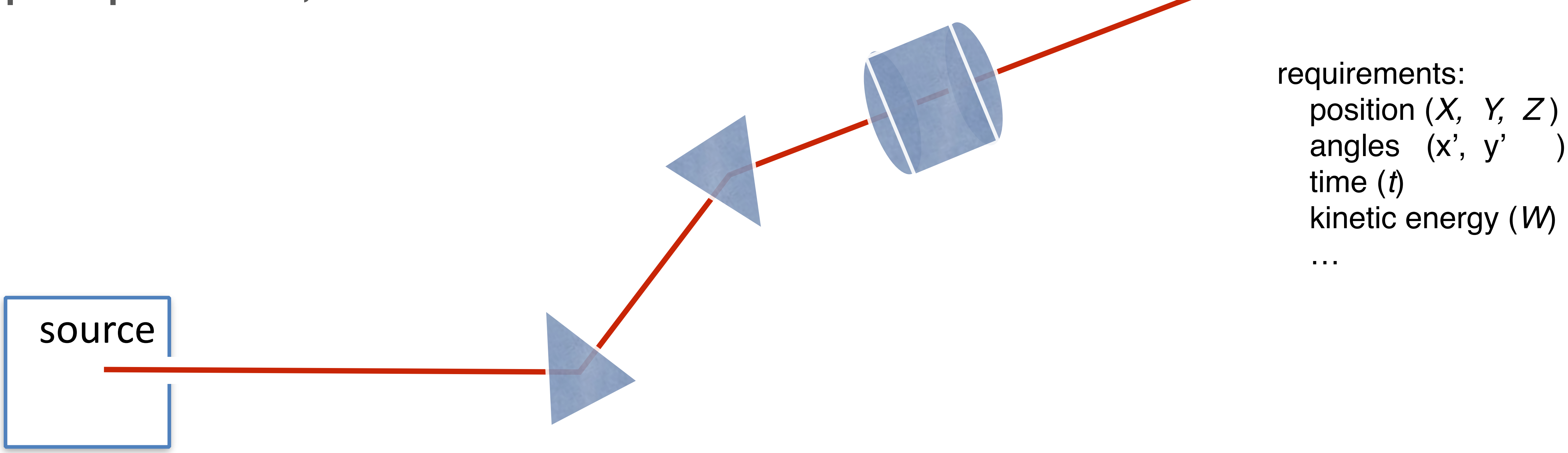
April 2020

syphers@fnal.gov

The Problem

- 1927: Lord Rutherford requested a “copious supply” of projectiles “more energetic than natural alpha and beta particles” for use in scientific studies of nature

- For given type of particle, create an ideal system to provide particles to a final location with desired trajectory, desired kinetic energy per particle, at the desired time



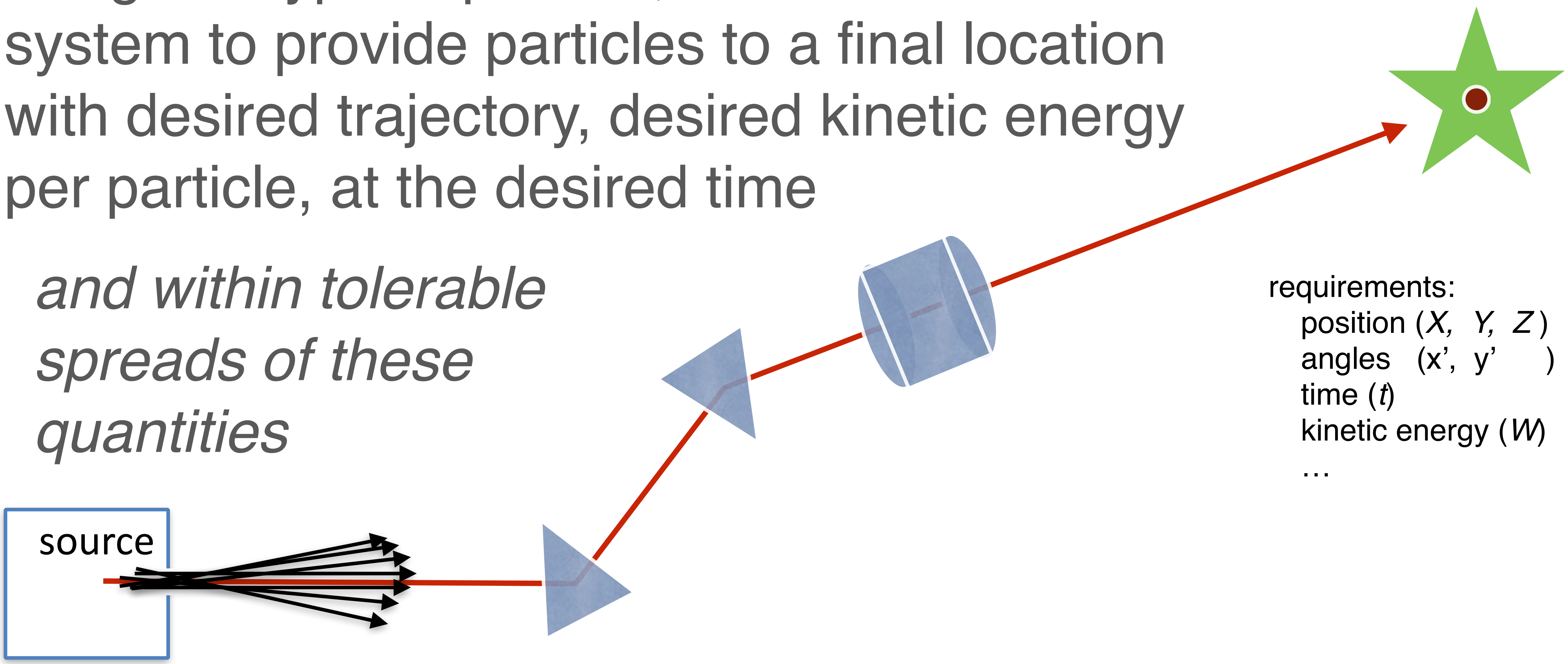
requirements:
position (X, Y, Z)
angles (x', y')
time (t)
kinetic energy (W)
…

The Problem

- 1927: Lord Rutherford requested a "copious supply" of projectiles "more energetic than natural alpha and beta particles" for use in scientific studies of nature

- For given type of particle, create an ideal system to provide particles to a final location with desired trajectory, desired kinetic energy per particle, at the desired time

  and within tolerable spreads of these quantities



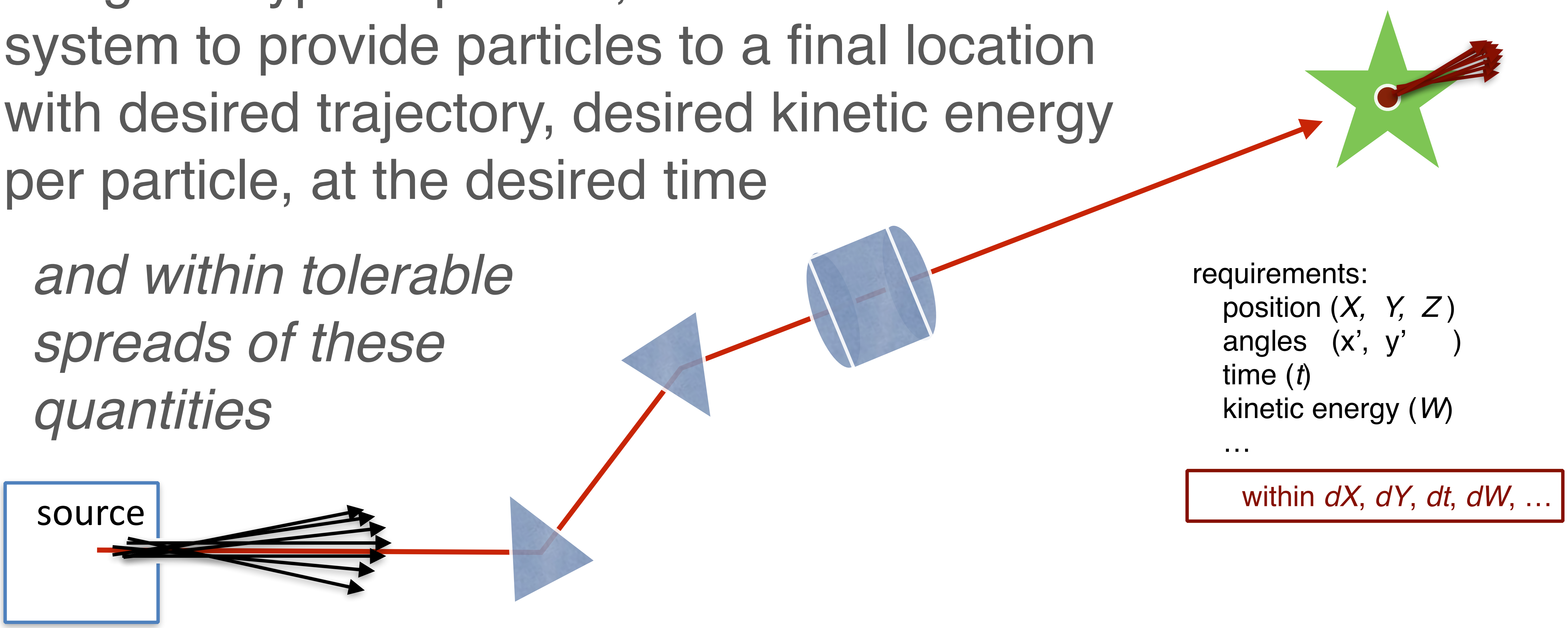
requirements:
position (X, Y, Z)
angles (x', y')
time (t)
kinetic energy (W)
…

The Problem

- 1927: Lord Rutherford requested a “copious supply” of projectiles “more energetic than natural alpha and beta particles” for use in scientific studies of nature

- For given type of particle, create an ideal system to provide particles to a final location with desired trajectory, desired kinetic energy per particle, at the desired time

  and within tolerable spreads of these quantities



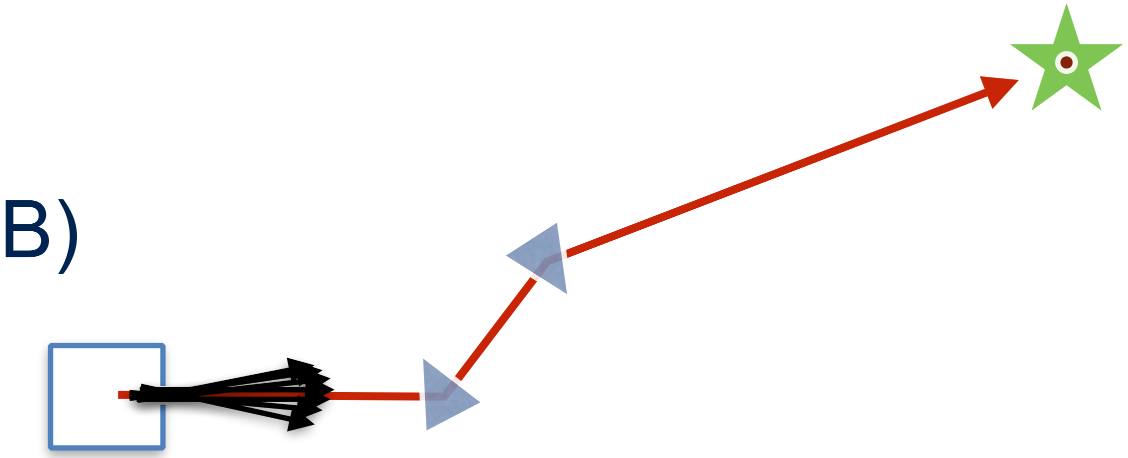
A Few Words on Particle Sources…

- Electrons — relatively easy
  - filaments; photocathodes, laser driven plasmas,…
- Protons — not “too” hard
  - ionized hydrogen gas, plasma sources,…
- Ions — similar techniques
  - ovens, plasma sources, ECRs — plus, separation
- Even more exotic particles: target, separate, collect
  - heavy ion isotopes
  - pions, muons, antiprotons, neutrinos,…
- Also polarized sources, …

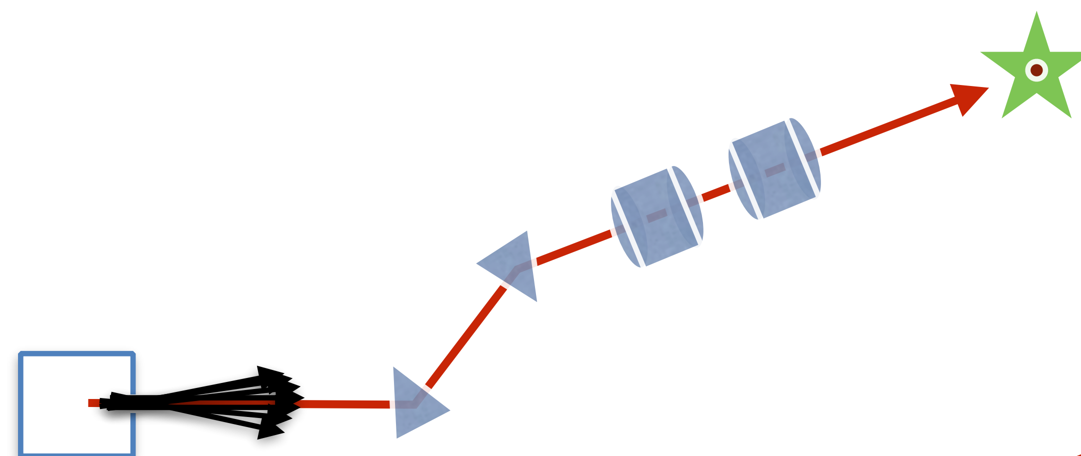
Single-Pass vs. Repetitive Systems

Northern Illinois University

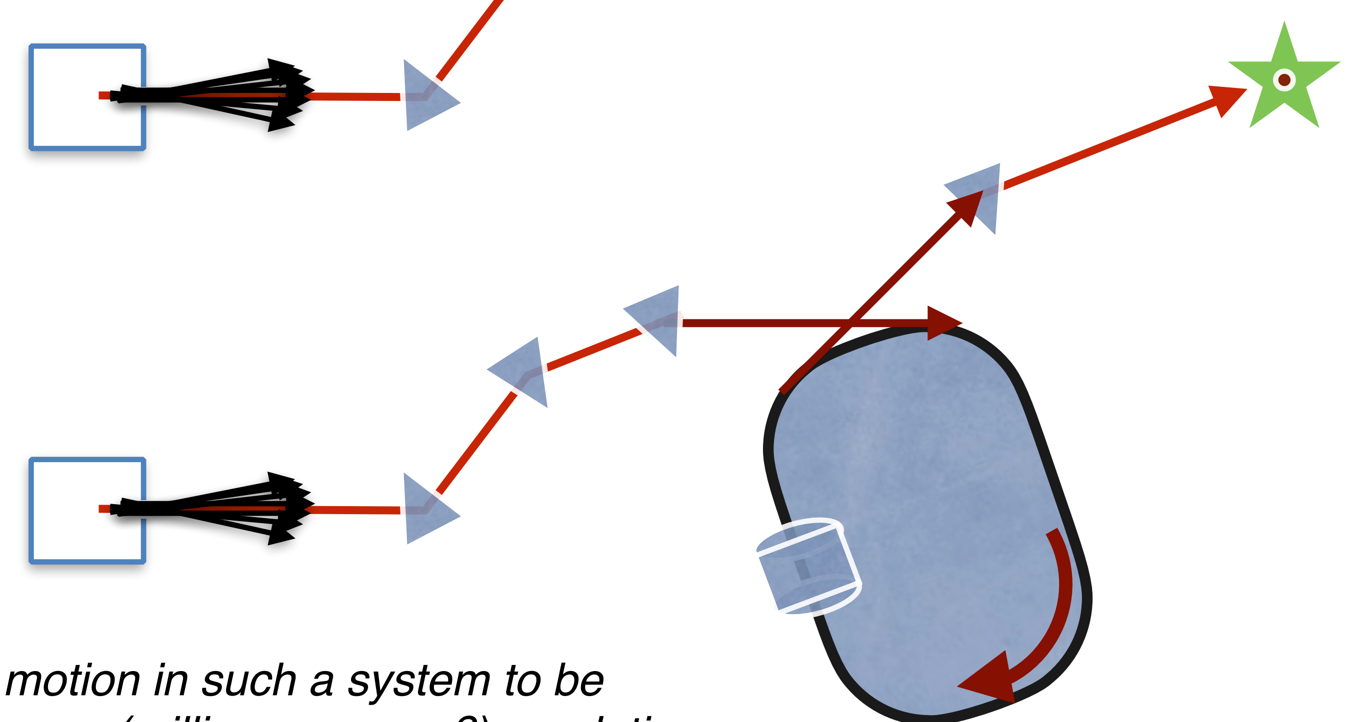
- Beam Transport (from point A to point B)



- Acceleration along the way
  - single-pass with acceleration



  - multi-pass acceleration



may need motion in such a system to be stable for many (millions or more?) revolutions

Stability of Motion Near the Ideal

Northern Illinois University

- Not all particles (any??) begin "on" the design trajectory with *exactly* the ideal energy/momentum
- We wish to have a system that will keep particles near the ideal conditions as they are transported (and possibly accelerated) through the system
- Particles emerge from their "source" with a slight divergence and will need to be guided back toward the ideal trajectory
- Also, particles with different energies/momenta will travel at different speeds, and hence may not arrive at cavities, experiments, etc., at the ideal time

Reduction of the Problem

- Will treat transverse motion of particles through the accelerator as independent of the longitudinal motion, and study these two cases separately. Must show along the way that this is viable approach.
- Certainly not always be the case...
  - electric fields used for focusing at low energies can also accelerate the particles as well;
  - fields in the gaps of cavities will have focusing effects; etc.
- However, much of the "cross talk" can be minimized, and for much of the particle's journey, especially at higher energies, the major transverse focusing can be performed by magnetic fields -- particle's energy not changed
- Look at "linear" fields, *i.e.* linear restoring forces

Outline

Northern Illinois University

- Today: Particle Transport, Beams, and Emittance
  - Investigate transverse motion of single particle in magnetic fields
  - Discuss properties of a “beam” of particles
  - Phase space and the beam “ellipse”
  - ***Transport*** of beam phase space properties

- Courant-Snyder Approach to Linear Transverse Dynamics

- Acceleration and Longitudinal Dynamics

- Linear and Nonlinear Imperfections and Adjustments

Equations of Motion

- Lorentz Force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

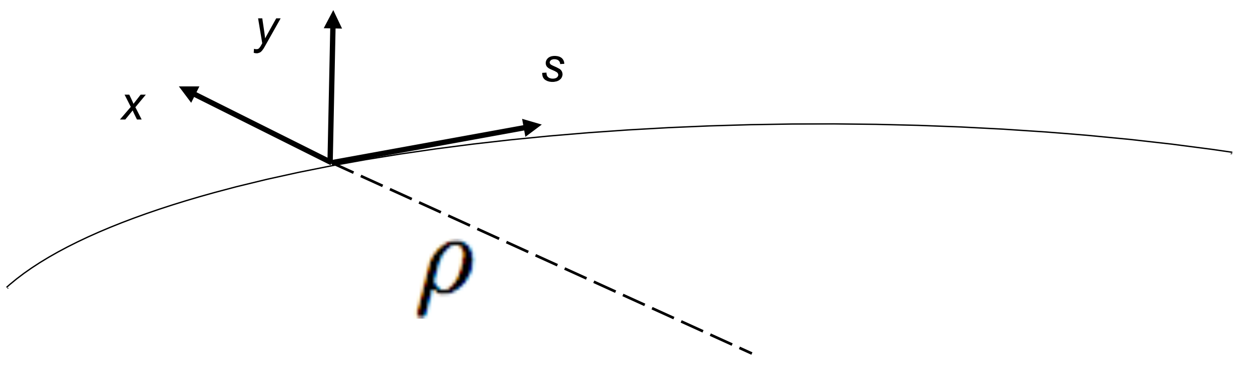
- Magnetic Rigidity
  - particle of unit charge, q = e:

$$\frac{(\gamma m)v^2}{\rho} = qvB \quad \longrightarrow \quad \gamma m v / q = B \cdot \rho$$

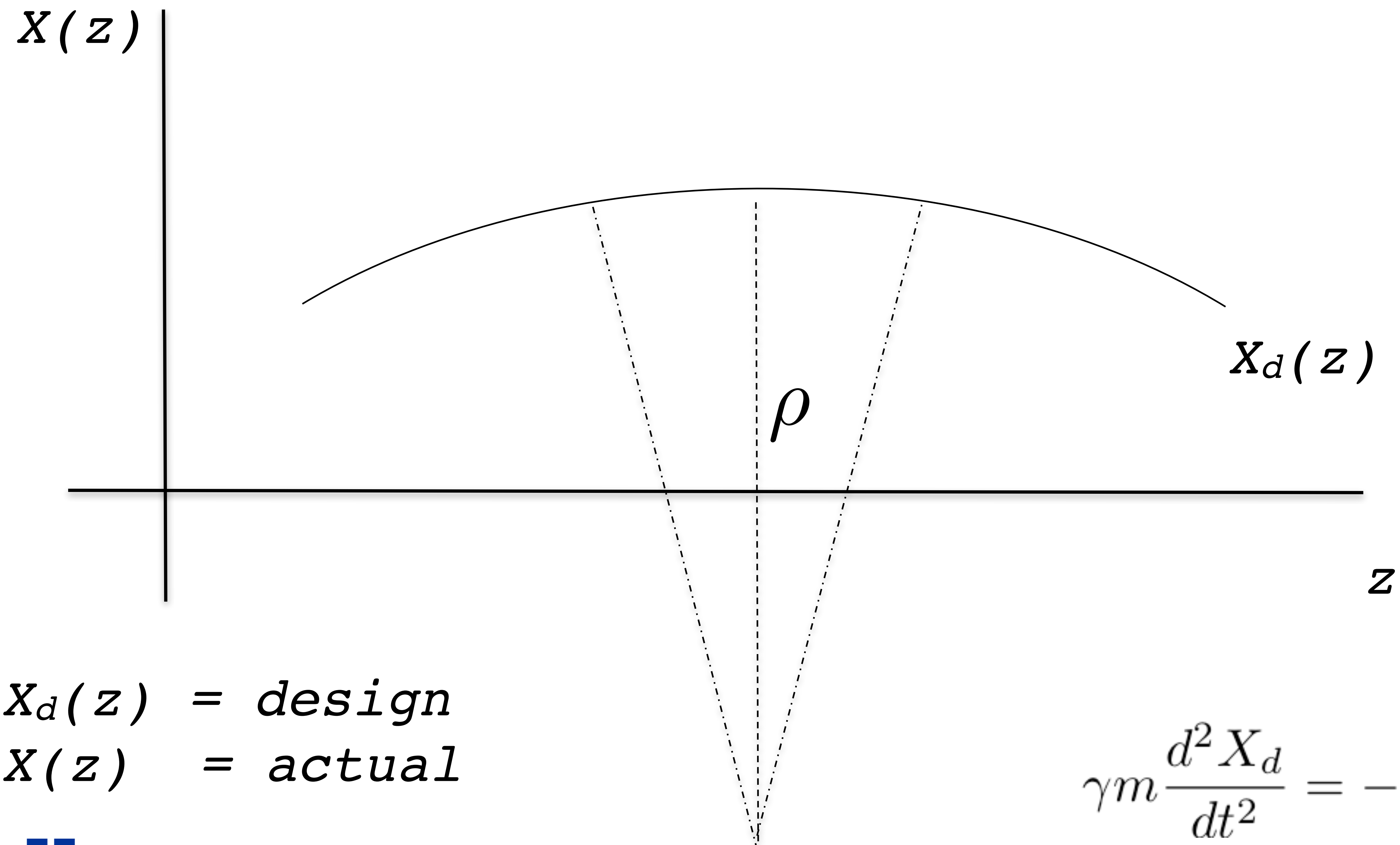
$$\boxed{B\rho \equiv \frac{p}{q}} = \frac{p}{e}$$

$$B\rho[\mathrm{T \cdot m}] = (10/2.9979) \cdot p[\mathrm{GeV/c}]$$

- Reference Trajectory
  - Local Coordinate System



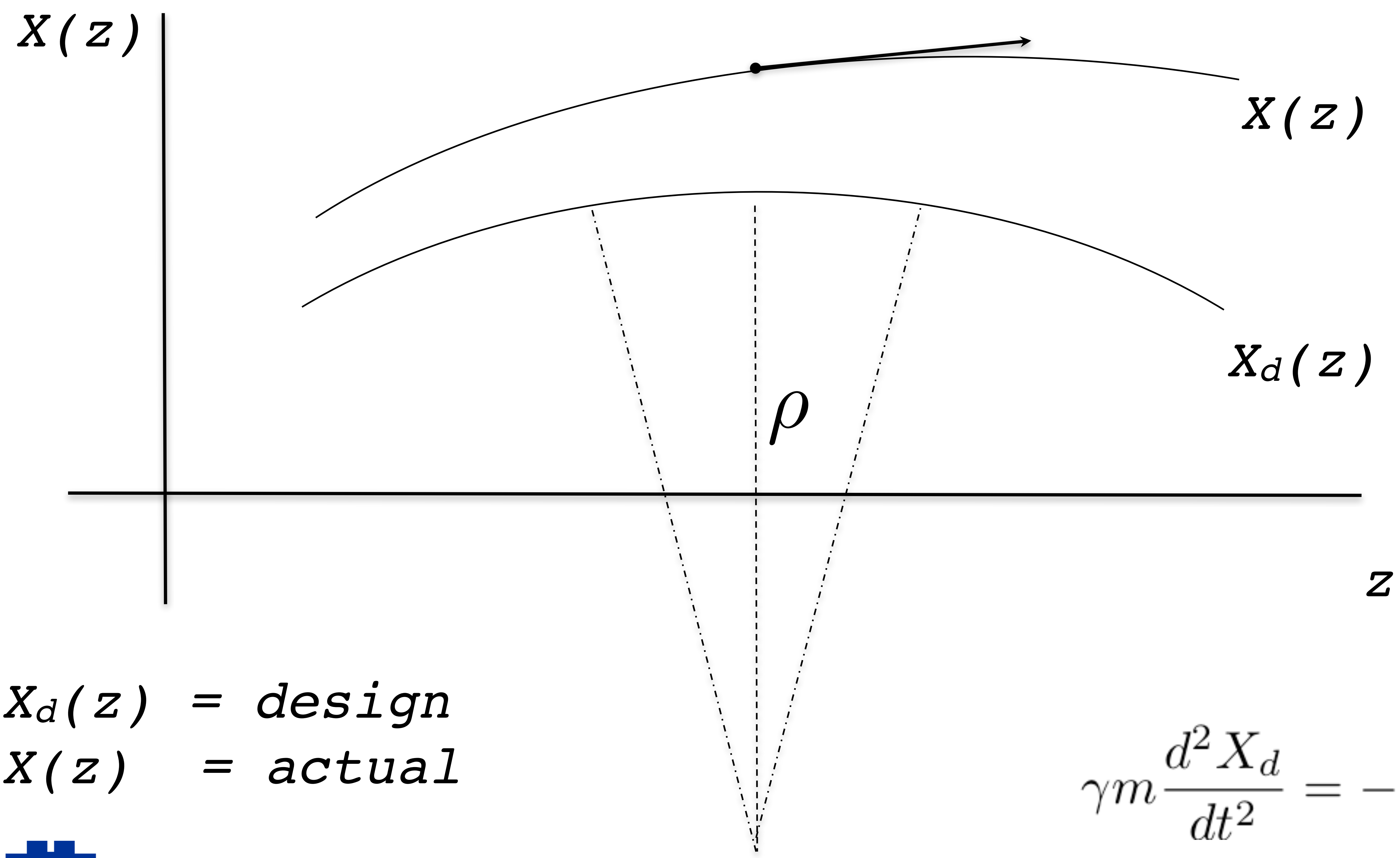
Linear Magnetic Fields for Guiding & Focusing



$X_d(z)$ = *design*
$X(z)$ = *actual*

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Linear Magnetic Fields for Guiding & Focusing

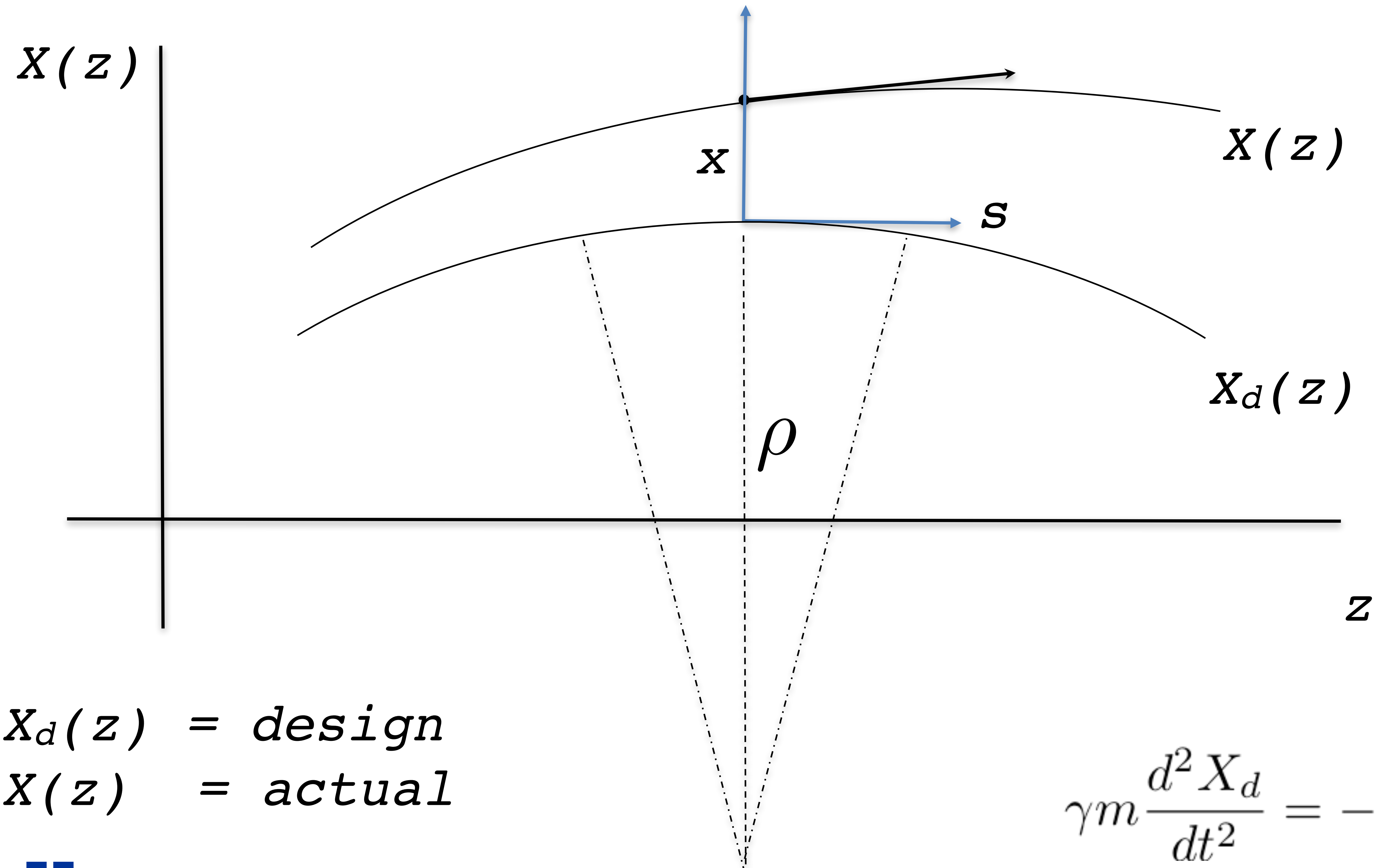


$X_d(z)$ = *design*

$X(z)$ = *actual*

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Linear Magnetic Fields for Guiding & Focusing

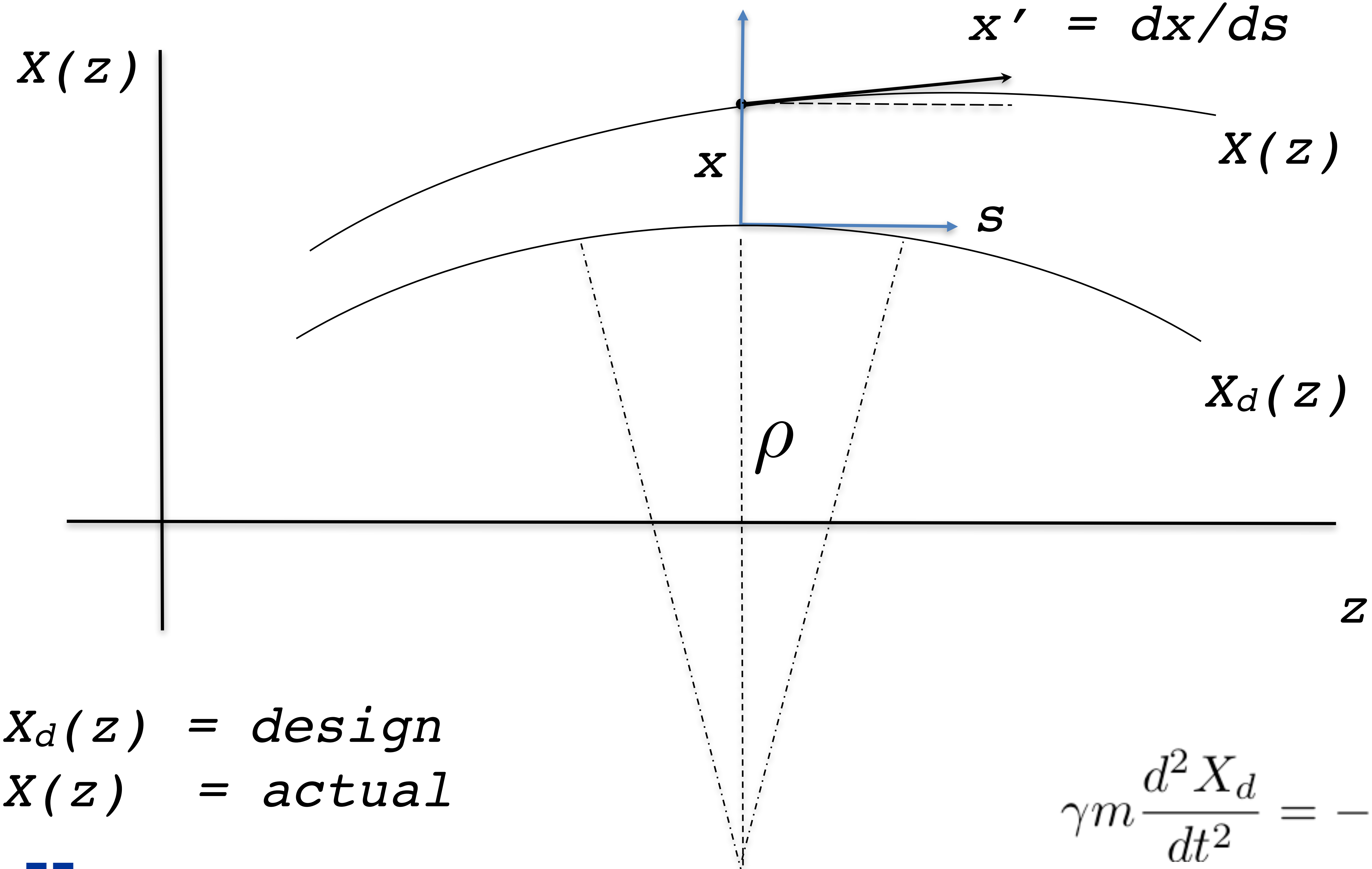


$X_d(z)$ = *design*

$X(z)$ = *actual*

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Linear Magnetic Fields for Guiding & Focusing



$X_d(z)$ = design
$X(z)$ = actual

$$\gamma m \frac{d^2 X_d}{dt^2} = -e v_s B_0$$

Linear Transverse Fields ($B_z = 0$)

Northern Illinois University

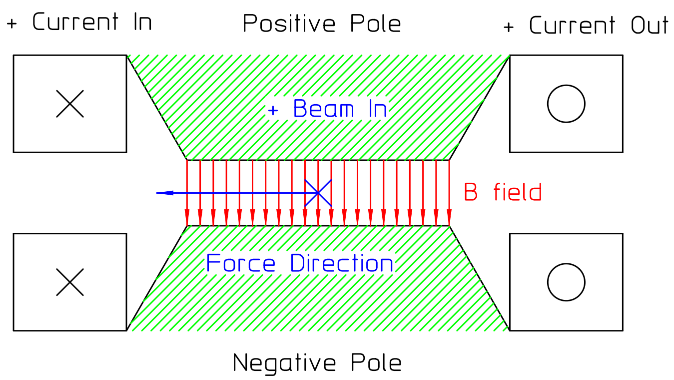
Think "*simple harmonic oscillator*": $F \propto -x \longrightarrow \ddot{x} + k \cdot x = 0 \longrightarrow x \sim a\sin(\omega t) + b\cos(\omega t)$

Drift Space	$B_x = 0$	$B_y = 0$
Bending Region (dipole magnet)	$B_x = 0$	$B_y = B_0$
Focusing Region (quadrupole magnet)	$B_x = B'y$	$B_y = B'x$
(electrostatic quadrupole)	$E_y = -E'y$	$E_x = E'x$
Combined Function Magnet	$B_x = B'y$	$B_y = B_0 + B'x$
Uniform magnet + ES quad (*a' la* Muon g-2 ring)	$B_x = 0$, $E_y = -E'y$	$B_y = B_0$, $E_x = E'x$

Note: *B'*, *E'* are "gradients" — change in field per distance: T/m, (V/m)/m

The Quadrupole Magnet

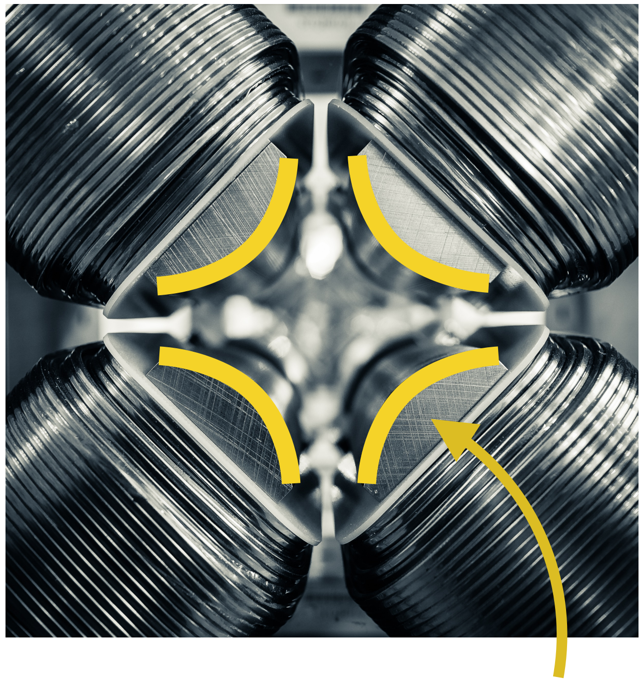
- Dipole magnet:
  - uniform bend field



Dipole, FNAL

Field: unit of Tesla (T)

- Quadrupole magnet:
  - field = 0 on longitudinal axis
  - varies linearly with transverse position



$$\vec{B} = \nabla \Phi_m \longrightarrow \nabla^2 \Phi_m = 0$$

$\Phi_m = constant$

$$\Phi_m = C r^n \sin n\phi$$

2n-pole magnet
$n = 2$ (quadrupole):

$$B_r = nCr^{n-1} \sin n\phi$$

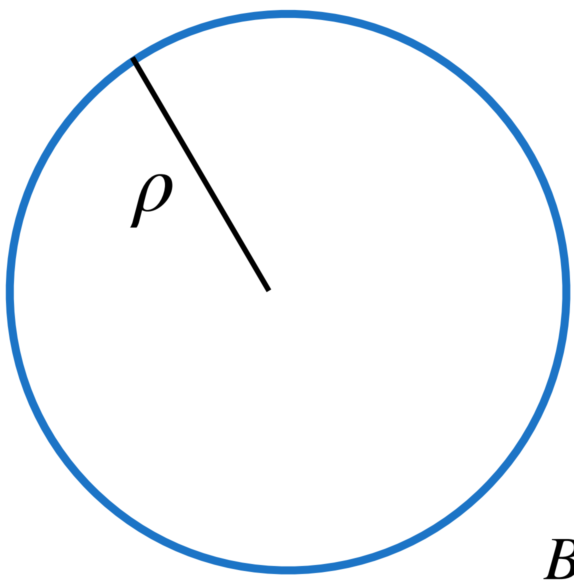
$$B_\phi = nCr^{n-1} \cos n\phi$$

$$B_x = 2C\ y \equiv B' y$$

$$B_y = 2C\ x \equiv B' x$$

B' = Gradient of Field: unit of T/m

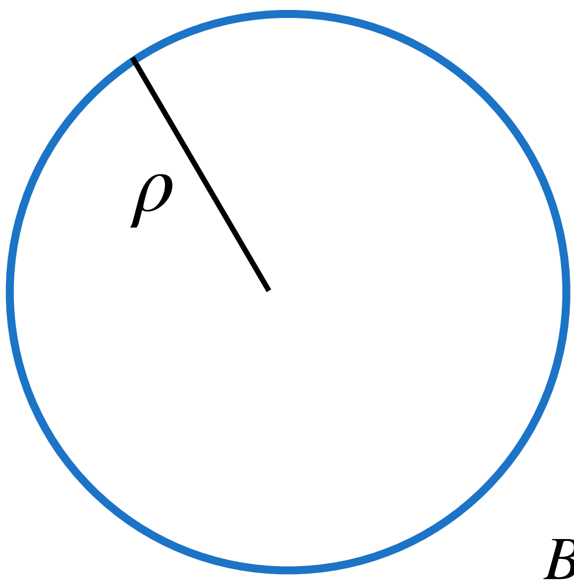
Linear Restoring Forces — Short Version



$$\frac{(\gamma m)v^2}{\rho} = evB_0$$

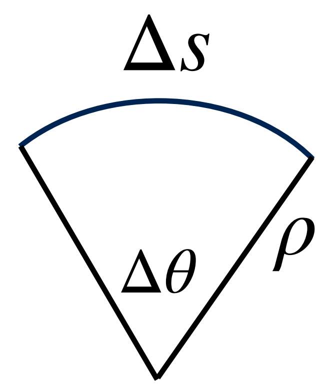
$$B_0\rho = \frac{(\gamma m)v}{e} = p/e$$

Linear Restoring Forces — Short Version



$$\frac{(\gamma m)v^2}{\rho} = evB_0$$

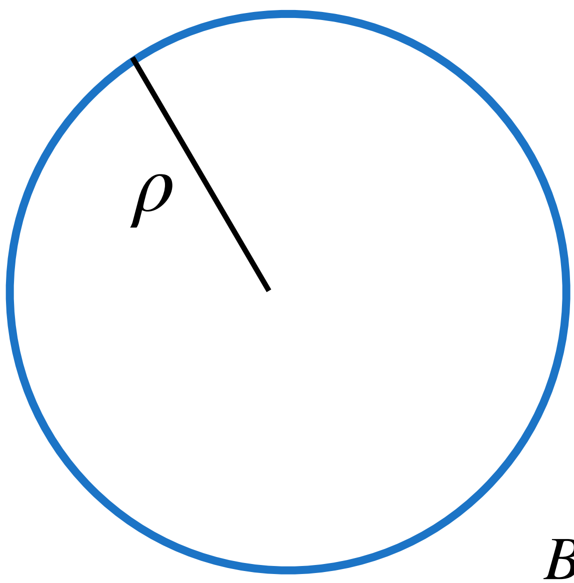
$$B_0\rho = \frac{(\gamma m)v}{e} = p/e$$



$$\Delta\theta = \frac{\Delta s}{\rho} = \frac{B \cdot \Delta s}{B\rho}$$

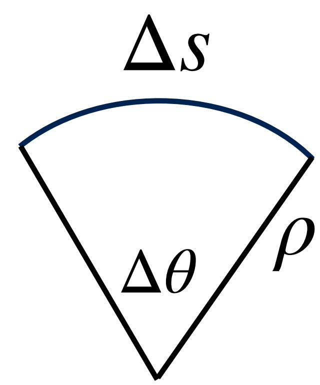
Linear Restoring Forces — Short Version

Northern Illinois University



$$\frac{(\gamma m)v^2}{\rho} = evB_0$$

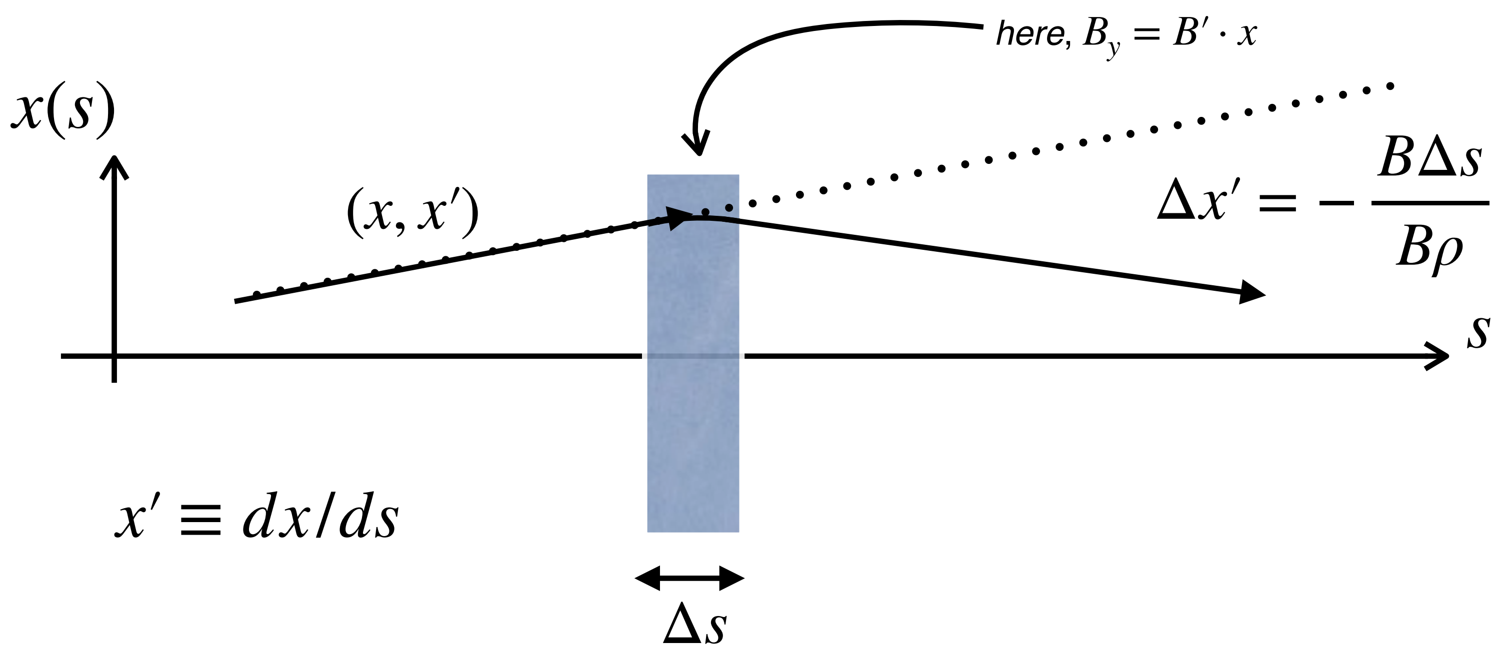
$$B_0\rho = \frac{(\gamma m)v}{e} = p/e$$



$$\Delta\theta = \frac{\Delta s}{\rho} = \frac{B \cdot \Delta s}{B\rho}$$

suppose: $B = B' \cdot x$

Look relative to the *ideal* trajectory:

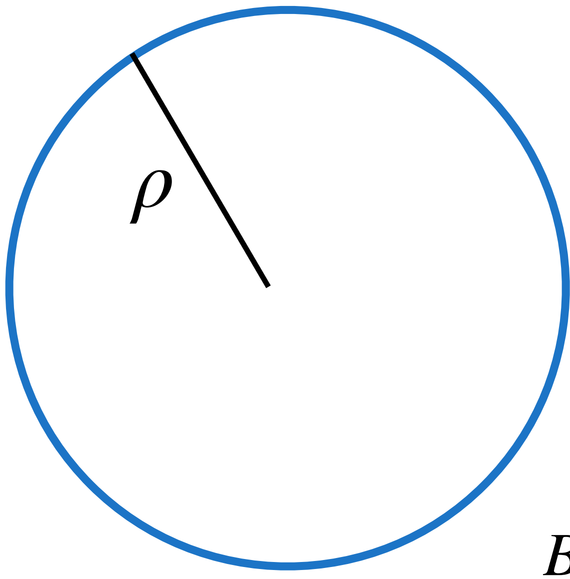


$$\Delta x' = -\frac{B\Delta s}{B\rho} = -\frac{B'\Delta s}{B\rho} \cdot x$$

$x' \equiv dx/ds$

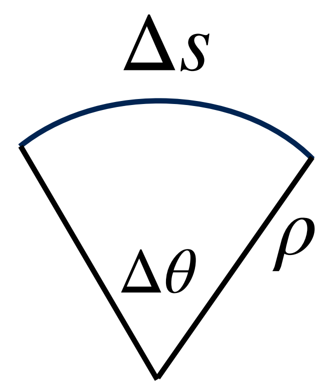
Linear Restoring Forces — Short Version

Northern Illinois University



$$\frac{(\gamma m)v^2}{\rho} = evB_0$$

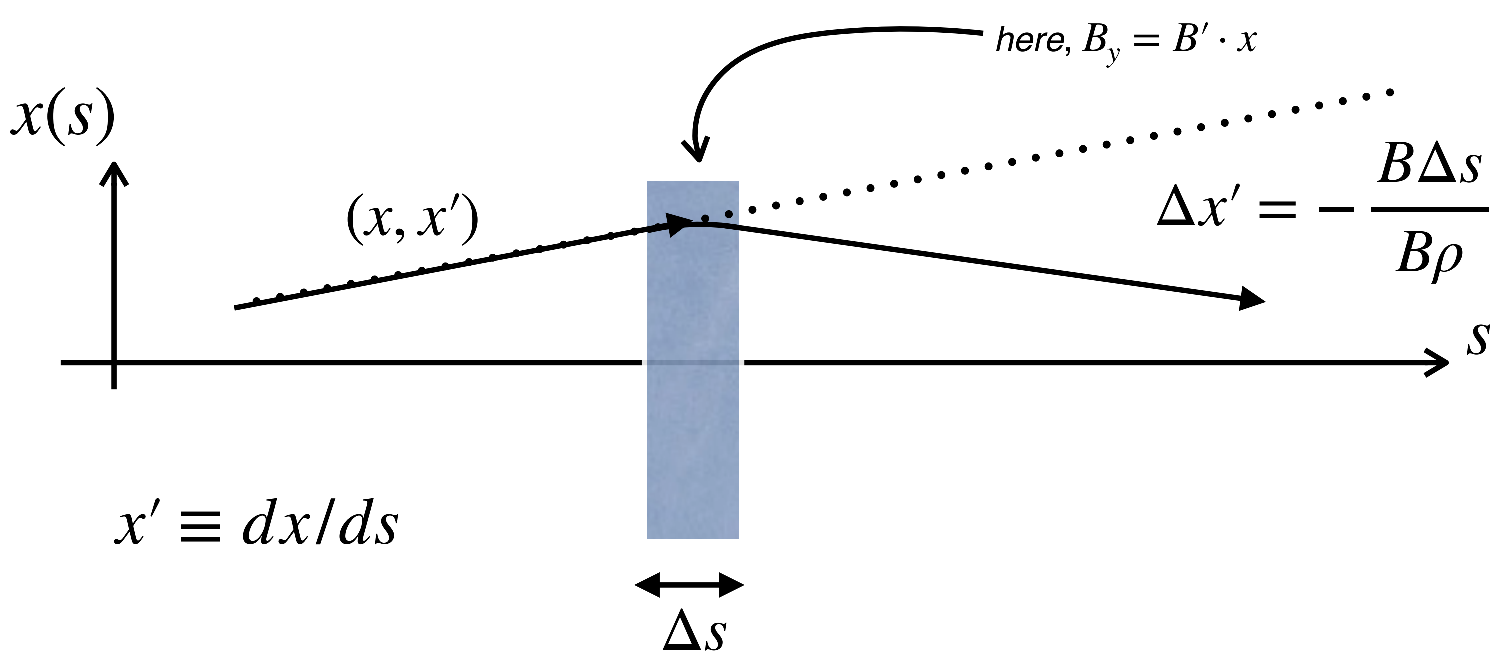
$$B_0\rho = \frac{(\gamma m)v}{e} = p/e$$



$$\Delta\theta = \frac{\Delta s}{\rho} = \frac{B \cdot \Delta s}{B\rho}$$

suppose: $B = B' \cdot x$

Look relative to the *ideal* trajectory:



$$\Delta x' = -\frac{B\Delta s}{B\rho} = -\frac{B'\Delta s}{B\rho} \cdot x$$

$$\frac{\Delta x'}{\Delta s} \equiv x'' = -\frac{B'}{B\rho} \cdot x$$

$x' \equiv dx/ds$

$$x'' + \frac{B'(s)}{B\rho}\, x = 0$$

Linear Restoring Forces - Details

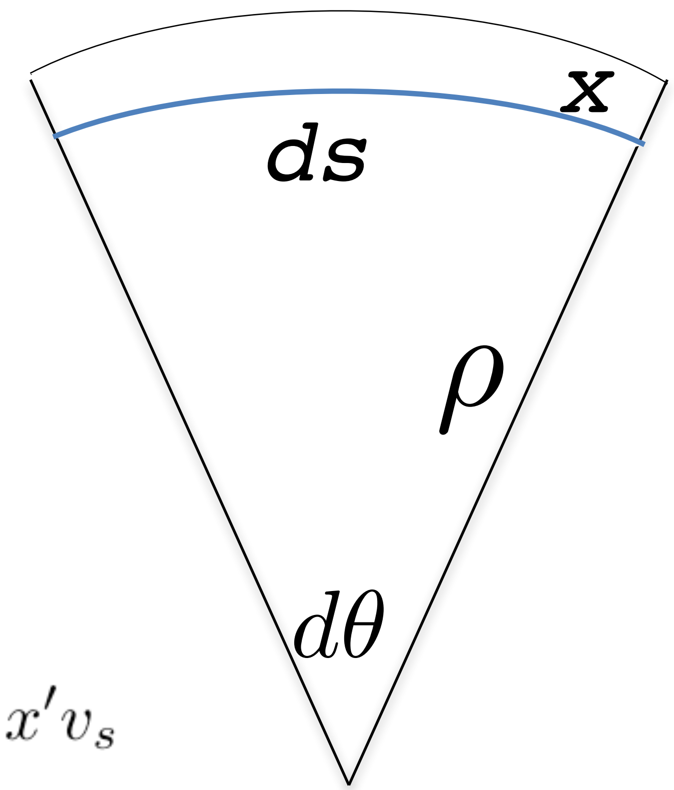
- Assume general linear guide fields: --
- Look at radial motion:

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

$$dt = \frac{\rho + x}{\rho}\frac{ds}{v_s}$$

$$\frac{ds}{dt} = v_s \frac{\rho}{r}$$

$$\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x' v_s$$



$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}$$

Use cylindrical coordinates

$$r = \rho + x, \quad \theta, \quad y$$

acceleration:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{e v_s^2 B_y}{p}$$

$$\ddot{y} = \frac{e v_s^2 B_x}{p}$$

Northern Illinois University

Linear Restoring Forces - Details

Northern Illinois University

- Assume general linear guide fields: --

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

- Look at radial motion:

$$\ddot{r} - r\dot{\theta}^2 = -\frac{ev_s^2 B_y}{p}$$

$$r = \rho + x$$

$$v_s^2 \left(\frac{\rho}{r}\right)^2 x'' - (\rho + x)\left(\frac{v_s}{r}\right)^2 = -\frac{ev_s^2 B_y}{p}$$

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{eB_y}{p}\left(\frac{r}{\rho}\right)^2$$

linearize…

$$x'' - \frac{1}{\rho} - \frac{x}{\rho^2} = -\frac{B_0 + B'x}{B\rho}\left(1 + \frac{2x}{\rho} + \frac{x^2}{\rho^2}\right)$$

$$x'' + \left(\frac{1}{\rho^2} + \frac{B'}{B\rho}\right) x = 0$$

curved coordinate system — *local focusing field*

Note: both ρ and B' are functions of location s

Hill's Equation

Northern Illinois University

- Now, for vertical motion:

$$\ddot{y} = \frac{e v_s^2 B_x}{p} \qquad y'' = \frac{eB_x}{p}\left(\frac{r}{\rho}\right)^2 \qquad y'' - \frac{eB_x}{p}\left(1+\frac{x}{\rho}\right)^2 = 0$$

$$\begin{aligned} B_y &= B_0 + B'x \\ B_x &= B'y \end{aligned}$$

linearize…

$$y'' - \frac{eB'y}{p} = 0$$

- So we have,
  - to lowest order,

$$\begin{aligned} x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right)x &= 0 \\ y'' - \left(\frac{B'}{B\rho}\right)y &= 0 \end{aligned}$$

Hill's Equation

General Form

$$x'' + K_x(s)\,x = 0$$

and $y'' + K_y(s)\,y = 0$

Determining *K* — *Examples*

Northern Illinois University

- Quadrupole Magnets

$$K_x = \frac{B'}{B\rho}$$

$$K_y = -\frac{B'}{B\rho}$$

ex: in Main Injector, $K_x \approx 0.04$ / m²

- Sector Bend Dipole Magnets

$$K_x = \frac{1}{\rho^2}$$
$$K_y = 0$$

ex: in Main Injector, $\rho \approx$ *400* m;
so, $K_x \approx few \times 10^{-6}$ / m² ≈ 0

$$x'' + K_x\, x = 0$$
$$y'' + K_y\, y = 0$$

- Combined Function Magnet:

$$K_x = \frac{1}{\rho^2} + \frac{B'}{B\rho}$$

$$K_y = \quad -\frac{B'}{B\rho}$$

- Sector Bends with Electrostatic Focusing

(*g-2 arrangement*)

$$K_x = \frac{1}{\rho^2} - \frac{E'}{v(B\rho)}$$

$$K_y = \frac{E'}{v(B\rho)}$$

- Other Considerations
  - Rectangular Bend
  - Bend with arbitrary Edge Angles

Piecewise Method of Solution

- Hill's Equation: $$x'' + K(s)x = 0$$

(Here, *x* could be *x or y,* and *K* could be +/-)

- Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$

- *K*

- *K*

Piecewise Method of Solution

Northern Illinois University

- Hill's Equation: $$x'' + K(s)x = 0$$

(Here, *x* could be *x or y,* and *K* could be +/-)

- Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$

- *K* > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$

- *K*

Piecewise Method of Solution

- Hill's Equation: $$x'' + K(s)x = 0$$

(Here, *x* could be *x or y,* and *K* could be +/-)

- Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: $$x'' = 0 \quad \longrightarrow \quad x(s) = x_0 + x_0' s$$

- *K* > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$

- *K* < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

Piecewise Method of Solution

Northern Illinois University

- Hill's Equation: $$x'' + K(s)x = 0$$

(Here, *x* could be *x or y,* and *K* could be +/-)

- Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: drift, or dipole $$x'' = 0 \longrightarrow x(s) = x_0 + x_0' s$$

- *K* > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$

- *K* < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

Piecewise Method of Solution

Northern Illinois University

- Hill's Equation: $x'' + K(s)x = 0$

(Here, *x* could be *x or y,* and *K* could be +/-)

- Though *K*(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, dipole mag, quad, ...)

- *K* = 0: drift, or dipole $$x'' = 0 \longrightarrow x(s) = x_0 + x_0' s$$
- *K* > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$
- *K* < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

Quad, Gradient Magnet, ...

Piecewise Method -- Matrix Formalism

Northern Illinois University

- Write solution to each piece in matrix form
  - for each, assume K = const. from s=0 to s=L

- $K = 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- $K > 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

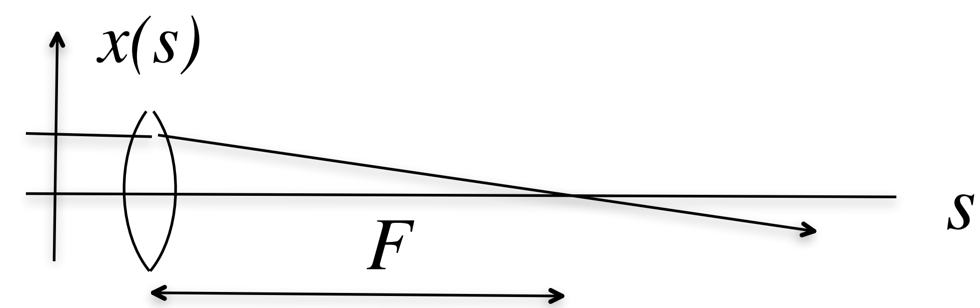
- $K < 0$:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

“Thin Lens” Quadrupole

Northern Illinois University

- If quadrupole magnet is short enough, particle’s offset through the quad does not change by much, but the slope of the trajectory does -- acts like a “thin lens” in geometrical optics



- Take limit as *L* --> 0, while *KL* remains finite
  - (similarly, for defocusing quadrupole)

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

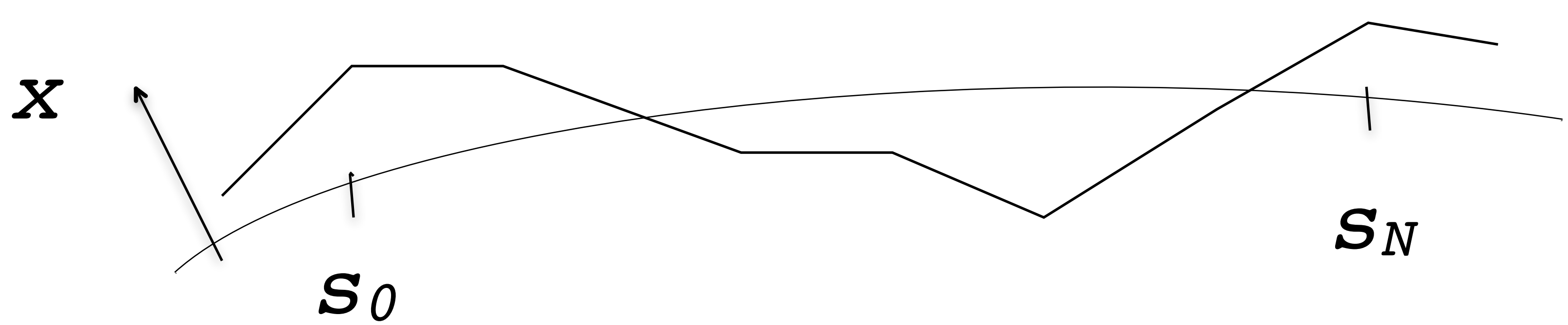
- Valid approx., if *F* >> L

$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Example: A Beam Line Calculation

Northern Illinois University

- Will consider two particle trajectories, starting with
  - (x,x') = (0, 0.5 mrad), and (x,x') = (5 mm, 0)
- A distance 6 m later, the trajectories enter a thin lens quadrupole of focal length F = 3 m. This is followed by a second quadrupole of focal length -F, a distance 1 m later.

- Find the trajectories (x,x') for each case at the exit of the second quad

Example: A Beam Line Calculation

Northern Illinois University

- Will consider two particle trajectories, starting with
  - (x,x') = (0, 0.5 mrad), and (x,x') = (5 mm, 0)
- A distance 6 m later, the trajectories enter a thin lens quadrupole of focal length F = 3 m. This is followed by a second quadrupole of focal length -F, a distance 1 m later.

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{3\ m} & 1 \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 1\ m \\ \frac{1}{3\ m} & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 1 & 6\ m \\ -\frac{1}{3\ m} & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

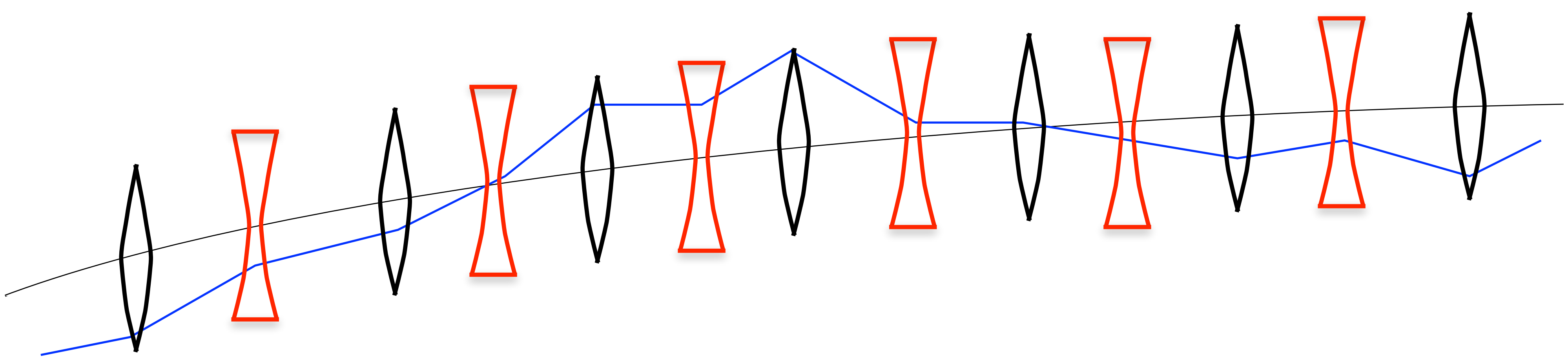
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 5\ m \\ -\frac{1}{9\ m} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}$$

x_0 = 0 mm, x'_0 = 0.5 mr —> x = 2.5 mm, x' = 0.33 mr

x_0 = 5 mm, x'_0 = 0.0 mr —> x = 3.3 mm, x' = -0.56 mr

Can now make LARGE accelerators!

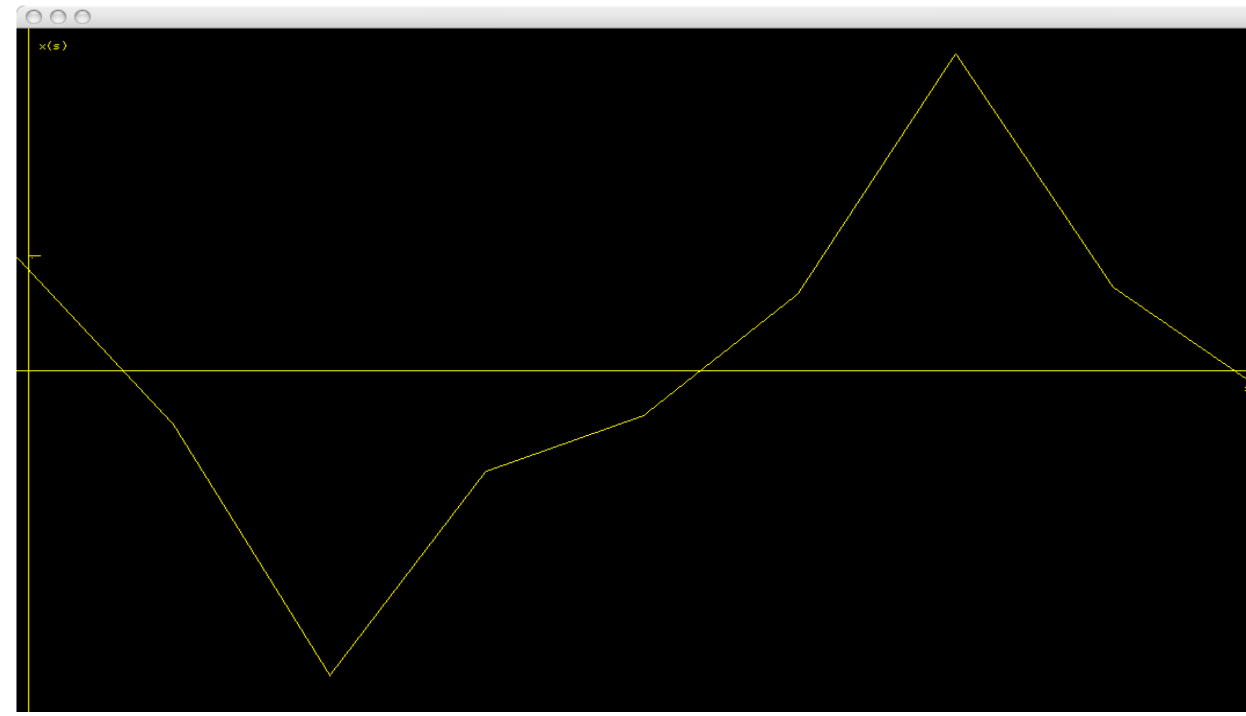
- Since the lens spacing can be made arbitrarily short, with corresponding focusing field strengths, then in principal can make beam transport systems (and linacs and synchrotrons, for instance) of arbitrary size



- Instrumental in paving the way for very large accelerators, both linacs and especially synchrotrons, where the bending and focusing functions can be separated into distinct magnet types

The Notion of an Amplitude Function…

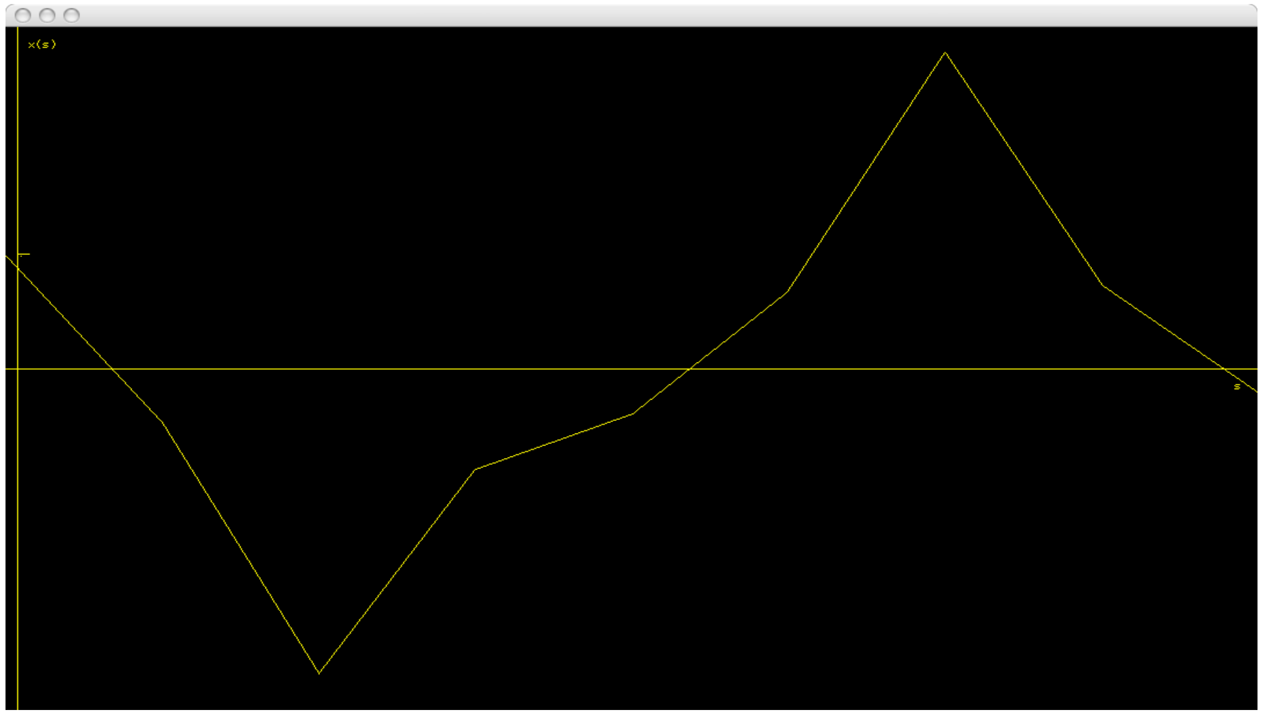
- Can trace single particle trajectories through a periodic system



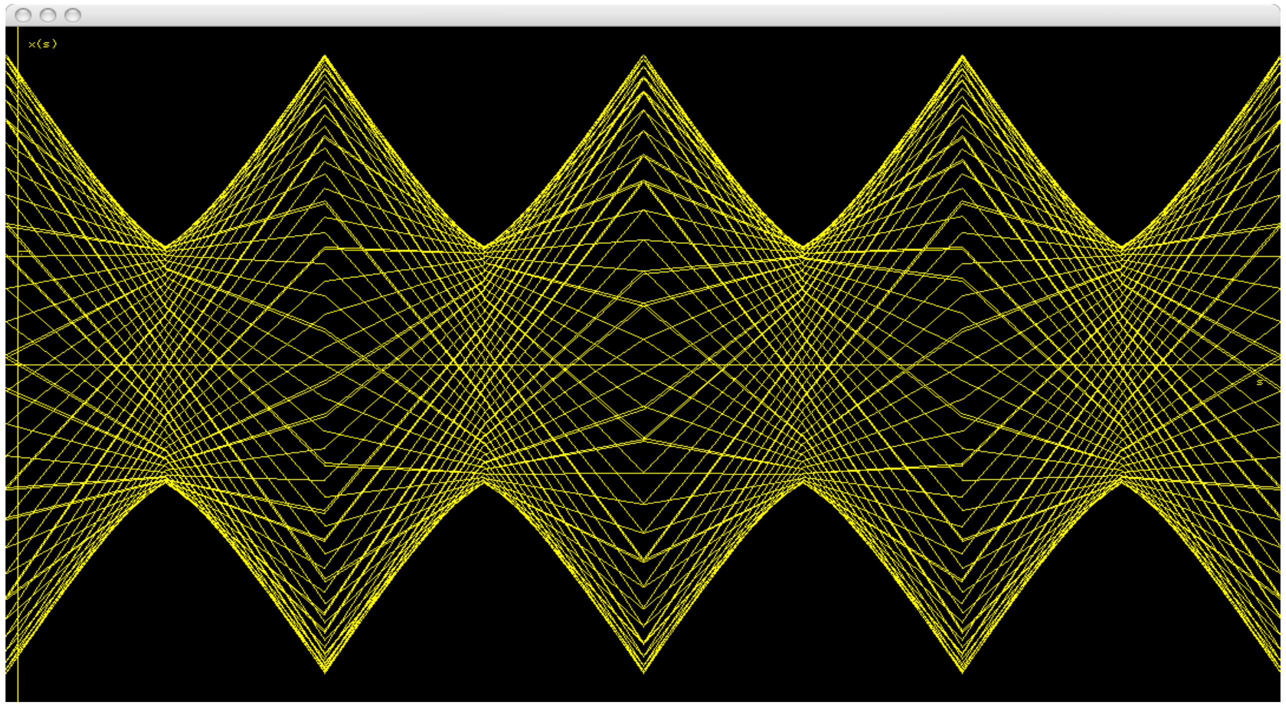
- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line

The Notion of an Amplitude Function…

- Can trace single particle trajectories through a periodic system



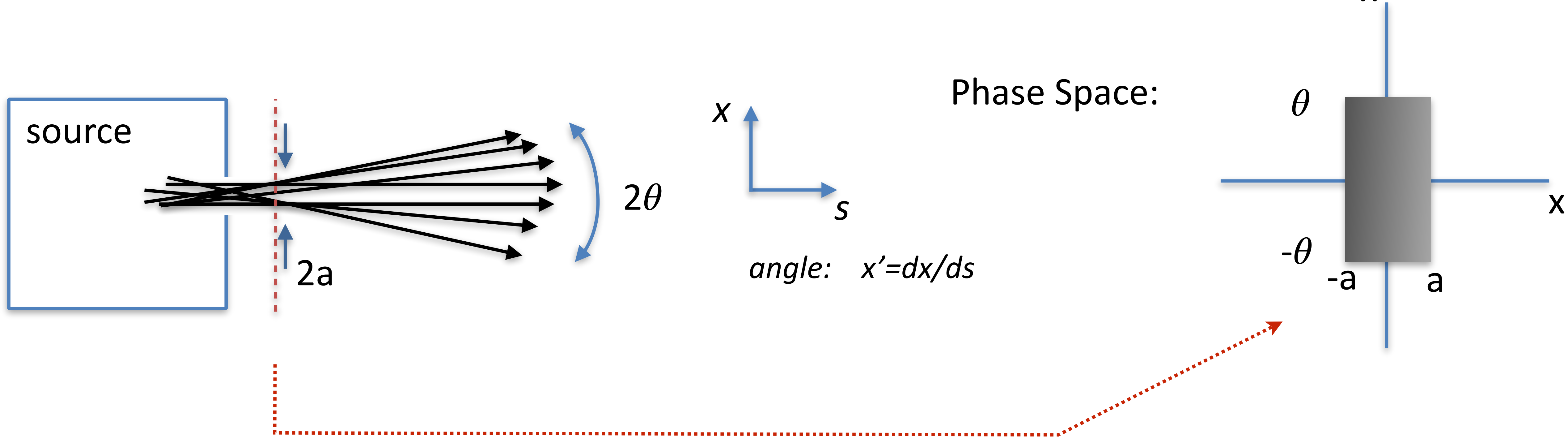
- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line



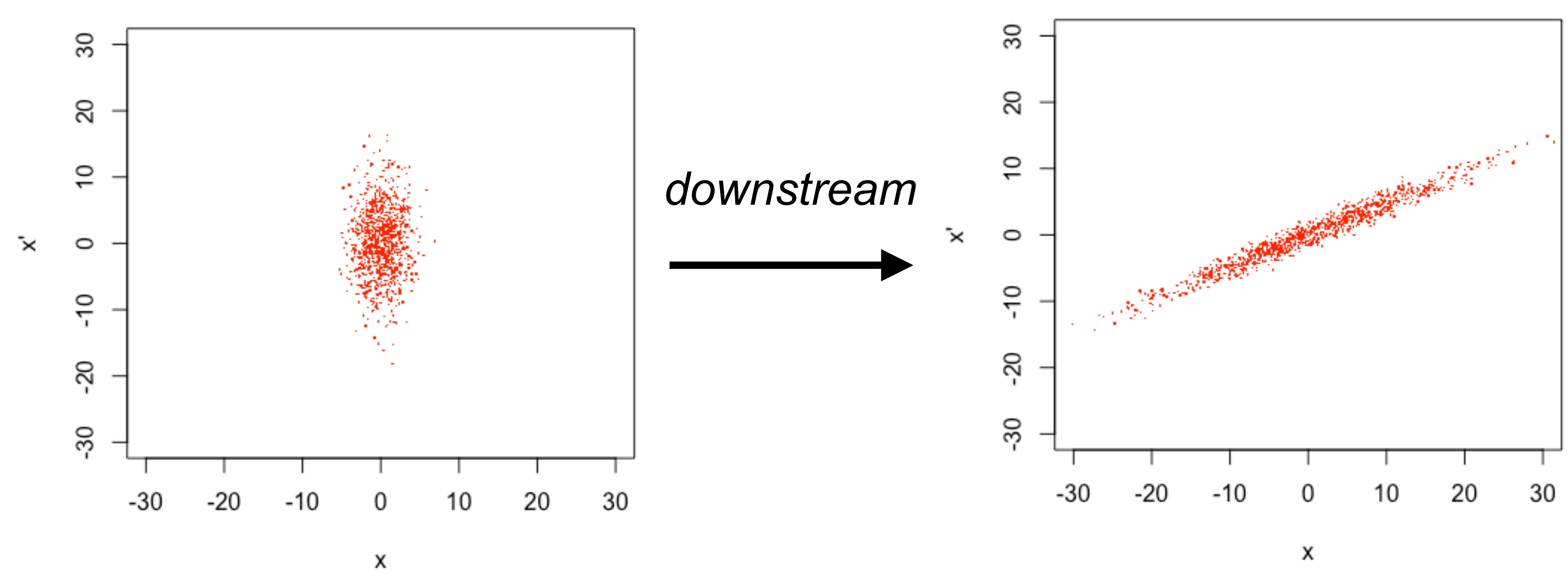
Particle Beams and Phase Space

Northern Illinois University

Transverse coordinates:



Shape, orientation of distribution in "phase space" will change as particles progress downstream, but effective "area" of distribution will remain constant (*Liouville*); correlations will naturally develop

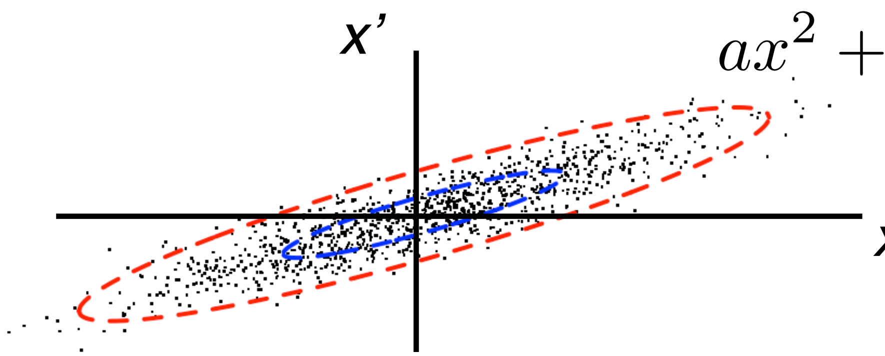


Emittance in Terms of Moments

see: http://nicadd.niu.edu/~syphers/tutorials/ellipseNotes.html

Northern Illinois University

- Considering the general equation of an ellipse, the area enclosed by the ellipse is related to its coefficients by:



$$ax^2 + bxx' + cx'^2 = 1$$

area of ellipse:

$$\mathcal{A} = \frac{2\pi}{\sqrt{4ac - b^2}}$$

- Can define quantities scaled by an area, ϵ, of our elliptical distribution:

$$\alpha \equiv -\frac{\langle xx' \rangle}{\epsilon/\pi} \qquad \beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \qquad \gamma \equiv \frac{\langle x'^2 \rangle}{\epsilon/\pi}$$

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

the "rms emittance"

α, β, γ collectively are called the ***Courant-Snyder* parameters**, or *Twiss* parameters

So, equation of the **blue** curve above:

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon/\pi$$

The ellipse (red curve above) that contains ~95% has area ~6ϵ

(*for Gaussian distribution*)

Linear Optics

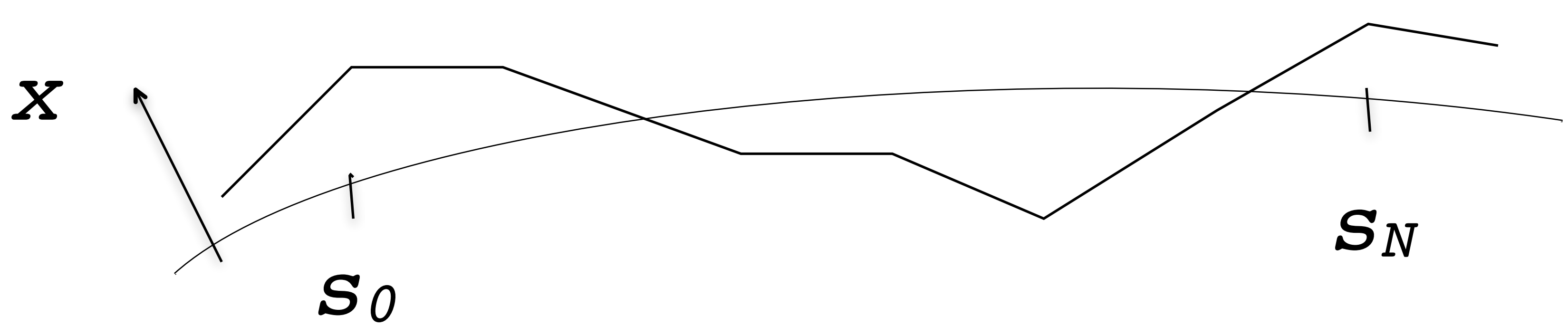
Northern Illinois University

- Let *x* be the transverse (horizontal, say) displacement of a particle from the ideal beam trajectory. Let the angle it makes to the ideal trajectory be *x'* = *dx*/*ds*, where *s* is the distance along the ideal trajectory. Transport through a magnetic element is then described by a matrix *M*, such that

$$\vec{X} = M\vec{X}_0 \qquad \vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix}$$

- An arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication for elements all along the beam line…

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



TRANSPORT of Beam Moments

Northern Illinois University

- Transport of particle state vector downstream from position 0

$$\vec{X} = \begin{pmatrix} x \\ x' \end{pmatrix} \qquad \vec{X} = M\vec{X}_0$$

- Create a "covariance matrix" of the resulting vector…

$$\vec{X}\vec{X}^T = \begin{pmatrix} x^2 & xx' \\ x'x & x'^2 \end{pmatrix} = M\vec{X}_0(M\vec{X}_0)^T = M\vec{X}_0\vec{X}_0^T M^T$$

- … then, by averaging over all the particles in the distribution,

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

we get: $$\Sigma = M\Sigma_0 M^T$$

TRANSPORT of Beam Moments

Northern Illinois University

- So, since

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x' x \rangle & \langle x'^2 \rangle \end{pmatrix} \sim \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = \epsilon \cdot K$$

- where

$$K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\boxed{K = M \; K_0 \; M^T}$$

- If know matrices *M*, then can "transport" beam parameters from one point to any point downstream, which determines beam distribution along the way.

$$\beta \equiv \frac{\langle x^2 \rangle}{\epsilon/\pi} \quad \longrightarrow \quad x_{rms}(s) = \sqrt{\epsilon\beta(s)/\pi}$$

Conservation of Emittance

- Note that from

$$\Sigma = M \Sigma_0 M^T$$

$$\Sigma = \epsilon \cdot K \qquad\qquad K \equiv \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

- then,

$$\det \Sigma = \det M \;\; \det \Sigma_0 \;\; \det M^T = \det \Sigma_0$$

- and

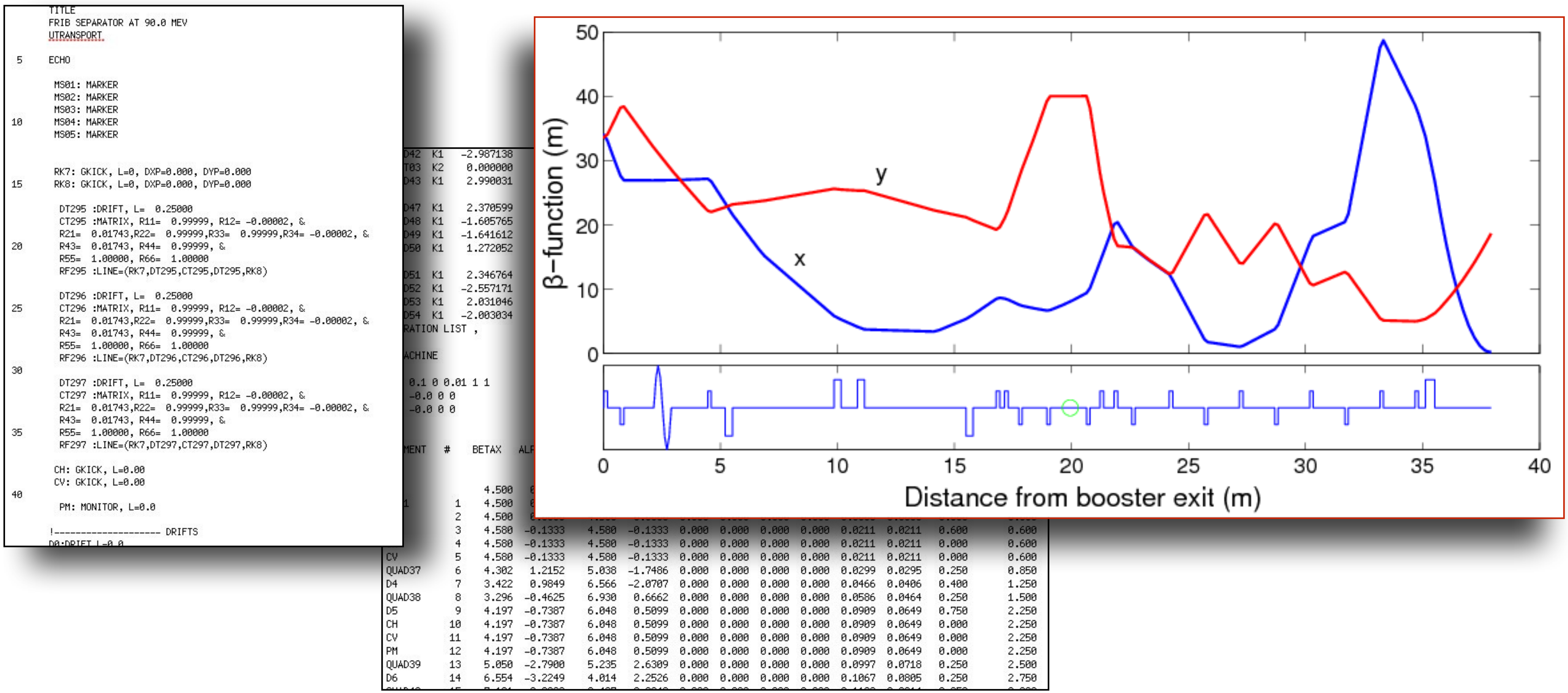
note: det$M$ = 1

$$\det \Sigma = \epsilon^2 \det K = \epsilon^2(\beta\gamma - \alpha^2) = \epsilon^2$$

- Thus, the emittance is conserved upon transport through the system

Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT, MAD, DIMAD, TRACE, TRACE3D, COSY, SYNCH, CHEF, many more ...



Let's Think About the Numbers & Units…

Northern Illinois University

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

- If $<x^2>$ ~ mm^2, and $<x'^2>$ ~ mrad^2,
  - then the emittance can have units of **mm-mrad** (*also* = μm)
- Courant-Snyder parameters

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon}$$

mm^2/(mm-mrad) ~ mm/mrad = m

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

(mm-mrad)/(mm-mrad) = dimensionless

$$\gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

mrad^2/(mm-mrad) ~ mrad/mm = 1/m

The "π" comes from our definition of emittance as an area in phase space; emittance is often expressed in units of "π mm-mrad"

Summary

- Given an initial particle distribution in phase space at the input to a beam transport system, can describe that distribution (sometimes not all that well, but we try…) using Courant-Snyder parameters:

$$\epsilon = \pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$$

$$\beta = \frac{\pi\langle x^2\rangle}{\epsilon} \qquad \gamma = \frac{\pi\langle x'^2\rangle}{\epsilon}$$

$$\alpha = -\frac{\pi\langle xx'\rangle}{\epsilon}$$

- The C-S parameters can then be computed downstream, using

$$\Sigma = M\Sigma_0 M^T$$