Northern Illinois University, PHY 571, Fall 2006

Part III: Particle Dynamics Electromagnetic Fields

Last updated on October 5, 2006 (report errors to piot@fnal.gov)

Lagrangian & Hamiltonian formulation

Classical mechanics, Given $\overrightarrow{x}(x^1, x^2, x^3)$ in K and $\overrightarrow{\dot{x}}$ system is characterized by a Lagrangian: $\mathcal{L}(x^i, \dot{x}^i, t)$. The action

$$\mathcal{A} \equiv \int_{t_1}^{t_2} \mathcal{L}(x^i, \dot{x}^i, t) dt \tag{1}$$

is a functional of $\overrightarrow{x}(t)$, $\forall \overrightarrow{x}(t)$ defined for $t \in [t_1, t_2]$.
The least action principle states that $\mathcal{A}$ is a stationary function for any small variation $\delta \overrightarrow{x}(t)$ verifying $\delta \overrightarrow{x}(t_1) = \delta \overrightarrow{x}(t_2) = 0$.
The equation of motion then follow from Euler-Lagrange equations:

$$\begin{aligned} P^i &= \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \\ \frac{dP^i}{dt} &= \frac{\partial \mathcal{L}}{\partial x^i} \end{aligned}$$

Case of a free relativistic particle

Equation of motion must be referential-invariant $\Rightarrow$ the least action princple $\delta\mathcal{A} = 0$ must have the same form in different referential $\Rightarrow$ $\mathcal{A}$ must be a scalar invariant.
$\mathcal{A}$ is a sum of infinitesimal elements along a universe line $x^i(t)$
$\Rightarrow$ $\mathcal{L}dt$ associated to a small displacement must be a scalar invariant.
$\Rightarrow \mathcal{L}dt = \alpha ds = \alpha\sqrt{1 - \frac{V^2}{c^2}}dt$ also,

$$\lim_{V \ll c} \mathcal{L} = \frac{1}{2}mV^2 + \text{const} = \alpha\left(1 - \frac{V^2}{c^2} + \mathcal{O}((V/c)^4)\right) \tag{2}$$

so $\alpha = -mc$, and the relativistic Lagrangian of a free particle is

$$\mathcal{L}_{free} = -mc^2\sqrt{1 - \frac{V^2}{c^2}} = -\frac{mc}{\gamma}\sqrt{u^\alpha u_\alpha}, \tag{3}$$

where $u^\alpha = (\gamma c, \gamma\overrightarrow{v})$ is the four-velocity. One can check:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\overrightarrow{\dot{x}}} = \frac{d}{dt}(m\gamma\overrightarrow{\dot{x}}) = 0$$

Lagrangian of a relativistic particle in e.m. field

The Lagrangian now takes the form $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$, where $\mathcal{L}_{int}$ is the interaction potential.
In the nonrelativistic limit $\mathcal{L}_{int}^{NR} = -e\Phi = -eA^0$ so let's try

$$
\begin{aligned}
\mathcal{L}_{int} &= -\frac{e}{\gamma c} u_\alpha A^\alpha \\
&= -\frac{e}{\gamma c} g_{\alpha\beta} u^\beta A^\alpha \\
&= -\frac{e}{\gamma c}\left(\gamma c \Phi - \gamma \overrightarrow{V}.\overrightarrow{A}\right) \\
\mathcal{L}_{int} &= -e\Phi + e\overrightarrow{\beta}\,\overrightarrow{A}. \qquad (4)
\end{aligned}
$$

The total Lagrangian is

$$\mathcal{L} \;=\; -mc^2\sqrt{1-\frac{V^2}{c^2}}+\frac{e}{c}\overrightarrow{V}.\overrightarrow{A}(\overrightarrow{x})-e\Phi(\overrightarrow{x}). \tag{5}$$

Let's check this gives the equation of motion, by calculating

$$\frac{\partial\mathcal{L}}{\partial\overrightarrow{x}}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\overrightarrow{x}}}=0 \tag{6}$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\overrightarrow{V}} &= \frac{d}{dt}\left(\gamma m\overrightarrow{V}+\frac{e}{c}\overrightarrow{A}\right)\\ &= \frac{d(\gamma m\overrightarrow{V})}{dt}+\frac{e}{c}\left(\frac{\partial\overrightarrow{A}}{\partial t}+\frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i}\overrightarrow{A}\right)=\frac{d(\gamma m\overrightarrow{V})}{dt}+\frac{e}{c}\left(\frac{\partial\overrightarrow{A}}{\partial t}+(\overrightarrow{V}.\overrightarrow{\nabla})\overrightarrow{A}\right)\end{aligned}$$

$$\frac{\partial}{\partial\overrightarrow{x}}\mathcal{L}=\frac{e}{c}\overrightarrow{\nabla}.(\overrightarrow{V}.\overrightarrow{A})-e\overrightarrow{\nabla}\Phi=\frac{e}{c}\left[(\overrightarrow{V}.\overrightarrow{\nabla})\overrightarrow{A}+\overrightarrow{V}\times(\overrightarrow{\nabla}\times\overrightarrow{A})\right]-e\overrightarrow{\nabla}\Phi$$

With $\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$, one finally has:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \overrightarrow{V}} - \frac{\partial \mathcal{L}}{\partial \overrightarrow{x}} = \frac{d}{dt}(\gamma m \overrightarrow{V}) + \frac{e}{c}\frac{\partial \overrightarrow{A}}{\partial t} + e\overrightarrow{\nabla}\Phi - \frac{e}{c}\overrightarrow{V} \times \overrightarrow{B} = 0$$

which gives the Lorentz force equation (in Gauss units!):

$$\frac{d}{dt}(\gamma m \overrightarrow{V}) = e\overrightarrow{E} + \frac{e}{c}(\overrightarrow{V} \times \overrightarrow{B}) \tag{7}$$

Let's check the Lagrangian verifies the "least action principle"
The total Lagrangian can be written

$$\mathcal{L} = -\frac{mc}{\gamma}\sqrt{u^\alpha u_\alpha} - \frac{q}{\gamma c}u_\alpha A^\alpha(x^\beta). \tag{8}$$

define $\tilde{\mathcal{L}} \equiv \gamma \mathcal{L}$. The action is $\mathcal{A} = \int_{\tau_1}^{\tau_2} d\tau \tilde{\mathcal{L}}$.

least action principle $\delta\mathcal{A} = 0$.

$$\delta\mathcal{A} = \delta\left[\int_{\tau_1}^{\tau_2} d\tau\tilde{\mathcal{L}}\right] = \int_{\tau_1}^{\tau_2} d\tau\delta\tilde{\mathcal{L}} \tag{9}$$

$$-\delta\tilde{\mathcal{L}} = mc\frac{1}{2}\frac{1}{\sqrt{u^\alpha u_\alpha}}\left[\frac{\partial(u^\alpha u_\alpha)}{\partial u^\beta}\right]\delta u^\beta + qA_\alpha\delta u^\alpha + qu^\alpha\frac{\partial A_\alpha}{\partial x^\beta}\delta x^\beta. \tag{10}$$

One has $\delta u^\alpha = \delta(\frac{\partial x^\alpha}{\partial\tau}) = \frac{\partial}{\partial\tau}(\delta x^\alpha)$, and

$$\frac{\partial(u^\alpha u_\alpha)}{\partial u^\beta} = g_{\alpha\gamma}\frac{\partial(u^\alpha u^\gamma)}{\partial u^\beta} = g_{\alpha\gamma}\left(\delta^\alpha_\beta u^\gamma + \delta^\gamma_\beta u^\alpha\right) = 2u_\beta \tag{11}$$

using $\delta\frac{dx^\alpha}{d\tau} = \frac{d(\delta x^\alpha)}{d\tau}$ (commutation of δ and d operators), one gets:

$$-c\delta\tilde{\mathcal{L}} = (mcu_\beta + qA_\beta)\frac{d(\delta x^\beta)}{d\tau} + qu^\alpha\partial_\beta A_\alpha\delta x^\beta. \tag{12}$$

Evaluating the integral by part and noting that $\delta x^\beta(\tau_1) = \delta x^\beta(\tau_2) = 0$ gives:

$$\delta \mathcal{A} = -\int_{\tau_1}^{\tau_2} d\tau \left[-mc\frac{du_\beta}{d\tau} - q(\partial_\alpha A_\beta)u^\alpha + qu^\alpha \partial_\beta A_\alpha \right] \delta x^\beta, \qquad (13)$$

and $\delta\mathcal{A} = 0 \Rightarrow [...] = 0$ (linear independence argument) gives the equation of motion $m\frac{d}{d\tau}u_\beta = \frac{q}{c}F_{\alpha\beta}u^\alpha$

The canonical momentum $\overrightarrow{P}$ conjugate to $\overrightarrow{x}$ is, by definition,

$$\begin{aligned} \overrightarrow{P} &= \frac{\partial \mathcal{L}}{\partial \overrightarrow{V}} = \gamma m \overrightarrow{\dot{x}} + \frac{e}{c}\overrightarrow{A} \\ \overrightarrow{P} &= \overrightarrow{p} + \frac{e}{c}\overrightarrow{A} \end{aligned} \qquad (14)$$

and the hamiltonian is defined as:

$$\mathcal{H} \equiv \overrightarrow{P}.\overrightarrow{V} - \mathcal{L} \qquad (15)$$

Relativistic Hamiltonian

Use $\overrightarrow{P} = \gamma m \overrightarrow{v} + \frac{e}{c}\overrightarrow{A}$ and calculate $\mathcal{H}$ then express $\mathcal{H}$ only as a function of $\overrightarrow{P}$ and $\overrightarrow{x}$. On can do the algebra (namely explicit $\overrightarrow{v}$ as a function $\overrightarrow{P}$ and replace in the expression of $\mathcal{H}$.

$$\begin{aligned} \mathcal{H} &= \overrightarrow{v}.\left(\gamma m \overrightarrow{v} + \frac{e}{c}\overrightarrow{A}\right) + \gamma m c^2 \frac{1}{\gamma} + e\Phi - \frac{e}{c}\overrightarrow{A}\overrightarrow{v} \qquad (16) \\ &= \gamma m v^2 + \frac{mc^2}{\gamma} + e\Phi = \gamma m c^2 + e\Phi. \qquad (17) \end{aligned}$$

We note that the relation between $\overrightarrow{P} - \frac{e}{c}\overrightarrow{A}$ and $\mathcal{H} - e\Phi$ is the same as between $\mathcal{H}$ and $\overrightarrow{p}$ for the case of zero-field so we have:

$$(\mathcal{H} - e\Phi)^2 = \left(\overrightarrow{P} - \frac{e}{c}\overrightarrow{A}\right)^2 c^2 + m^2c^4 \qquad (18)$$

So finally,

$$\mathcal{H} = \sqrt{\left(\overrightarrow{P}c - e\overrightarrow{A}\right)^2 + m^2c^4} + e\Phi \qquad (19)$$

Motion of a particle in a constant uniform E-field

Let $\mathcal{E}$ be the total energy: $\mathcal{E} = \sqrt{(pc)^2 + (mc^2)^2} = \gamma mc^2$
$\Rightarrow \gamma = \frac{\mathcal{E}}{mc^2}$. Thus,

$$\overrightarrow{p} = \gamma m \overrightarrow{v} = \frac{\mathcal{E}}{c^2}\overrightarrow{v} \Rightarrow \overrightarrow{v} = \frac{c^2}{\mathcal{E}}\overrightarrow{P}. \tag{20}$$

Let's consider the case of a particle of charge q interacting with the field $\overrightarrow{E} = E\widehat{x}$, and with initial conditions $p(t=0) = p_0\widehat{y}$. Lorentz Force gives:

$$\dot{p}_x = qE, \text{ and, } \dot{p}_y = 0 \tag{21}$$

which yields:

$$p_x = qEt, \text{ and, } p_y = p_0 \tag{22}$$

and $p^2 = (qEt)^2 + p_0^2$.

So the total energy at time t is:

$$\mathcal{E}^2(t) = c^2 \left[(qEt)^2 + p_0^2\right] + m^2c^4 = (cqEt)^2 + \mathcal{E}_0 \qquad (23)$$

where $\mathcal{E}_0 \equiv \mathcal{E}(t=0)$. The velocity is:

$$v_x = \frac{dx}{dt} = c\frac{cqEt}{\sqrt{(cqEt)^2 + \mathcal{E}_0^2}} \qquad (24)$$

note that $\lim_{t\to\infty} = c$. Performing a time integration yields:

$$x(t) = \frac{1}{qE}\sqrt{(cqEt)^2 + \mathcal{E}_0^2}. \qquad (25)$$

For y-axis we have:

$$\frac{dy}{dt} = \frac{c^2 p_0}{\sqrt{(cqEt)^2 + \mathcal{E}_0^2}}, \qquad (26)$$

and $\lim_{t\to\infty} \frac{dy}{dt} = 0$.

A time integration gives:

$$y = \frac{p_0 c}{qE} \sinh^{-1}\left(\frac{cqEt}{\mathcal{E}_0}\right). \tag{27}$$

remember: $\int_0^{\xi} \frac{d\tilde{\xi}}{\tilde{\xi}^2+1} = \sinh^{-1}(\xi)$. Expliciting t as a function of y:

$$cqEt = \sinh\left(\frac{qEy}{p_0 c}\right), \tag{28}$$

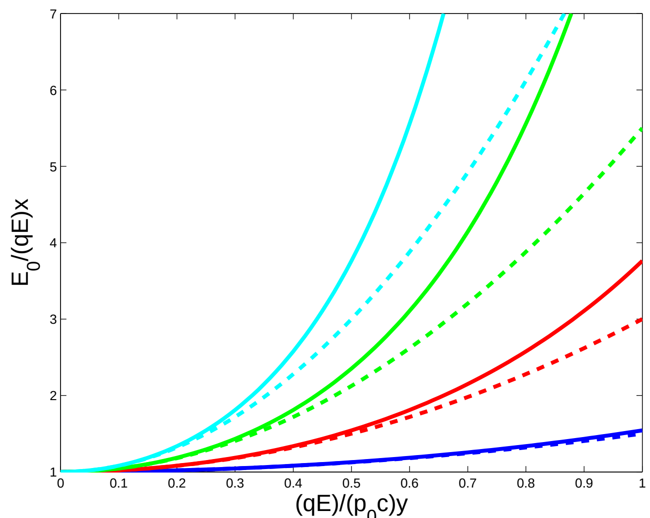
and substituting in x, we have the trajectory equation in (x, y) plane:

$$\begin{aligned} x &= \frac{\mathcal{E}_0}{qE}\sqrt{\sinh^2\left(\frac{qEy}{p_0 c}\right) + 1} \\ &= \frac{\mathcal{E}_0}{qE}\cosh\left(\frac{qEy}{p_0 c}\right). \end{aligned} \tag{29}$$

The nonrelativistic limit ($v \ll c$) is given by setting $\mathcal{E}_0 = mc^2$, $p_0 = mv_0$:

$$x = \frac{mc^2}{qE}\cosh\left(\frac{qEy}{mv_0c}\right) \simeq \frac{qE}{2mv_0^2}y^2 + \text{const.} \tag{30}$$

the familiar parabola. The expansion $\cosh(x) = 1 + x^2/2! + \mathcal{O}(x^4)$ was used.



Trajectories (in normalized coordinate) in uniform constant E-field: $\hat{x} = \cosh(\kappa\hat{y})$, with $\kappa = 1, 2, 3, 4$. dashed are corresponding parabolic approximation $\hat{x} = 1 + \frac{1}{2}(\kappa\hat{y})^2$.

Motion of a particle in a constant uniform B-field

Lorentz force gives (CGS!): $\dot{\overrightarrow{p}} = \frac{q}{c}\overrightarrow{v} \times \overrightarrow{B}$; $\overrightarrow{p} = \frac{\mathcal{E}}{c^2}\overrightarrow{v} \Rightarrow \dot{\overrightarrow{v}} = \frac{cq}{\mathcal{E}}\overrightarrow{v} \times \overrightarrow{B}$.
$\overrightarrow{B}$ changes the direction of $\overrightarrow{v}$ but not its magnitude so W, and γ are constants. Consider for simplicity $\overrightarrow{B} = B\hat{z}$, then

$$\overrightarrow{v} \times \overrightarrow{B} = v_y B\hat{x} - v_x B\hat{y},$$

which gives

$$\begin{aligned} \dot{v}_x &= \frac{cqB}{\mathcal{E}} v_y, \\ \dot{v}_y &= -\frac{cqB}{\mathcal{E}} v_x, \\ \dot{v}_z &= 0. \end{aligned} \tag{31}$$

So we have to solve a system of coupled ODE of the form:

$$\dot{v}_x = \omega v_y, \;\; \dot{v}_y = -\omega v_x, \;\; \dot{v}_z \;\; = \;\; 0. \tag{32}$$

where $\omega \equiv \frac{cqB}{\mathcal{E}}$. Let's cast the transverse equation of motions:

$$\frac{d}{dt}(v_x + iv_y) = -i\omega(v_x + iv_y), \tag{33}$$

the solution is of the form $v_x + iv_y = v_\perp e^{-i(\omega t+\alpha)}$. Let $v_\parallel = v_z$. With these notations we can write:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_\perp \cos(\omega t + \alpha) \\ -v_\perp \sin(\omega t + \alpha) \\ v_\parallel \end{pmatrix}; \text{ with } v_\perp = \sqrt{v_x^2 + v_y^2}, \text{ and,} \tag{34}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + R\sin(\omega t + \alpha) \\ y_0 + R\cos(\omega t + \alpha) \\ z_0 + v_\parallel t \end{pmatrix}; \text{ with } R \equiv \frac{v_\perp}{\omega} = \frac{v_\parallel \mathcal{E}}{cqB}. \tag{35}$$

So the trajectory is a helix whose axis is along $\hat{z}$, with radius R. The frequency ω is the rotation frequency of the trajectory when projected in a plan orthogonal to the helix axis.

R is called the gyroradius, $R = \frac{v_\| \mathcal{E}}{cqB} = \frac{p_\perp c}{qB}$. $\omega = \frac{qcB}{\mathcal{E}} = \frac{qcB}{\gamma m c^2} \Rightarrow \frac{qB}{\gamma mc}$ is the gyrofrequency ($\frac{v_\perp}{R}$).

The gyroradius and gyrofrequency arise in all calculations involving particle motion in magnetic fields. Note that in SI units:

$$\omega = \frac{qB}{\gamma m}, \text{ and } R = \frac{\gamma m v_\perp}{qB}. \tag{36}$$

Motion of a particle in a constant uniform magnetic and electric field

We now consider the case where both $\overrightarrow{E}$ and $\overrightarrow{B}$ fields are present in some arbitrary orientation. The idea is to directly solve the equation of motion

$$\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^\alpha_\beta u^\beta; \tag{37}$$

the treatment follows G. Muñoz, *Am. J. Phys.* **65**, 429 (1997). Let $\theta \equiv \frac{q\tau}{mc}$, and rewrite the equation of motion in matrix form:

$$\frac{dU}{d\theta} = Fu \quad \text{with solution } u = e^{\theta F} u(0), \tag{38}$$

where,

$$e^{\theta F} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} F^n. \tag{39}$$

Now, recall the identity (see handout end of part **II**) $F^2 = \mathcal{F}^2 - 2\mathcal{I}_1 I$. Because of this, every power of F can be written as a linear combination of I, F, $\mathcal{F}$, and F^2, e.g.:

$$\begin{aligned} F^3 &= FF^2 = F\mathcal{F}^2 - 2\mathcal{I}_1 F = -\mathcal{I}_2\mathcal{F} - 2\mathcal{I}_1 F; \\ F^4 &= -\mathcal{I}_2 F\mathcal{F} - 2\mathcal{I}_1 F^2 = \mathcal{I}_2^2 I - 2\mathcal{I}_1 F^2; \\ F^5 &= \mathcal{I}_2^2 F - 2\mathcal{I}_1 F^3 = (4\mathcal{I}_1^2 + \mathcal{I}_2^2)F + 2\mathcal{I}_1\mathcal{I}_2\mathcal{F}; \\ &\text{etc...} \end{aligned} \tag{40}$$

This means,

$$e^{\theta F} = \alpha I + \beta F + \gamma\mathcal{F} + \delta F^2. \tag{41}$$

To find the constants α, β, γ, and δ, consider the following traces (note that the trace of odd power of F and $\mathcal{F}$ are zero:

$$
\begin{aligned}
t_0 &\equiv \frac{1}{4}\mathrm{Tr}[e^{\theta F}] = \alpha - \mathcal{I}_1\delta, \\
t_1 &\equiv \frac{1}{4}\mathrm{Tr}[Fe^{\theta F}] = -\mathcal{I}_1\beta - \mathcal{I}_2\gamma, \\
t_2 &\equiv \frac{1}{4}\mathrm{Tr}[F^2e^{\theta F}] = -\mathcal{I}_1\alpha + (2\mathcal{I}_1^2 + \mathcal{I}_2^2)\delta, \\
t_3 &\equiv \frac{1}{4}\mathrm{Tr}[F^3e^{\theta F}] = 2(\mathcal{I}_1^2 + \mathcal{I}_2^2)\beta + \mathcal{I}_1\mathcal{I}_2\gamma.
\end{aligned}
$$

Solving this system of equation for α, β, γ, and δ, yields:

$$
\begin{aligned}
\alpha &= \tfrac{(2\mathcal{I}_1^2+\mathcal{I}_2^2)t_0+\mathcal{I}_1t_2}{\mathcal{I}_1^2+\mathcal{I}_2^2}; \quad \beta = \frac{t_3 + \mathcal{I}_1t_1}{\mathcal{I}_1^2 + \mathcal{I}_2^2}; \\
\gamma &= -\tfrac{(2\mathcal{I}_1^2+\mathcal{I}_2^2)t_1+\mathcal{I}_1t_3}{\mathcal{I}_2(\mathcal{I}_1^2+\mathcal{I}_2^2)}; \quad \delta = \frac{t_2 + \mathcal{I}_1t_0}{\mathcal{I}_1^2 + \mathcal{I}_2^2}.
\end{aligned}
$$

The traces are found upon diagnonalization of $e^{\theta F} \rightarrow e^{\theta F'}$:

$$\mathrm{Tr}[e^{\theta F}] = \mathrm{Tr}[e^{\theta F'}] = \sum_{i=1}^{4} e^{\theta \lambda_i}, \tag{42}$$

where λ_i are the eigenvalues of F: $\lambda_1 = -\lambda_2 = \lambda_-$, and $\lambda_3 = -\lambda_4 = i\lambda_+$ where $\lambda_\pm = \sqrt{\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2} \pm \mathcal{I}_1}$.
Thus

$$\begin{aligned} t_0 &= \frac{1}{4}\mathrm{Tr}[e^{\theta F}] = \frac{1}{2}[\cosh(\theta\lambda_-) + \cos(\theta\lambda_+)] \\ t_k &= \frac{1}{4}\mathrm{Tr}[F^k e^{\theta F}] = \frac{\partial^k t_0}{\partial \theta^k} \end{aligned} \tag{43}$$

So

$$
\begin{aligned}
t_1 &= \frac{1}{2}[\lambda_- \sinh(\theta\lambda_-) - \lambda_+ \sin(\theta\lambda_+)] \\
t_2 &= \frac{1}{2}[\lambda_-^2 \cosh(\theta\lambda_-) - \lambda_+^2 \cos(\theta\lambda_+)] \\
t_3 &= \frac{1}{2}[\lambda_-^3 \cosh(\theta\lambda_-) + \lambda_+^3 \sin(\theta\lambda_+)]
\end{aligned}
$$

Substitute and simplify to finally obtain the values:

$$
\begin{aligned}
\alpha &= \frac{\lambda_+^2 \cosh(\theta\lambda_-)+\lambda_-^2 \cos(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}; \quad \beta = \frac{\lambda_- \sinh(\theta\lambda_-) + \lambda_+ \sin(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}}; \\
\gamma &= \frac{|\mathcal{I}_2|}{\mathcal{I}_2}\frac{\lambda_- \sin(\theta\lambda_+)-\lambda_+ \sinh(\theta\lambda_-)}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}; \quad \delta = \frac{\cosh(\theta\lambda_-) - \cos(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}}.
\end{aligned}
$$

Substitute into the power expansion for $e^{\theta F}$ to find:

$$
\begin{aligned}
u(\theta) &= \frac{1}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}\Big[(\lambda_+^2 I + F^2)\cosh(\theta\lambda_-) + (\lambda_-^2 I - F^2)\cos(\theta\lambda_+) \\
&+ \left(\lambda_- F - \frac{|\mathcal{I}_2|}{\mathcal{I}_2}\lambda_+\mathcal{F}\right)\sinh(\theta\lambda_-) + \left(\lambda_+ F + \frac{|\mathcal{I}_2|}{\mathcal{I}_2}\lambda_-\mathcal{F}\right)\sin(\theta\lambda_+)\Big] u(0).
\end{aligned}
$$

Note that $u(\theta) = \frac{2}{mc}\frac{dx}{d\theta}$, so integrate over $\theta \in [0,\theta)$ to get

$$
\begin{aligned}
x(\tau) &= x(0) + \frac{mc}{q\mathcal{I}_2}\mathcal{F}u(0) + \frac{mc}{2q\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}\left[\left(F - \frac{\lambda_+^2}{\mathcal{I}_2}\mathcal{F}\right)\cosh(\theta\lambda_-)\right. \\
&- \left.\left(F + \frac{\lambda_-^2}{\mathcal{I}_2}\mathcal{F}\right)\cos(\theta\lambda_+) + \frac{\lambda_+^2 I + F^2}{\lambda_-}\sinh(\theta\lambda_-) + \frac{\lambda_-^2 I - F^2}{\lambda_+}\sin(\theta\lambda_+)\right] u(0).
\end{aligned}
$$

which is the final result.

Consider the special case of $\vec{E} = E\hat{x}$, $\vec{B} = B\hat{y}$ then $\vec{E} \perp \vec{B} \Rightarrow \mathcal{I}_2 = 0$. Taking the limit $\mathcal{I}_2 \to 0$ gives:
$\lambda_- \to 0$; $\lambda_+ \to \sqrt{2\mathcal{I}_1}$, $\cosh(\theta\lambda_-) \to 1$ and $\sinh(\theta\lambda_-)/\lambda_- \to \theta$.
Consider the case $\mathcal{I}_1 = \frac{1}{2}(B^2 - E^2) > 0$ and let's take $x(0) = 0$. Then:

$$x(\tau) = \frac{mc}{q\mathcal{I}_2}\mathcal{F}u(0) + \frac{mc}{2q\mathcal{I}_1}\left[\left(F - \frac{2\mathcal{I}_1}{\mathcal{I}_2}\mathcal{F}\right) - F\cos(\theta\lambda_+) + (2\mathcal{I}_1 I + F^2)\theta - \frac{1}{\sqrt{2\mathcal{I}_1}}F^2\sin(\theta\sqrt{2\mathcal{I}_1})\right]u(0). \tag{44}$$

Define $\Omega \equiv \frac{q}{mc}\sqrt{2\mathcal{I}_1}$ then

$$x(\tau) = (I + \frac{F^2}{2\mathcal{I}_1}u(0)\tau + \frac{mc}{2q\mathcal{I}_1}\left(1 - \cos\Omega\tau - \frac{F}{\sqrt{2\mathcal{I}_1}}\sin\Omega\tau\right)Fu(0) \tag{45}$$

$$Fu(0) = \gamma_0 c \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \beta_{0x} \\ \beta_{0y} \\ \beta_{0z} \end{pmatrix} = \gamma_0 c \begin{pmatrix} E \\ E - \beta_{0z}B \\ 0 \\ \beta_{0x}B \end{pmatrix} ; F^2 u(0) = \gamma_0 c \begin{pmatrix} E(E - \beta_{0z}B) \\ -2\mathcal{I}_1\beta_{0x} \\ 0 \\ B(E - \beta_{0z}B) \end{pmatrix}$$

and so:

$$\begin{aligned}
x &= \frac{\gamma_0 mc^2}{2q\mathcal{I}_1}\left[(E - B\beta_{0z})(1 - \cos\Omega\tau) + \sqrt{2\mathcal{I}_1}\beta_{0x}\sin\Omega\tau\right] \\
y &= \gamma_0 v_{0y}\tau \\
z &= \frac{\gamma_0 cE}{2\mathcal{I}_1}(B - E\beta_{0z})\tau + \frac{\gamma_0 mc^2 B}{2q\mathcal{I}_1}\left[\beta_{0x}(1 - \cos\Omega\tau) - \frac{E - B\beta_{0z}}{\sqrt{2\mathcal{I}_1}}\sin\Omega\tau\right] \\
t &= \frac{\gamma_0 B}{2\mathcal{I}_1}(B - E\beta_{0z})\tau + \frac{\gamma_0 mcE}{2q\mathcal{I}_1}\left[\beta_{0x}(1 - \cos\Omega\tau) - \frac{E - B\beta_{0z}}{\sqrt{2\mathcal{I}_1}}\sin\Omega\tau\right]
\end{aligned}$$

Note that the particle has a velocity perpendicular to $\overrightarrow{E}$ and $\overrightarrow{B}$ fields. The so-called $E \times B$ drift. The drift velocity is $v_d = cE/B$.

Non uniform magnetic field and adiabatic invariance

Suppose the magnetic field is non uniform but changes "slowly" compared to the "gyroperiod" of the charge particle (charge=q) under its influence. This is a so-called "adiabatic change. The action integral is conserved:

$$J = \oint \overrightarrow{P}_{\perp}.\overrightarrow{dl} \tag{46}$$

$\overrightarrow{dl}$ is the line element along the particle trajectory. Expliciting $P_{\perp}$:

$$\begin{aligned} J &= \oint (\gamma m \overrightarrow{v}_{\perp} + \frac{q}{c}\overrightarrow{A}).\overrightarrow{dl} \\ &= (\gamma m \omega_B a)(2\pi a) + \frac{q}{c}\int_S \overrightarrow{B}.\hat{n} dS \end{aligned} \tag{47}$$

$$\Rightarrow J = 2\pi\gamma m\omega_B a^2 - \frac{q}{c}\pi B a^2 \tag{48}$$

since B is anti-parallel to $\widehat{n}$. Also, $\gamma m\omega_B = \frac{q}{c}B$ so that:

$$J = \frac{q}{c}\pi B a^2. \tag{49}$$

This means the magnetic flux

$$\Phi_B = \int_S \overrightarrow{B}.\overrightarrow{dS} = \pi B a^2 \tag{50}$$

is an adiabatic invariant.

Non uniform magnetic field without adiabatic invariance: the solenoid

A B_r component of magnetic field impart a p_θ to a charge particle coming off a cathode immersed in a B-field. Let $\overrightarrow{B}(z=0) \equiv B_c$.

$$\begin{aligned} F_\theta &= \frac{q}{c} v_z B_r = \frac{dp_\theta}{dt};\ p_\theta(t=0) = p_\theta(t=0) = 0 \\ &\Rightarrow p_\theta = \frac{q}{c}\int_0^\infty B_r v_z dt = \frac{q}{c}\int_0^\infty B_r dz \end{aligned} \tag{51}$$

But

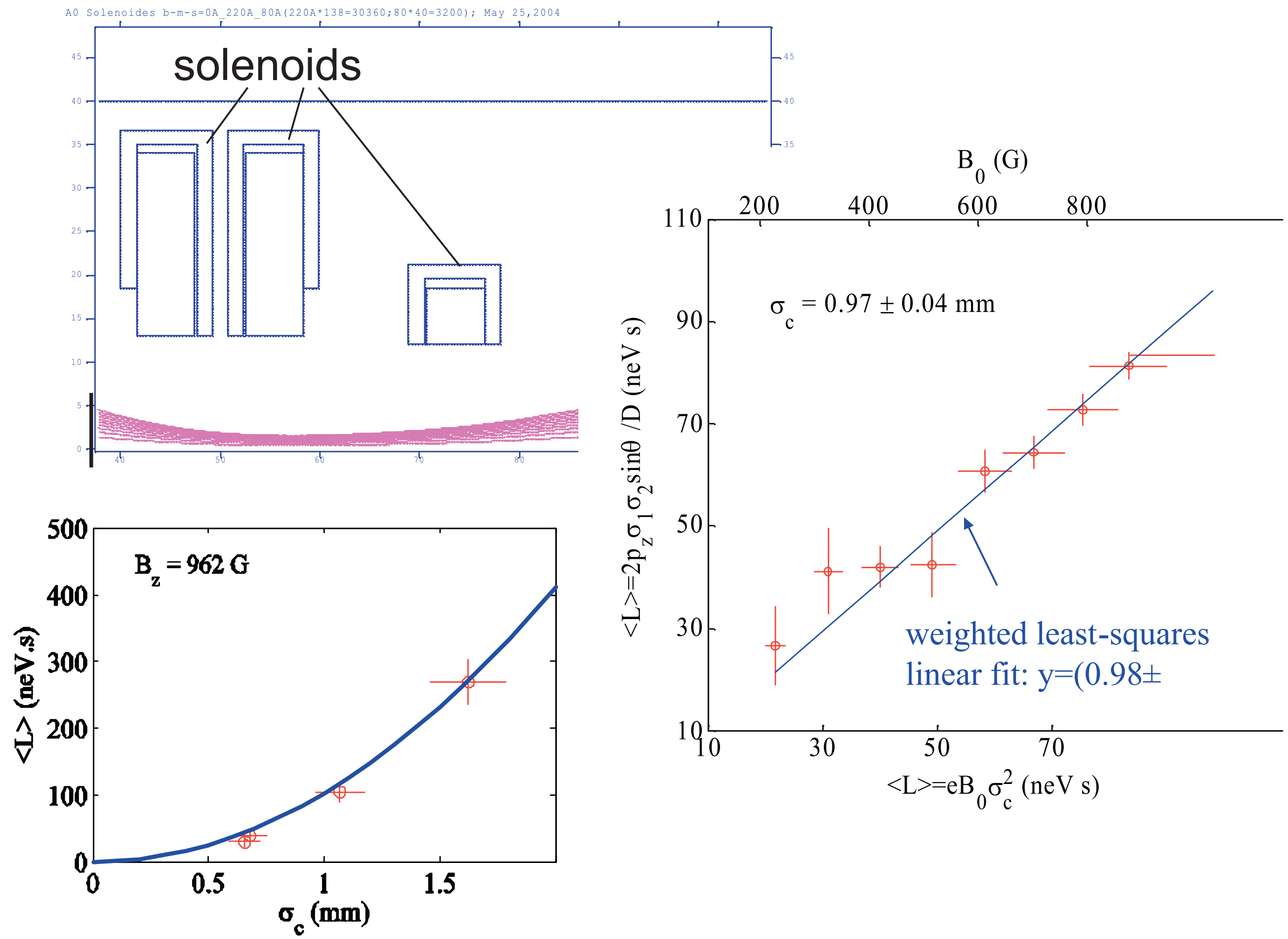
$$\begin{aligned} \int \overrightarrow{B}\overrightarrow{dS} = 0 &= -\pi r^2 B_c + 2\pi r \int_0^\infty Brdz \\ &\Rightarrow \int_0^\infty B_r dz = \frac{1}{2} B_c r. \end{aligned} \tag{52}$$

Consequently the charge q picks-up a total angular momentum $p_\theta = \frac{q}{2c} B_c r$. Note that

$$\frac{p_\theta}{p_c} = \frac{1}{2} \frac{q B_c}{p_c c} r = \frac{r}{2\rho} \tag{53}$$

where $\rho^{-1} \equiv \frac{q B_c}{p_c c}$. This tells what fraction of initial momentum got converted to angular momentum. ρ is the gyroradius the electron would have had if it was orthogonal to $\overrightarrow{B}_c$.
Note that for a particle originating external to the solenoid, $p_\theta = 0$ by symmetry.

example of application: generation of “magnetized e- beam”



see http://prst-ab.aps.org/abstract/PRSTAB/v7/i12/e123501