Northern Illinois University, PHY 571, Fall 2006

Part II: Special Relativity

Last updated on September 26, 2006 (report errors to
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EM field of point charge moving at constant velocity *

Start with Maxwell's equations:

$$\vec{\nabla}.\vec{D} = \rho, \;\; \vec{\nabla}.\vec{H} = 0,$$
$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0, \text{ and } \vec{\nabla} \times \vec{H} - \partial_t \vec{D} = \vec{J}.$$

Write in terms of electromagnetic potentials, $\vec{A}$ and Φ:

$$\begin{aligned} \vec{B} &= \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \times (\vec{E} + \partial_t \vec{A}) = 0 \Rightarrow \vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A} \\ \frac{1}{\mu}\vec{\nabla} \times \vec{B} - \epsilon \partial_t \vec{E} &= \vec{J} \Rightarrow \vec{\nabla} \times \vec{B} - \mu\epsilon\partial_t \vec{E} = \mu \vec{J} \\ &\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) + \mu\epsilon(\vec{\nabla}\partial_t\Phi + \partial_t^2 \vec{A}) = \mu \vec{J} \end{aligned}$$

Note $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2 \vec{A}$

*for pretty movies of moving charge check Shintake-san's homepage SCSS-FEL: http://www-xfel.spring8.or.jp

$$-\nabla^2\overrightarrow{A} + \overrightarrow{\nabla}(\overrightarrow{\nabla}.\overrightarrow{A} + \mu\epsilon\partial_t\Phi) + \mu\epsilon\partial_t^2\overrightarrow{A} = \mu\overrightarrow{J}$$

$\overrightarrow{\nabla}.\overrightarrow{A} + \mu\epsilon\partial_t\Phi = 0$ in Lorenz gauge.

$$\nabla^2\overrightarrow{A} - \mu\epsilon\partial_t^2\overrightarrow{A} = -\mu\overrightarrow{J} \quad \text{[JDJ, Eq. (6.16)]} \tag{1}$$

$$\overrightarrow{\nabla}.\overrightarrow{D} = \rho \Rightarrow -\nabla^2\Phi - \partial_t\overrightarrow{\nabla}.\overrightarrow{A} = \frac{\rho}{\epsilon}$$

$$\nabla^2\overrightarrow{\Phi} - \mu\epsilon\partial_t^2\Phi = -\frac{\rho}{\epsilon} \quad \text{[JDJ, Eq. (6.15)]} \tag{2}$$

For a source moving at constant velocity, $\overrightarrow{v}$: $\rho = \rho(\overrightarrow{x} - \overrightarrow{v}t)$ and $\overrightarrow{J} = \overrightarrow{v}\rho(\overrightarrow{x} - \overrightarrow{v}t)$. We then have to solve a set of inhomogeneous d'Alembert equations: $\Box f = g(\overrightarrow{x} - \overrightarrow{v}t)$.

Consider the case $\overrightarrow{v} = v\hat{z} \Rightarrow f(\overrightarrow{x} - \overrightarrow{v}t) = (x, y, z - vt) = f(x, y, \zeta)$ with $\zeta \equiv z - vt$. Then

$$\partial_z f \quad \rightarrow \quad \frac{\partial \zeta}{\partial z} \partial_\zeta f = \partial_\zeta f \tag{3}$$

$$\partial_t f \quad \rightarrow \quad \frac{\partial \zeta}{\partial t} \partial_\zeta f = -v \partial_\zeta f \tag{4}$$

$$\Rightarrow \Box f \rightarrow \left(\partial_x^2 + \partial_y^2 + \partial_\zeta^2 - \mu\epsilon v^2 \partial_\zeta^2\right) f = \left(\partial_x^2 + \partial_y^2 + \gamma^{-2} \partial_\zeta^2\right) f. \tag{5}$$

with $\gamma \equiv \frac{1}{\sqrt{1-\mu\epsilon v^2}}$. Let $z' = \gamma\zeta \Rightarrow \partial_\zeta = \frac{\partial z'}{\partial \zeta} \partial_{z'} = \gamma \partial_{z'}$:

$$\left(\partial_x^2 + \partial_y^2 + \partial_{z'}^2\right) f(x, y, \gamma^{-1} z') = g(x, y, \gamma^{-1} z'). \tag{6}$$

<u>Point</u> charge $\Rightarrow \rho(\overrightarrow{x} - \overrightarrow{v}t) \rightarrow \delta(x)\delta(y)\delta(\gamma^{-1} z') = \gamma\delta(x)\delta(y)\delta(z') = \gamma\delta(\overrightarrow{x}')$.

Results:
$\overrightarrow{A} \to A\hat{z}$ $(A_x = A_y = 0)$;

$$\nabla^2_{x'} A = -\gamma \mu q v \delta(\overrightarrow{x'}), \; \nabla^2_{x'} \Phi = -\gamma \frac{q}{\epsilon} \delta(\overrightarrow{x'}). \tag{7}$$

Solve by inspection:

$$\nabla^2_{x'} \left(\frac{1}{|\overrightarrow{x'}|} \right) = -4\pi \delta(\overrightarrow{x'}) \Rightarrow \begin{cases} A = \frac{\gamma \mu}{4\pi} \frac{qv}{R}, \\ \Phi = \frac{\gamma}{4\pi\epsilon} \frac{q}{R}, \end{cases} \tag{8}$$

where $R \equiv \sqrt{x^2 + y^2 + \gamma^2 (z - vt)^2}$.
Now we can calculate $\overrightarrow{E} = -\overrightarrow{\nabla} \Phi - \partial_t \overrightarrow{A}$:

$$\begin{aligned} \overrightarrow{E} &= -\frac{\gamma q}{4\pi\epsilon} (\overrightarrow{\nabla} + \mu \epsilon v \partial_t \hat{z}) \frac{1}{R} \\ &= \frac{\gamma q}{4\pi\epsilon R^3} \left[x\hat{x} + y\hat{y} + \gamma^2 (z - vt)(1 - \mu\epsilon v^2)\hat{z} \right] \end{aligned} \tag{9}$$

$$\vec{E} = \frac{\gamma q}{4\pi\epsilon R^3}\left[x\hat{x} + y\hat{y} + (z - vt)\hat{z}\right] \tag{10}$$

Convert to spherical coordinates:

$x^2 + y^2 = r^2 \sin^2\theta,\ z - vt = r\cos\theta.$

$$\begin{aligned}
\Rightarrow R^2 &= r^2(\sin^2\theta + \gamma^2\cos^2\theta) \\
&= \gamma^2 r^2\left(1 + \frac{1-\gamma^2}{\gamma^2}\sin^2\theta\right) = \gamma^2 r^2(1 - \mu\epsilon v^2 \sin^2\theta),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow E &= \frac{\gamma q}{4\pi\epsilon}\frac{r}{\gamma^3 r^3(1 - \mu\epsilon v^2\sin^2\theta)^{3/2}} \\
&= \frac{q}{4\pi\epsilon r^2}\frac{1 - \mu\epsilon v^2}{(1 - \mu\epsilon v^2\sin^2\theta)^{3/2}}. \qquad (11)
\end{aligned}$$

Note: In vacuum, take $\mu\epsilon \to \mu_0\epsilon_0 = c^{-2}$, and then

$$\overrightarrow{E} = \frac{q}{4\pi\epsilon r^2}\frac{\overrightarrow{r}}{\gamma^2(1-\beta^2\sin^2\theta)^{3/2}}. \quad \text{[JDJ, Eq. (11.154)]} \tag{12}$$

Note that $E(\pi/2)/E(0) = \gamma^3 \Rightarrow$ field lines are "squashed" orthogonal to the direction of motion.

Also we can find $\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$:

$$\begin{aligned} \overrightarrow{A} = \mu\epsilon\Phi\overrightarrow{v} \quad &\Rightarrow \quad \overrightarrow{B} = \mu\epsilon\overrightarrow{\nabla} \times (\Phi\overrightarrow{v}) = \mu\epsilon[\overrightarrow{\nabla}\Phi \times \overrightarrow{v} + \Phi\overrightarrow{\nabla} \times \overrightarrow{v}] \\ &\Rightarrow \quad \overrightarrow{B} = \mu\epsilon\overrightarrow{\nabla}\Phi \times \overrightarrow{v}. \end{aligned}$$

$$\overrightarrow{v} \times \overrightarrow{E} = -\overrightarrow{v} \times (\overrightarrow{\nabla}\Phi + \partial_t\overrightarrow{A}) = \overrightarrow{\nabla}\Phi \times \overrightarrow{v}.$$

$$\overrightarrow{B} = \mu\epsilon\overrightarrow{v} \times \overrightarrow{E}, \text{ or } \overrightarrow{B} = \frac{\mu}{4\pi}\frac{\gamma q}{R^3}\overrightarrow{v}(x\hat{y} - y\hat{x}). \tag{13}$$

Further reductions [toward JDJ Eq. (11.152)]:

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\frac{\hat{r}}{\gamma^2(1-\beta^2\sin^2\theta)^{3/2}} \tag{14}$$

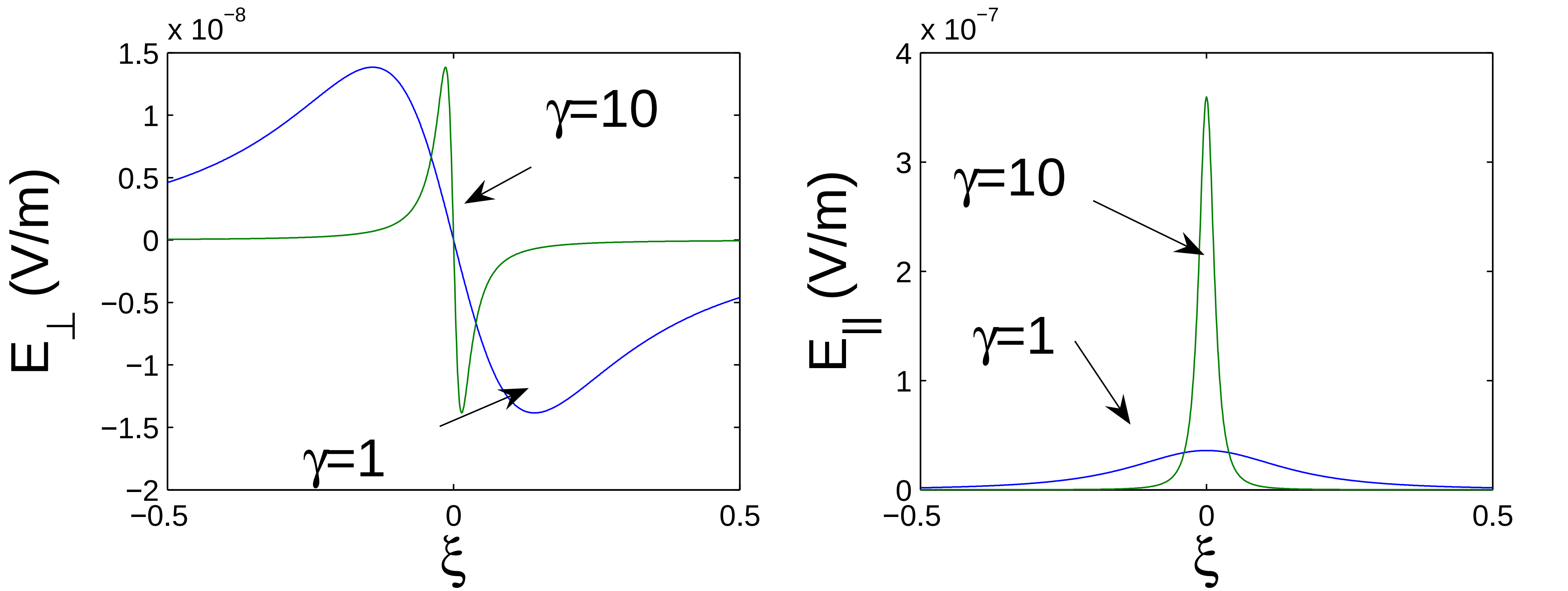
$\sin\theta = \frac{b}{r} = \frac{b}{\sqrt{b^2+v^2t^2}}$.

$$1-\beta^2\sin^2\theta = 1 - \frac{\beta^2 b^2}{b^2+(vt)^2} = \frac{b^2+v^2t^2-\beta^2b^2}{b^2+v^2t^2} = \frac{(1-\beta^2)b^2+v^2t^2}{b^2+v^2t^2}$$

$$1-\beta^2\sin^2\theta = \frac{b^2+\gamma^2v^2t^2}{\gamma^2r^2} \Rightarrow \gamma r\sqrt{1-\beta^2\sin^2\theta} = \sqrt{b^2+\gamma^2v^2t^2}$$

Finally

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\,\vec{r}}{(b^2+\gamma^2v^2t^2)^{3/2}} \Rightarrow \vec{E}_\perp = \frac{q}{4\pi\epsilon_0}\frac{\gamma b\hat{x}}{(b^2+\gamma^2v^2t^2)^{3/2}}. \tag{15}$$


x 10^−8
1.5
1
0.5
0
−0.5
−1
−1.5
−2
E⊥ (V/m)
γ=10
γ=1
−0.5
0
0.5
ξ
x 10^−7
4
3
2
1
0
E|| (V/m)
γ=10
γ=1
−0.5
0
0.5
ξ

Consider a charge q_0 comoving with q at velocity $\overrightarrow{v}$. The force imparted to q_0 by q is

$$
\begin{aligned}
F &= q_0(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}) \\
&= q_0[\overrightarrow{E} + \mu\epsilon \overrightarrow{v} \times (\overrightarrow{v} \times \overrightarrow{E})]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \overrightarrow{F} &= q_0 \left[(1 - \mu\epsilon v^2)\overrightarrow{E} + \mu\epsilon v^2 E_z \widehat{z}\right] \\
&= q_0 \left(\frac{1}{\gamma^2}\overrightarrow{E} + \frac{\gamma^2 - 1}{\gamma^2} E_z \widehat{z}\right) = q_0 \left[\frac{1}{\gamma^2}(\overrightarrow{E} - E_z \widehat{z}) + E_z \widehat{z}\right]
\end{aligned}
$$

$$
\Rightarrow \overrightarrow{F} = q_0 \left[\frac{1}{\gamma^2}\overrightarrow{E}_\perp + \overrightarrow{E}_\parallel\right] \tag{16}
$$

The self-magnetic field of q cancels its self-electric field to within a factor $1/\gamma^2$.

The squashing of the E-field of a moving charge, as it corresponds to the equation of motion, is suggestive of the Lorentz contraction, and thus indicative that electrodynamics is invariant under Lorentz transformations.

Invariance of proper time:
spherical waves propagate such that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2$.
If c is the same in all inertial reference frames (postulate), then

$$\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2 = c^2$$

So, we write:

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 \quad \text{for photons.} \tag{17}$$

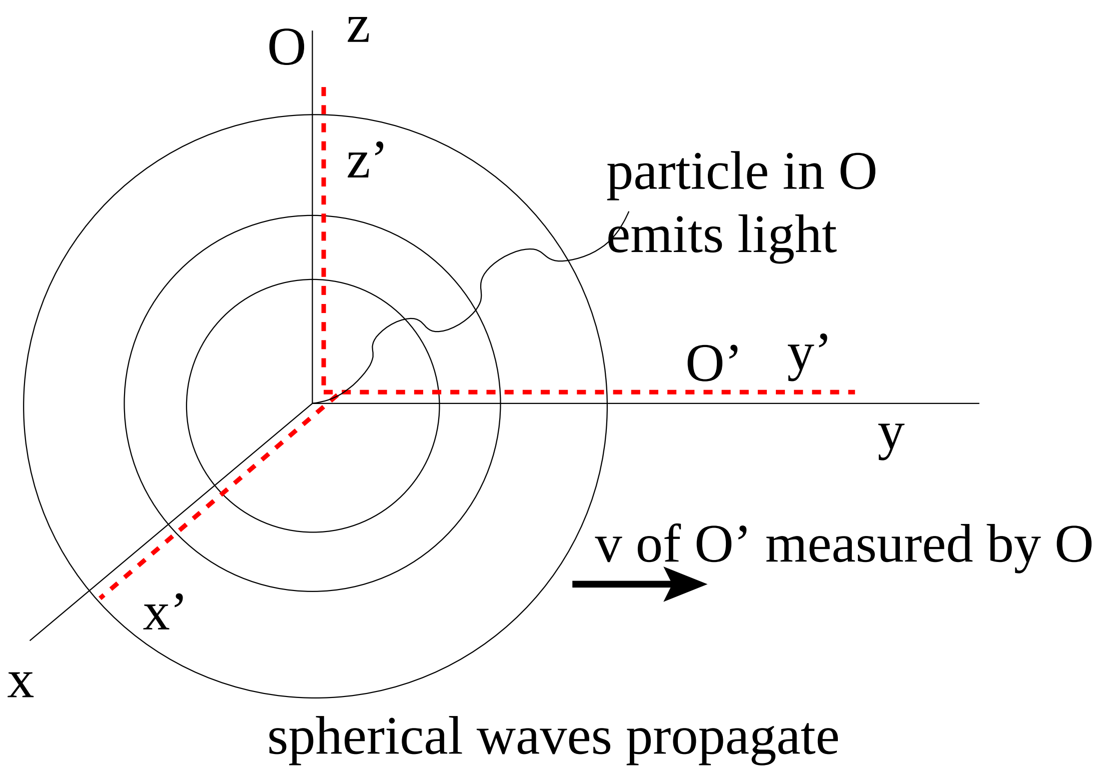
This holds true in any inertial coordinate system. More generally we can define the proper time:

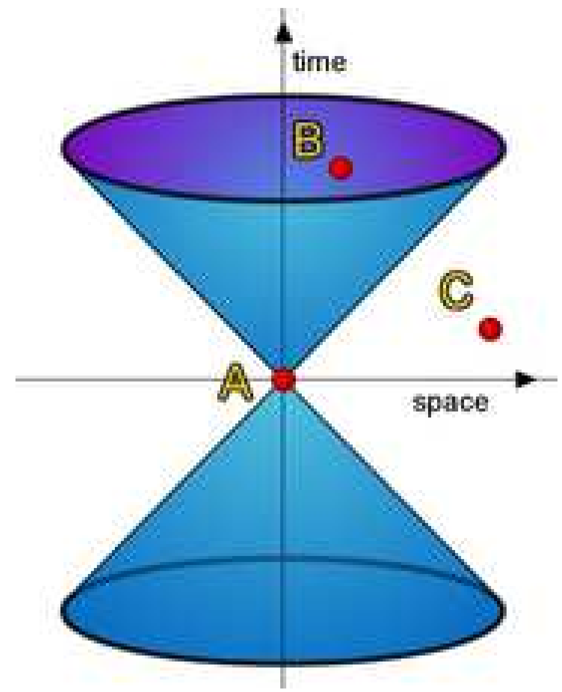
$$d\tau^2 \equiv dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2). \tag{18}$$

In SR, the proper time is an invariant – all inertial observers measure the same $d\tau$. Note that:

$$d\tau^2 = dt^2(1 - \beta^2) = \frac{1}{\gamma^2}dt^2; \tag{19}$$

$\overrightarrow{\beta} \equiv \frac{1}{c}\overrightarrow{v}$; $\overrightarrow{v}$ = velocity measured in lab frame ($\mathcal{O}$), dt = period between “ticks” of clock in lab frame.
When $\overrightarrow{v} = 0$, $d\tau = dt \Rightarrow d\tau$ = period between “ticks” of clock comoving with $\mathcal{O}'$. Every inertial observer measure the same value for this time interval: it is a scalar – a fixed physical quantity!





left: notation for previous slides. right: light cone, $[AB]$ is time-like $[AC]$ is space-like.

If δt represents the period between ticks of $\mathcal{O}'$'s clock, then $\mathcal{O}$ sees it ticks with period:

$$dt = \gamma \delta t \tag{20}$$

This is "time dilatation": $\mathcal{O}$ thinks $\mathcal{O}'$'s clock runs slow.

<u>Minkowski metric and Lorentz transformations:</u>

Let $x^0 \equiv ct$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$ [so $\overrightarrow{x}^i \equiv \overrightarrow{X}$ (i=1,2,3)]. Then we can write:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{21}$$

with $\alpha, \beta = 0, 1, 2, 3$ and $g_{\alpha\beta}$ is the Minkowski metric:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{22}$$

standard convention: Use Greek indices to represent sums from 0-3 and Latin indices for sum from 1-3.

The Lorentz transformation matrix from stationary observer $\mathcal{O}$ to moving observer $\mathcal{O}'$ is the "boost matrix" [JDJ, Eq.(11.98)] ($\Lambda^{\alpha}_{\gamma}\Lambda^{\beta}_{\delta}g_{\alpha\beta} = g_{\gamma\delta}$:

$$\Lambda^{\nu}_{\mu} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\left(\frac{\beta_x}{\beta}\right)^2(\gamma-1) & \frac{\beta_x\beta_y}{\beta^2}(\gamma-1) & \frac{\beta_x\beta_z}{\beta^2}(\gamma-1) \\ -\gamma\beta_y & \frac{\beta_x\beta_y}{\beta^2}(\gamma-1) & 1+\left(\frac{\beta_y}{\beta}\right)^2(\gamma-1) & \frac{\beta_y\beta_z}{\beta^2}(\gamma-1) \\ -\gamma\beta_z & \frac{\beta_x\beta_z}{\beta^2}(\gamma-1) & \frac{\beta_y\beta_z}{\beta^2}(\gamma-1) & 1+\left(\frac{\beta_z}{\beta}\right)^2(\gamma-1) \end{pmatrix}, \quad (23)$$

provided the coordinates of $\mathcal{O}$ and $\mathcal{O}'$ are aligned. The the Lorentz transformation from $\mathcal{O}$ and $\mathcal{O}'$ is:

$$x'^{\alpha} = \Lambda^{\alpha}_{\beta}x^{\beta}. \quad (24)$$

Note $\Lambda^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}}$. If the coordinate axes are not aligned then the transformation is the product of Λ^{α}_{β} and a rotation matrix.

The principle of SR is : All laws of physics must be invariant under Lorentz transformations. "Invariant" $\leftrightarrow$ Laws retain the same mathematical form and numerical constant (scalar) retain the same value.

Particle dynamics in SR

Define the "4-velocity": $u^\alpha \equiv \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds}$:

$$u^0 = c\frac{dt}{d\tau} = \gamma c \quad \text{and } u^i = \frac{1}{c}\frac{dx^i}{d\tau} = c\frac{dt}{d\tau}\frac{dx^i}{dt} = c\gamma\beta^i \tag{25}$$

Then

$$u_\alpha u^\alpha = g_{\alpha\beta}u^\beta u^\alpha = \gamma^2 - \gamma^2\beta^2 = c^2 \tag{26}$$

is an invariant.

Moreover since $d\tau$ is an invariant and x^α conforms to Lorentz transformation, then

$$u'^\alpha = \Lambda^\alpha_\beta u^\beta \tag{27}$$

$\Rightarrow u^\alpha$ satisfies the Principle of SR.
Define the 4-momentum of a particle:

$$P_\alpha \equiv m u^\alpha \tag{28}$$

$\Rightarrow P^0 = \gamma m c = E/c$, $P^i = p^i$; $E =$ total energy, $p^i =$ ordinary 3-momentum, $m =$ particle's rest mass. Then

$$P_\alpha P^\alpha = m^2 u_\alpha u^\alpha = m^2 c^2 = E/c^2 \tag{29}$$

is an invariant. The fundamental dynamical law for particle interactions in SR is that 4-momentum is conserved in any Lorentz frame.

Note that

$$P'^{\alpha} = \Lambda^{\alpha}_{\beta} P^{\beta} \tag{30}$$

also one has:

$$P_{\alpha} P^{\alpha} = g_{\alpha\beta} P^{\beta} P^{\alpha} = E^2/c^2 - p^2 \tag{31}$$

$$E^2/c^2 - p^2 = (mc)^2 \tag{32}$$

$$\Rightarrow \;\; E = \sqrt{(pc)^2 + (mc^2)^2}.$$

The kinetic energy of a particle is $T = E - mc^2$:

$$T = \sqrt{(pc)^2 + (mc^2)^2} - mc^2 \tag{33}$$

Example: Consider the reaction (one neutron at rest)

$$n + n \to n + n + n + \bar{n}$$

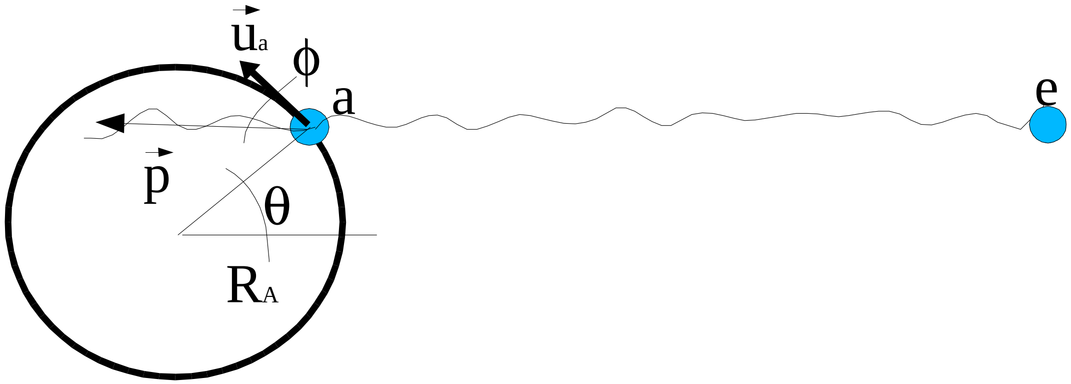
What is the minimum required energy for the incoming n that will enable the reaction to proceed?
At threshold the four neutron are at rest in the lab frame, so that the 4-momentum conservation requires:

$$\begin{aligned}
& P_1^\alpha + P_2^\alpha = P_f^\alpha && (34)\\
\Rightarrow\ & (P_1^\alpha + P_2^\alpha)(P_{1\alpha} + P_{2\alpha}) = P_f^\alpha P_{f\alpha} = 16(m_n c)^2 \\
& P_1^\alpha P_{1\alpha} + 2P_1^\alpha P_{2\alpha} + P_2^\alpha P_{2\alpha} = 2(m_n c)^2 + 2P_1^\alpha P_{2\alpha} \\
& \Rightarrow P_1^\alpha P_{2\alpha} = 7(m_n c)^2. && (35)
\end{aligned}$$

$$P_1^\alpha P_{2\alpha} = g_{\alpha\beta} P_1^\alpha P_2^\beta = g_{00} P_1^0 P_2^0 = m_n c \frac{E}{c}$$

$$E = 7 m_n c^2. \tag{36}$$

Photon emission and absorption:



Let $u^\alpha_{e,a}$ = 4-velocity of emitter, absorber, respectively. $E_{e,a}$ = photon energy measured by emitter, absorber, respectively.
P^α = 4-momentum of photon.

Then look at

$$\begin{aligned} P_\alpha u^\alpha &= g_{\alpha\beta} P^\beta u^\alpha \\ &= P^0 u^0 - P^i u^i = c P^0 = E. \end{aligned}$$

1st term $u^0 = c$, 2nd term $u^i = 0$ in either emitter's or absorber's frame.

So $E = p_\alpha u^\alpha$ is the photon energy measured by an observer with 4-velocity u^α. The expression is the same in any frame, including accelerating frame! So:

$$E_e = P_\alpha u_e^\alpha \text{ and, } E_a = P_\alpha u_a^\alpha$$

Example: "Absorber" is rotating with angular velocity Ω on a circle of radius R_A. Emitter is stationary – Let's find E_a/E_e

In emitter's frame: $c^2 d\tau = g_{\alpha\beta} dx^\alpha dx^\beta$, the emitter is stationary so $u_e^\alpha = (c, 0, 0, 0)$.

In absorber frame:

$$\begin{aligned} c^2(d\tau)^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= c^2 dt^2 - v^2 dt^2 = c^2 dt^2 - R_A^2 d\phi^2 \\ d\tau^2 &= dt^2 - \frac{R_A^2}{c^2} d\phi^2 \end{aligned} \tag{37}$$

From previous slide (two slides ago) we have:

$$\frac{E_a}{E_e} = \frac{P_\alpha u_a^\alpha}{P_\alpha u_e^\alpha} = \frac{P_0 u_a^0 - \overrightarrow{p}\,\overrightarrow{u_a^i}}{P_0 c}$$
$$= \frac{P_0 u_\alpha^0 - |\overrightarrow{p}||\overrightarrow{u_a^i}|\cos\theta}{P_0 c} \tag{38}$$

But $\cos\theta = \sin\phi$, and for photons $P_\alpha P^\alpha = (P^0)^2 - |\overrightarrow{P}|^2 = 0 \Rightarrow |\overrightarrow{P}| = P^0$. Thus

$$\frac{E_a}{E_e} = \frac{u_a^0 - |\overrightarrow{u}_a|\sin\phi}{c}. \tag{39}$$

But,

$$|\overrightarrow{u}_a| = R_A\frac{d\phi}{d\tau} = \frac{R_A\Omega}{\sqrt{1-(R_A\Omega/c)^2}};\ u_a^0 = \frac{c}{\sqrt{1-(R_A\Omega/c)^2}} \tag{40}$$

$$\frac{E_a}{E_e} = \frac{\lambda_e}{\lambda_a} \Rightarrow \frac{\lambda_e}{\lambda_a} = \frac{1 - (R_A\Omega/c)\sin\phi}{\sqrt{1 - (R_A\Omega/c)^2}} \tag{41}$$

Doppler shift ($\phi = 90^\circ$):

$$\frac{\lambda_e}{\lambda_a} = \frac{1 - (R_A\Omega/c)}{\sqrt{1 - (R_A\Omega/c)^2}} = \sqrt{\frac{1 - (R_A\Omega/c)}{1 + (R_A\Omega/c)}} \tag{42}$$

Covariance of Electrodynamics

We wish to proceed in keeping with Jackson's notation, which involves switching from SI units to Gaussian units.

$$
\begin{array}{cc}
SI & G \\
\hline\hline
\begin{array}{c}
\overrightarrow{\nabla}.\overrightarrow{D} = \rho \\
\overrightarrow{\nabla}\times\overrightarrow{H} - \partial_t\overrightarrow{D} = \overrightarrow{J} \\
\overrightarrow{\nabla}\times\overrightarrow{E} + \partial_t\overrightarrow{B} = 0 \\
\overrightarrow{\nabla}\overrightarrow{B} = 0 \\
\overrightarrow{F} = q(\overrightarrow{E} + \overrightarrow{v}\times\overrightarrow{B}) = 0
\end{array}
&
\begin{array}{c}
\overrightarrow{\nabla}.\overrightarrow{D} = 4\pi\rho \\
\overrightarrow{\nabla}\times\overrightarrow{H} - \frac{1}{c}\partial_t\overrightarrow{D} = \frac{4\pi}{c}\overrightarrow{J} \\
\overrightarrow{\nabla}\times\overrightarrow{E} + \frac{1}{c}\partial_t\overrightarrow{B} = 0 \\
\overrightarrow{\nabla}\overrightarrow{B} = 0 \\
\overrightarrow{F} = q(\overrightarrow{E} + \frac{\overrightarrow{v}}{c}\times\overrightarrow{B})
\end{array}
\end{array}
\tag{43}
$$

Conversions:

$\frac{\overrightarrow{E}^G}{\sqrt{4\pi\epsilon_0}} = \overrightarrow{E}^{SI}$; $\sqrt{\frac{\epsilon_0}{4\pi}}\overrightarrow{D}^G = \overrightarrow{D}^{SI}$; $\sqrt{4\pi\epsilon_0}\rho^G(\overrightarrow{J}^G, q^G) = \rho^{SI}(\overrightarrow{J}^{SI}, q^{SI})$; $\sqrt{\frac{\mu_0}{4\pi}}\overrightarrow{B}^G = \overrightarrow{B}^{SI}$; $\frac{\overrightarrow{H}^G}{\sqrt{4\pi\mu_0}} = \overrightarrow{H}^{SI}$; $\epsilon_0\epsilon^G = \epsilon^{SI}$; $\mu_0\mu^G = \mu^{SI}$; $c = (\mu_0\epsilon_0)^{-1/2}$.

As one check, look at the Lorentz force:

$$
\begin{aligned}
\overrightarrow{F}^{G} &= q^{G}(\overrightarrow{E}^{G} + \frac{1}{c}\overrightarrow{v} \times \overrightarrow{B}^{G}) \\
\Rightarrow \overrightarrow{F}^{SI} &= \frac{q^{SI}}{\sqrt{4\pi\epsilon_0}}\left[\sqrt{4\pi\epsilon_0}\,\overrightarrow{E}^{SI} + \sqrt{\mu_0\epsilon_0}\,\overrightarrow{v} \times \sqrt{\frac{4\pi}{\mu_0}}\overrightarrow{B}^{SI}\right] \\
&= q^{SI}(\overrightarrow{E}^{SI} + \overrightarrow{v} \times \overrightarrow{B}^{SI}).
\end{aligned}
$$

The conversion from "Maxwell G" to "Maxwell SI" works the same way. So we do have a prescription to go from Gaussian results to SI results and vice versa.

Current density as a 4-vector:
Consider a system of particles with positions $\vec{x}_n(t)$ and charges q_n.
The current and charge densities are:

$$\begin{aligned}\vec{J}(\vec{x},t) &= \sum_n q_n \delta^3(\vec{x}-\vec{x}_n(t))\vec{\dot{x}_n}(t),\\ \rho(\vec{x},t) &= \sum_n q_n \delta^3(\vec{x}-\vec{x}_n(t))\end{aligned}$$

Note that for any smooth function $f(\vec{x})$, δ^3 acts as:

$$\int_{-\infty}^{\infty} f(\vec{x})\delta^3(\vec{x}-\vec{y}) = f(\vec{y}) \tag{44}$$

if we define $J^0 \equiv c\rho$ and $J^i(\vec{x}) = \sum_n q_n \delta^3(x^i - x_n^i(t)) d_t x_n^i(t)$, then using δ^4 function we can write:

$$J^\alpha(x) = \int \sum_n q_n \delta^4(x^\alpha - x_n^\alpha(t)) dx^0 \, \frac{dx_n^\alpha(t)}{dt} \tag{45}$$

J^α is a function of $x^\alpha \rightarrow$ it is a Lorentz invariant; J^α is a 4-vector.
$J^\alpha \equiv (c\rho, \vec{J})$. Also note $J^\alpha \equiv \rho u^\alpha$

Equation of charge continuity:

$$
\begin{aligned}
\overrightarrow{\nabla}.\overrightarrow{J}(\overrightarrow{x},t) &= \sum_n q_n \frac{\partial}{\partial x^i}\delta^3(\overrightarrow{x}-\overrightarrow{x}_n(t))\frac{dx_n^i(t)}{dt} \\
&= -\sum_n q_n \frac{\partial}{\partial x_n^i}\delta^3(\overrightarrow{x}-\overrightarrow{x}_n(t))\frac{dx_n^i(t)}{dt} \\
&= -\sum_n q_n \partial_t \delta^3(\overrightarrow{x}-\overrightarrow{x}_n(t)) \\
&= -\partial_t \rho(\overrightarrow{x},t) = -\partial_0[c\rho(\overrightarrow{x},t)].
\end{aligned} \tag{46}
$$

So the equation of charge continuity writes as $\partial^\alpha J_\alpha = 0$

4-gradient

In the previous slide we use the operator ∂_α. It is defined as

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}. \tag{47}$$

This operator transforms as:

$$\partial'_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu_\mu \frac{\partial_\nu}{.} \tag{48}$$

Note that $\partial_\mu = (\partial_0, \overrightarrow{\nabla})$.

We can "upper" the indice and define

$$\partial^\mu = g^{\mu\nu}\partial_\nu = (\partial_0, -\overrightarrow{\nabla}) \tag{49}$$

Finally we can define the d'Alembertian: $\Box \equiv \partial^\alpha \partial_\alpha$.

Potential as a 4-vector:

$$A^\alpha \equiv (\phi, \overrightarrow{A}) \tag{50}$$

Lorentz Gauge then write $\partial_\alpha A^\alpha = 0$. We also have

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha, \tag{51}$$

or in SI units

$$\Box A^\alpha = \mu_0 J^\alpha, \qquad \text{[SI]} \tag{52}$$

this is the equation we wrote when deriving the field induced by a charge moving at constant velocity.

Returning to Maxwell Equation

Define the matrix $F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = g^{\alpha\delta}\partial_\delta A^\beta - g^{\beta\delta}\partial_\delta A^\alpha$:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{53}$$

Look at:

$\partial_\alpha F^{\alpha\beta} = \partial_0 F^{0\beta} + \partial_1 F^{1\beta} + \partial_2 F^{2\beta} + \partial_3 F^{3\beta}$:

$$\begin{aligned} \partial_\alpha F^{\alpha 0} &= \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} \\ &= \partial_i E^i = \vec{\nabla}.\vec{E} = 4\pi\rho = \frac{4\pi}{c} J^0. \end{aligned} \tag{54}$$

Similarly,

$$\begin{aligned} \partial_\alpha F^{\alpha 1} &= \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} \\ &= \frac{1}{c}\partial_t(-E_x) + \partial_x(0) + \partial_y(-B_z) - \partial_z(B_y) = -\frac{1}{c}\partial_t(E_x) + [\vec{\nabla} \times \vec{B}]_x \\ &= [\vec{\nabla} \times \vec{B}]_x - \frac{1}{c}\partial_t E_x = \frac{4\pi}{c} J^1 \end{aligned} \tag{55}$$

...The same for component 2, and 3. So we cast these equations under:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \tag{56}$$

This corresponds to the inhomogeneous Maxwell's equations. In SI units $F^{\alpha\beta}$ is obtained by replacing $\overrightarrow{E}$ by $\overrightarrow{E}/c$.

How do we get the homogenous Maxwell's equations?
Let's introduce the Levi-Civita (rank 4) tensor as:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if} \alpha,\beta,\gamma,\delta \text{ are } \textbf{even} \text{ permutation of } 0,1,2,3 \\ -1 & \text{if} \alpha,\beta,\gamma,\delta \text{ are } \textbf{odd} \text{ permutation of } 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}, \tag{57}$$

and consider the quantity $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\delta\gamma}$; with $F_{\delta\gamma} = g_{\gamma\alpha} g_{\delta\beta} F^{\alpha\beta}$.

$$F_{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \tag{58}$$

$F_{\gamma\delta}$ is obtained from $F^{\alpha\beta}$ by doing the change $\overrightarrow{E} \rightarrow -\overrightarrow{E}$. Now consider the component "0" of the 4-vector $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta}$:

$$\begin{aligned} \epsilon^{0\beta\gamma\delta}\partial_\beta F_{\gamma\delta} &= \epsilon^{0123}\partial_1 F_{23} + \epsilon^{0132}\partial_1 F_{32} + \\ &\quad \epsilon^{0213}\partial_2 F_{13} + \epsilon^{0231}\partial_2 F_{31} + \epsilon^{0312}\partial_3 F_{12} + \epsilon^{0321}\partial_3 F_{21} \\ &= \partial_1 F_{23} - \partial_1 F_{32} - \partial_2 F_{13} + \partial_2 F_{31} + \partial_3 F_{12} - \partial_3 F_{21} \\ &= \partial_x(-B_x) - \partial_x(B_x) - \partial_y(B_y) + \partial_y(-B_y) + \partial_z(-B_z) - \partial_z(B_z) \\ &= -2\overrightarrow{\nabla}.\overrightarrow{B}(= 0) \end{aligned} \tag{59}$$

now let's compute the "1" component

$$
\begin{aligned}
\epsilon^{1\beta\gamma\delta}\partial_\beta F_{\gamma\delta} &= \epsilon^{1023}\partial_0 F_{23} + \epsilon^{1032}\partial_0 F_{32} + \epsilon^{1302}\partial_3 F_{02} + \epsilon^{1320}\partial_3 F_{20} \\
&\quad + \epsilon^{1203}\partial_2 F_{03} + \epsilon^{1230}\partial_2 F_{30} \\
&= -\partial_0 F_{23} + \partial_0 F_{32} - \partial_3 F_{02} + \partial_3 F_{20} + \partial_2 F_{03} - \partial_2 F_{30} \\
&= 2(D_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20}) \\
&= 2\left(\frac{1}{c}\partial_t B_x - \partial_z E_y + \partial_y E_z\right) \\
&= 2\left[(\overrightarrow{\nabla} \times \overrightarrow{E})_x + \frac{1}{c}\partial_t B_x\right](= 0)
\end{aligned} \tag{60}
$$

It is common to define the **dual tensor** of $F_{\gamma\delta}$ as $\mathcal{F}^{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$. With such a definition the homogeneous Maxwell equations can be casted in the expression:

$$
\partial_\alpha \mathcal{F}^{\alpha\beta} = 0. \tag{61}
$$

Note: $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta}(\overrightarrow{E} \to \overrightarrow{B}, \overrightarrow{B} \to -\overrightarrow{E})$.

To include $\overrightarrow{H}$ and $\overrightarrow{D}$, one defines the tensor $G^{\alpha\beta} = F^{\alpha\beta}(\overrightarrow{E} \to \overrightarrow{D}, \overrightarrow{B} \to \overrightarrow{H})$, and then Maxwell's equations write:

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \text{ and } \partial_\alpha \mathcal{F}^{\alpha\beta} = 0. \tag{62}$$

Due to covariance of $F^{\alpha\beta}$, it is a tensor, the calculation of em field from one Lorentz frame to another is made easy. Just consider:

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}, \tag{63}$$

or in matrix notation

$$F' = \tilde{\Lambda} F \Lambda = \Lambda F \Lambda \tag{64}$$

Example: Consider a boost along the $\hat{z}$-axis, then

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{65}$$

Plug the F matrix associated to $F^{\gamma\delta}$ in Eq. 64, the matrix multiplication yields:

$$F'^{\gamma\delta} = \begin{pmatrix} 0 & \gamma(E_x - \beta B_y) & \gamma(E_y + \beta B_x) & E_z \\ -\gamma(E_x - \beta B_y) & 0 & B_z & -\gamma(B_y - \beta E_x) \\ -\gamma(E_y + \beta B_x) & -B_z & 0 & \gamma(B_x + \beta E_y) \\ -E_z & \gamma(B_y - \beta E_x) & -\gamma(B_x + \beta E_y) & 0 \end{pmatrix} \tag{66}$$

by inspection we obtain the same equation as [JDJ, Eq. (11.148)].

Fundamental Invariant of the electromagnetic field tensor: *
Note that the quantities

$$F^{\mu\nu}F_{\mu\nu} = 2(E^2 - B^2), \text{ and } F^{\mu\nu}\mathcal{F}_{\mu\nu} = 4\vec{E}.\vec{B}, \tag{67}$$

are invariants. Usually one redefines these two invariants as:

$$\mathcal{I}_1 \equiv -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(B^2 - E^2), \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4}F^{\mu\nu}\mathcal{F}_{\mu\nu} = -\vec{E}.\vec{B}. \tag{68}$$

Note that these invariants may be rewritten as:

$$\mathcal{I}_1 \equiv -\frac{1}{4}\mathrm{tr}(F^2) \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4}\mathrm{tr}(F\mathcal{F}), \tag{69}$$

where $F \equiv F_\mu^\nu = F^{\mu\alpha}g_{\alpha\nu}$ and $\mathcal{F} \equiv \mathcal{F}_\mu^\nu = \mathcal{F}^{\mu\alpha}g_{\alpha\nu}$.
Finally note the identities:

$$F\mathcal{F} = \mathcal{F}F = -\mathcal{I}_2 I, \text{ and } F^2 - \mathcal{F}^2 = -2\mathcal{I}_1 I \tag{70}$$

*adapted from G. Muñoz, Am. J. Phys. **65** (5), May 1997

Eigenvalues of F (for later!):
Look for eigenvalue λ associated to eigenvector Ψ:

$$F\Psi = \lambda\Psi \Rightarrow \mathcal{F}F\Psi = \lambda\mathcal{F}\Psi \Rightarrow \mathcal{F}\Psi = -\frac{\mathcal{I}_2}{\lambda}\Psi. \tag{71}$$

$$(F^2 - \mathcal{F}^2)\Psi = -2I\mathcal{I}_1\Psi = [\lambda^2 - (\mathcal{I}_2/\lambda)^2]\Psi, \tag{72}$$

So characteristic polynomial is: $\lambda^4 + 2\mathcal{I}_1\lambda^2 - \mathcal{I}_2^2 = 0$.
Solutions are:

$$\lambda_\pm = \sqrt{\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2} \pm \mathcal{I}_1} \tag{73}$$

$\lambda_1 = -\lambda_2 = \lambda_-$, $\lambda_3 = -\lambda_4 = i\lambda_+$.

Equation of motion:
The equation describing the dynamics of a relativistics particle of mass m and charge q moving under the influence of em field $F_{\alpha\beta}$ is:

$$\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^\alpha_\beta u^\beta. \tag{74}$$

with $u^\alpha = (\gamma c, \gamma \overrightarrow{v})$. Note that this is equivalent to introducing the "quadri-force"

$$f^\mu = F^{\mu\nu} u_\nu. \tag{75}$$