

Northern Illinois University, PHY 571, Fall 2006

Part I: Electromagnetic resonance
in cylindrical cavities and waveguides

Last updated on September 7, 2006 (report errors to
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Maxwell's Equations in MKSA units

$$\vec{\nabla} \cdot \vec{D} = \rho, \text{ Coulomb law} \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \text{ no magnetic charges} \quad (2)$$

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}, \text{ Faraday law} \quad (3)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \partial_t \vec{D}, \text{ Ampère- Maxwell law} \quad (4)$$

$\vec{D} \equiv \epsilon \vec{E} + \vec{P}$ is the electric displacement, ϵ the medium electric permittivity, \vec{E} the electric field \vec{P} is the polarization.

ρ , is the charge density.

\vec{B} is the induction, $\vec{H} \equiv \vec{B}/\mu - \vec{M}$ is the magnetic field, μ the magnetic permeability of the medium and \vec{M} the magnetization.

\vec{J} is the current.

Consider a perfectly conducting cavity which is cylindrical, and which is filled with a homogeneous, isotropic, non-conducting, non-dissipative medium. Then in the medium,

$$\vec{D} = \epsilon \vec{E}, \text{ and } \vec{B} = \mu \vec{H}. \quad (5)$$

The boundary condition at the cavity walls are:

$$\hat{n} \cdot \vec{B} = \hat{n} \times \vec{E} = 0. \quad (6)$$

Maxwells' equations (MKSA units):

$$\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow \vec{\nabla} \cdot \vec{E} = 0; \vec{\nabla} \cdot \vec{B} = 0; \quad (7)$$

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0; \quad (8)$$

$$\vec{\nabla} \times \vec{H} - \partial_t \vec{D} = 0 \Rightarrow \nabla \times \vec{B} - \mu \epsilon \partial_t \vec{E} = 0. \quad (9)$$

Derivation of JDJ Eqn (8.26)

$$\begin{bmatrix} \vec{E}(r, \phi, z, t) \\ \vec{B}(r, \phi, z, t) \end{bmatrix} = \begin{bmatrix} \vec{E}(r, \phi) \\ \vec{B}(r, \phi) \end{bmatrix} e^{\pm ikz - i\omega t}. \quad (10)$$

Separate transverse and axial components:

$$\begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} \vec{E}_t \\ \vec{B}_t \end{bmatrix} + \begin{bmatrix} E_z \\ B_z \end{bmatrix} \hat{z}; \quad (11)$$

$$\begin{bmatrix} E_z \\ B_z \end{bmatrix} = \hat{z} \cdot \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix}; \text{ and } \begin{bmatrix} \vec{E}_t \\ \vec{B}_t \end{bmatrix} = (\hat{z} \times \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix}) \times \hat{z}. \quad (12)$$

From Maxwell's equations,

$$[\hat{z} \times (\vec{\nabla} \times \vec{E})] \times \hat{z} - i\omega \vec{B}_t = 0, \quad (13)$$

$$[\hat{z} \times (\vec{\nabla} \times \vec{B})] \times \hat{z} + i\mu\epsilon\omega \vec{E}_t = 0. \quad (14)$$

In general,

$$\hat{z} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\hat{z} \cdot \vec{V}) - (\hat{z} \cdot \vec{\nabla}) \vec{V} = \vec{\nabla} V_z - \partial_z \vec{V} \quad (15)$$

$$[\hat{z} \times (\vec{\nabla} \times \vec{V})] \times \hat{z} = \vec{\nabla} V_z \times \hat{z} - \partial_z (\vec{V} \times \hat{z}) \quad (16)$$

$$= -\hat{z} \times \vec{\nabla}_t V_z + \hat{z} \times \partial_z \vec{V}_t. \quad (17)$$

So,

$$i\omega \vec{B}_t = \hat{z} \times \partial_z \vec{E}_t - \hat{z} \times \vec{\nabla}_t E_z, \quad (18)$$

$$-i\omega\mu\epsilon \vec{E}_t = \hat{z} \times \partial_z \vec{B}_t - \hat{z} \times \vec{\nabla}_t B_z. \quad (19)$$

Apply “ $\hat{z} \times \partial_z$ ” to 18:

$$i\omega \hat{z} \times \partial_z \vec{B}_t = \hat{z} \times (\hat{z} \times \partial_z^2 \vec{E}_t) - \hat{z} \times (\hat{z} \times \vec{\nabla}_t \partial_z E_z) \quad (20)$$

$$= -\partial_z^2 \vec{E}_t - (-) \vec{\nabla}_t \partial_z E_z \quad (21)$$

$$= +k^2 \vec{E}_t \pm ik \vec{\nabla}_t E_z. \quad (22)$$

$$\Rightarrow \hat{z} \times \partial_z \vec{B}_t = \frac{-i}{\omega} (k^2 \vec{E}_t \pm ik \vec{\nabla}_t E_z). \quad (23)$$

Similarly:

$$\hat{z} \times \partial_z \vec{E}_t = \frac{+i}{\mu \epsilon \omega} (k^2 \vec{B}_t \pm ik \vec{\nabla}_t B_z). \quad (24)$$

Insert 24 into 18:

$$i\omega B_t = \frac{i}{\mu\epsilon\omega} (k^2 \vec{B}_t \pm ik \vec{\nabla}_t B_z) - \hat{z} \times \vec{\nabla}_t E_z \quad (25)$$

$$i(\mu\epsilon\omega^2 - k^2) \vec{B}_t = \mp k \vec{\nabla}_t B_z + \mu\epsilon\omega \hat{z} \times \vec{\nabla}_t E_z \quad (26)$$

$$\Rightarrow \vec{B}_t = \frac{i}{\mu\epsilon\omega^2 - k^2} (\pm k \vec{\nabla}_t B_z + \mu\epsilon\omega \hat{z} \times \vec{\nabla}_t E_z). \quad (27)$$

Insert 23 into 19:

$$-i\mu\epsilon\omega E_t = \frac{-i}{\omega} (k^2 \vec{E}_t \pm ik \vec{\nabla}_t E_z) - \hat{z} \times \vec{\nabla}_t B_z \quad (28)$$

$$-i(\mu\epsilon\omega^2 - k^2) \vec{E}_t = \pm k \vec{\nabla}_t E_z - \omega \hat{z} \times \vec{\nabla}_t B_z \quad (29)$$

$$\Rightarrow \vec{E}_t = \frac{i}{\mu\epsilon\omega^2 - k^2} (\pm k \vec{\nabla}_t E_z - \omega \hat{z} \times \vec{\nabla}_t B_z). \quad (30)$$

Eqn 27 and 30 generally pertain as long as the transverse cross-section is z -independent as illustrated in JDJ Fig. 8.3.

Wave Equation

from Maxwell's equations:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \partial_t \vec{\nabla} \times \vec{B} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} + \mu\epsilon \partial_t^2 \vec{E} \quad (31)$$

$$= -\nabla_t^2 \vec{E} + k^2 \vec{E} - \mu\epsilon\omega^2 \vec{E} = 0 \quad (32)$$

$$[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)] \vec{E} = 0. \quad (33)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) - \mu\epsilon \partial_t \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} + \mu\epsilon \partial_t^2 \vec{B} \quad (34)$$

$$= -\nabla_t^2 \vec{B} + k^2 \vec{B} - \mu\epsilon\omega^2 \vec{B} = 0 \quad (35)$$

$$[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)] \vec{B} = 0. \quad (36)$$

33 and 36 can be casted as

$$[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)] \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = 0. \quad (37)$$

Recipe for EM-field calculations

1/ Find $E_z(r, \phi)$ and $B_z(r, \phi)$ from $\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2) \right] \begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = 0$.

2/ Find $\vec{E}_t(r, \phi)$ and $\vec{B}_t(r, \phi)$ from

$$\begin{bmatrix} \vec{E}_t(r, \phi) \\ \vec{B}_t(r, \phi) \end{bmatrix} = \frac{i}{\mu\epsilon\omega^2 - k^2} \left\{ \pm k \vec{\nabla}_t \begin{bmatrix} E_z(r, \phi) \\ B_z(r, \phi) \end{bmatrix} - \omega \hat{z} \times \vec{\nabla}_t \begin{bmatrix} B_z(r, \phi) \\ -\mu\epsilon E_z(r, \phi) \end{bmatrix} \right\}.$$

3/ The total field is:

$$\begin{bmatrix} \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} \vec{E}_t \\ \vec{B}_t \end{bmatrix} + \begin{bmatrix} E_z \\ B_z \end{bmatrix} \hat{z}.$$

4/ Incorporate the boundary conditions (S : cavity side surface):

$$\hat{n} \times \vec{E} = 0 \Rightarrow E_z|_S = 0 \text{ and, } \vec{E}_t = 0 \text{ at end plates}$$

$$\hat{n} \cdot \vec{B} = 0 \Rightarrow \hat{n} \cdot \vec{\nabla}_t B_z = 0 \Rightarrow \partial_n B_z|_S = 0 \text{ from Eq.19}$$

Boundary conditions at the cavity side \mathcal{S} for B_z (E_z straightforward)

$$-i\mu\epsilon\omega \vec{E}_t = \hat{z} \times \partial_z \vec{B}_t - \hat{z} \times \vec{\nabla}_t B_z = \hat{z} \times (\partial_z \vec{B}_t - \vec{\nabla}_t B_z) \quad (\text{from Eq.19}) \quad (38)$$

take “ $\hat{n} \times$ ”:

$$-i\mu\epsilon\omega \hat{n} \times \vec{E}_t = \hat{n} \times [\hat{z} \times (\partial_z \vec{B}_t - \vec{\nabla}_t B_z)] \quad (39)$$

I.h.s= 0 since $\hat{n} \times \vec{E} = 0$ and $E_z=0$ at the cavity walls.

use $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c}$:

$$[\hat{n} \cdot (\partial_z \vec{B}_t - \vec{\nabla}_t B_z)] \hat{z} - (\hat{n} \cdot \hat{z})(\partial_z \vec{B}_t - \vec{\nabla}_t B_z) = 0 \quad (40)$$

$\hat{n} \cdot \hat{z} = 0$ because $\hat{n} \perp \hat{z}$ on the cavity side

$$\Rightarrow \partial_z(\hat{n} \cdot \vec{B}_t) - \hat{n} \cdot \vec{\nabla}_t B_z = 0 \quad (41)$$

$$\hat{n} \cdot \vec{B}_t = \hat{n} \cdot \vec{B} - (\hat{n} \cdot \hat{z}) B_z = 0$$

$$\Rightarrow \partial_n B_z = 0 \quad (42)$$

Mode categories

The boundary conditions $E_z|_S = 0$ and $\partial_n B_z|_S = 0$ cannot generally be satisfied simultaneously. Consequently the fields divide themselves into two distinct categories:

Transverse Magnetic (TM): $B_z = 0$ everywhere; $E_z|_S = 0$.

Transverse Electric (TE): $E_z = 0$ everywhere; $\partial_n B_z|_S = 0$.

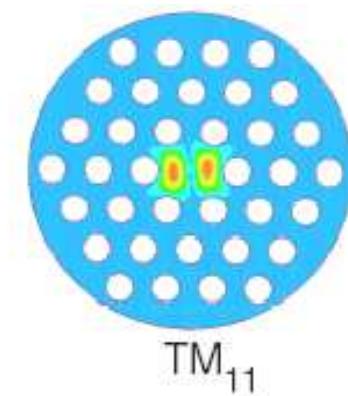
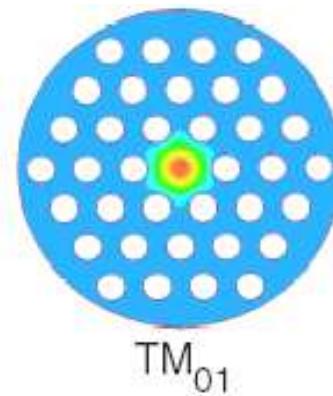
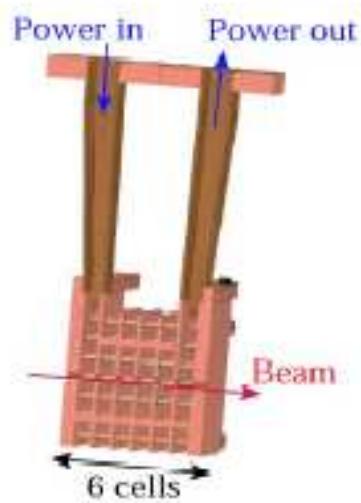
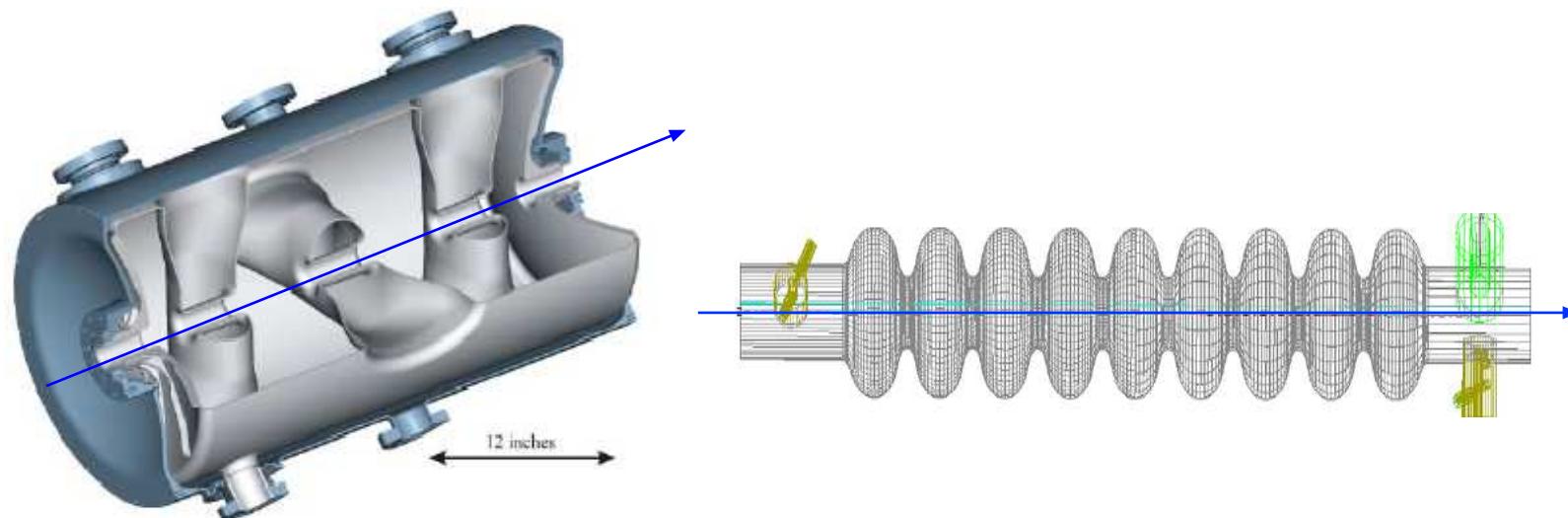
Consider the case of a resonant cavity with end plates located at $z = 0, L$ in cylindrical coordinate (r, ϕ, z) .

$\vec{E}_t = 0$ at end plates $\Rightarrow \vec{E}_t \propto \sin(kz)$ with $k = \frac{p\pi}{L}$ ($p \in \mathbb{N}$).

Let $\gamma^2 \equiv \mu\epsilon\omega^2 - \left(\frac{p\pi}{L}\right)^2$, then the longitudinal field are found from:

$$(\nabla_t^2 + \gamma^2)\Psi(r, \phi) = 0 \quad (43)$$

where $\Psi = E_z$ (TM) or B_z (TE)



example of TM-mode cavities used in particle accelerators.

The transverse field are then given by:

$$\text{TM: } \begin{bmatrix} \vec{E}_t(r, \phi) \\ \vec{B}_t(r, \phi) \end{bmatrix} = \frac{i}{\gamma^2} \begin{bmatrix} \pm k \vec{\nabla}_t \\ \mu \epsilon \omega \hat{z} \times \vec{\nabla}_t \end{bmatrix} \Psi(r, \phi), \quad (44)$$

$$\text{TE: } \begin{bmatrix} \vec{E}_t(r, \phi) \\ \vec{B}_t(r, \phi) \end{bmatrix} = \frac{i}{\gamma^2} \begin{bmatrix} -\omega \hat{z} \times \vec{\nabla}_t \\ \pm k \vec{\nabla}_t \end{bmatrix} \Psi(r, \phi), \quad (45)$$

Cylindrically symmetric solution of wave equation

$$\left[\frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_\phi^2 + \gamma^2 \right] \Psi(r, \phi) = 0; \quad \Psi(r, \phi) = \Psi(r) e^{\pm im\phi}; \quad (m \in \mathbb{N}).$$

$$(r^2 d_r^2 + r dr + r^2 \gamma^2 - m^2) \Psi(r) = 0 \quad \text{Bessel's Eq.} \quad (46)$$

solution of the form $\Psi(r) = AJ_m(\gamma_{mn}r) + BN_m(\gamma_{mn}r)$.

$B = 0$ because $\lim_{r \rightarrow 0} N_m = -\infty$ which is unphysical.

So finally:

$$\Psi(r, \phi) = AJ_m(\gamma_{mn}r)e^{\pm im\phi}; \gamma_{mn} \equiv \frac{x_{mn}}{R} \quad (47)$$

where x_{mn} is the n^{th} root of $J_m(x) = 0$, and R cavity radius. γ_{mn} is defined to insure $\Psi \rightarrow 0$ as $r \rightarrow R$.

Resonant frequencies:

$$\gamma_{mn}^2 = \mu\epsilon\omega_{mn}^2 - \left(\frac{p\pi}{L}\right)^2 = \left(\frac{x_{mn}}{R}\right)^2 \quad (48)$$

$$\omega_{mn} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2} \quad (49)$$

resonant frequency of the “ mnp ” mode.

TM-mode fields

$$E_z(r, \phi, z, t) = E_0 J_m(\gamma_{mn} r) e^{\pm i(kz + m\phi) - i\omega_{mn} t}; \quad B_z = 0. \quad (50)$$

$$\begin{aligned} \vec{E}_t(r, \phi) &= \pm \frac{ikE_0}{\gamma^2} \left(\partial_r \hat{r} + \frac{1}{r} \partial_\phi \hat{\phi} \right) J_m(\gamma r) e^{\pm im\phi}, \\ &= \pm i \frac{kE_0}{\gamma^2} e^{\pm im\phi} \left(\gamma_{mn} J'_m \hat{r} \pm \frac{im}{r} J_m \hat{\phi} \right). \end{aligned} \quad (51)$$

Insert $J'_m = \frac{m}{\gamma_{mn} r} J_m - J_{m+1}$:

$$\begin{aligned} \vec{E}_t(r, \phi, z, t) &= \frac{\pm ikE_0}{\gamma^2} \left\{ \left[\frac{m}{r} J_m(\gamma_{mn} r) - \gamma_{mn} J_{m+1}(\gamma_{mn} r) \right] \hat{r} \right. \\ &\quad \left. \pm \frac{im}{r} J_m(\gamma_{mn} r) \hat{\phi} \right\} e^{\pm im\phi} e^{\pm ikz} e^{-i\omega_{mn} t}. \end{aligned} \quad (52)$$

Note that:

- $\vec{E}_t = 0$ at $z = 0, L \Rightarrow \pm ie^{\pm ikz} \rightarrow -\sin(kz)$
- “time” is arbitrary: $e^{-i\omega t} \rightarrow \cos(\omega t)$
- ϕ is also arbitrary $\Rightarrow e^{\pm im\phi} \rightarrow \cos(m\phi)$ and $\pm ie^{\pm im\phi} \rightarrow -\sin(m\phi)$.

\vec{E}_t can be re-written in a real form:

$$\vec{E}_t = -\frac{kE_0 \sin(kz)}{\gamma_{mn}^2} \left\{ \left[\frac{m}{r} J_m(\gamma_{mn} r) - \gamma_{mn} J_{m+1}(\gamma_{mn} r) \right] \cos(m\phi) \hat{r} - \frac{m}{r} J_m(\gamma_{mn} r) \sin(m\phi) \hat{\phi} \right\} \cos(\omega_{mnt}). \quad (53)$$

where $E_0 \in \mathbb{R}$.

likewise,

$$E_z(r, \phi, z, t) = E_0 J_m(\gamma_{mn} r) \cos(m\phi) \cos(kz) \cos(\omega_{mnt}) \quad (54)$$

$$\begin{aligned} \vec{B}_t(r, \phi) &= \frac{i\mu\epsilon\omega}{\gamma^2} \hat{z} \times \left(\partial_r \hat{r} + \frac{1}{r} \partial_\phi \hat{\phi} \right) \Psi(r, \phi) \\ &= \frac{i\mu\epsilon\omega}{\gamma^2} \left(-\frac{1}{r} \partial_\phi \hat{r} + \partial_r \hat{\phi} \right) \Psi(r, \phi) \\ &= \frac{i\mu\epsilon\omega E_0}{\gamma^2} e^{\pm im\phi} \left[\mp \frac{im}{r} J_m \hat{r} + \left(\frac{m}{r} J_m - \gamma J_{m+1} \right) \hat{\phi} \right]. \end{aligned}$$

- $ie^{-i\omega t} \rightarrow \sin(\omega t)$ \vec{B} and \vec{E} are 90° out of phase;
- $e^{\pm im\phi} \rightarrow \cos(m\phi)$, $\mp ie^{\pm im\phi} \rightarrow \sin(m\phi)$, and $e^{ikz} \rightarrow \cos(kz)$.

$$\begin{aligned} B_z = 0; \vec{B}_t(r, \phi, z, t) &= \frac{\mu\epsilon\omega_{mn} E_0 \cos(kz)}{\gamma_{mn}^2} \left\{ \frac{m}{r} J_m(\gamma_{mn} r) \sin(m\phi) \hat{r} \right. \\ &\quad \left. + \left[\frac{m}{r} J_m(\gamma_{mn} r) - \gamma_{mn} J_{m+1}(\gamma_{mn} r) \right] \cos(m\phi) \hat{\phi} \right\} \sin(\omega_{mnt}) \quad (55) \end{aligned}$$

Shorthand notations:

$$k_p \equiv \frac{p\pi}{L}, J_{mn}(r) \equiv J_m(\gamma_{mn}r), \tilde{J}_{mn}(r) \equiv \frac{m}{r}J_m(\gamma_{mn}r) - \gamma_{mn}J_{m+1}(\gamma_{mn}r),$$

$$\begin{bmatrix} s_p(z) \\ c_p(z) \end{bmatrix} = \begin{bmatrix} \sin(k_p z) \\ \cos(k_p z) \end{bmatrix}, \begin{bmatrix} s_m(\phi) \\ c_m(\phi) \end{bmatrix} = \begin{bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{bmatrix}, \begin{bmatrix} s_{mnp}(t) \\ c_{mnp}(t) \end{bmatrix} = \begin{bmatrix} \sin(\omega_{mnp}t) \\ \cos(\omega_{mnp}t) \end{bmatrix}$$

The TM-fields are:

$$E_r^{TM}(r, \phi, z, t) = -E_0 \frac{k_p}{\gamma_{mn}^2} \tilde{J}_{mn}(r) c_m(\phi) s_p(z) c_{mnp}(t), \quad (56)$$

$$E_\phi^{TM}(r, \phi, z, t) = E_0 \frac{k_p}{\gamma_{mn}^2} \frac{m}{r} J_{mn}(r) s_m(\phi) s_p(z) c_{mnp}(t), \quad (57)$$

$$E_z^{TM}(r, \phi, z, t) = E_0 J_{mn}(r) c_m(\phi) c_p(z) c_{mnp}(t), \quad (58)$$

$$B_r^{TM}(r, \phi, z, t) = E_0 \frac{\mu \epsilon \omega_{mn}}{\gamma_{mn}^2} \frac{m}{r} J_{mn}(r) s_m(\phi) c_p(z) s_{mnp}(t), \quad (59)$$

$$B_\phi^{TM}(r, \phi, z, t) = E_0 \frac{\mu \epsilon \omega_{mn}}{\gamma_{mn}^2} \tilde{J}_{mn}(r) c_m(\phi) c_p(z) s_{mnp}(t), \quad (60)$$

$$B_z^{TM}(r, \phi, z, t) = 0. \quad (61)$$

The TE-fields now follow at once by inspection:

$$E_r^{TE}(r, \phi, z, t) = -B_0 \frac{\omega_{mn} m}{\gamma_{mn}^2 r} J_{mn}(r) s_m(\phi) s_p(z) c_{mnp}(t), \quad (62)$$

$$E_\phi^{TE}(r, \phi, z, t) = -B_0 \frac{\omega_{mn}}{\gamma_{mn}^2} \tilde{J}_{mn}(r) c_m(\phi) s_p(z) c_{mnp}(t), \quad (63)$$

$$E_z^{TE}(r, \phi, z, t) = 0, \quad (64)$$

$$B_r^{TE}(r, \phi, z, t) = -B_0 \frac{k_p}{\gamma_{mn}^2} \tilde{J}_{mn}(r) c_m(\phi) c_p(z) s_{mnp}(t), \quad (65)$$

$$B_\phi^{TM}(r, \phi, z, t) = B_0 \frac{k_p m}{\gamma_{mn}^2 r} J_{mn}(r) s_m(\phi) c_p(z) s_{mnp}(t), \quad (66)$$

$$B_z^{TE}(r, \phi, z, t) = -B_0 J_{mn}(r) c_m(\phi) s_p(z) s_{mnp}(t), \quad (67)$$

with $\partial_r B_z(r = R) = 0 \Rightarrow \gamma_{mn} = x'_{mn}/R$; x'_{mn} root of $J_{mn}(x) = 0$.

Summary of TM_{mnp} mode for a cylinder of length L and radius R :

$$\omega_{mnp}^{TM} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}; \begin{cases} m = 0, 1, 2, \dots \\ n = 1, 2, 3, \dots \\ p = 0, 1, 2, \dots \end{cases} \rightarrow \begin{cases} x_{0n} = 2.405, 5.520, 8.564, \dots \\ x_{1n} = 3.832, 7.016, 10.714, \dots \\ x_{2n} = 5.136, 8.417, 11.620, \dots \end{cases} .$$

$$\begin{aligned}
 E_r(r, \phi, z, t) &= -E_0 \left(\frac{p\pi}{x_{mn}} \frac{R}{L} \right) \left[\left(\frac{m}{x_{mn}} \frac{R}{r} \right) J_m \left(x_{mn} \frac{r}{R} \right) - J_{m+1} \left(x_{mn} \frac{r}{R} \right) \right] \cos(m\phi) \sin \left(p\pi \frac{z}{L} \right) \cos(\omega_{mnp}^{TM} t) \\
 E_\phi(r, \phi, z, t) &= E_0 \left(\frac{p\pi}{x_{mn}} \frac{R}{L} \right) \left(\frac{m}{x_{mn}} \frac{R}{r} \right) J_m \left(x_{mn} \frac{r}{R} \right) \sin(m\phi) \sin \left(p\pi \frac{z}{L} \right) \cos(\omega_{mnp}^{TM} t) \\
 E_z(r, \phi, z, t) &= E_0 J_m \left(x_{mn} \frac{r}{R} \right) \cos(m\phi) \cos \left(p\pi \frac{z}{L} \right) \cos(\omega_{mnp}^{TM} t) \\
 B_r(r, \phi, z, t) &= E_0 \sqrt{\mu\epsilon} \left(\frac{R}{x_{mn}} \sqrt{\mu\epsilon} \omega_{mnp}^{TM} \right) \left(\frac{m}{x_{mn}} \frac{R}{r} \right) J_m \left(x_{mn} \frac{r}{R} \right) \sin(m\phi) \cos \left(p\pi \frac{z}{L} \right) \sin(\omega_{mnp}^{TM} t) \\
 B_\phi(r, \phi, z, t) &= E_0 \sqrt{\mu\epsilon} \left(\frac{R}{x_{mn}} \sqrt{\mu\epsilon} \omega_{mnp}^{TM} \right) \left[\left(\frac{m}{x_{mn}} \frac{R}{r} \right) J_m \left(x_{mn} \frac{r}{R} \right) - J_{m+1} \left(x_{mn} \frac{r}{R} \right) \right] \cos(m\phi) \cos \left(p\pi \frac{z}{L} \right) \sin(\omega_{mnp}^{TM} t) \\
 B_z(r, \phi, z, t) &= 0
 \end{aligned}$$

Summary of TE_{mnp} mode for a cylinder of length L and radius R :

$$\omega_{mnp}^{TE} = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{x'_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}; \begin{cases} m = 0, 1, 2, \dots \\ n = 1, 2, 3, \dots \\ p = 1, 2, 3, \dots \end{cases} \rightarrow \begin{cases} x'_{0n} = 3.832, 7.016, 10.714, \dots \\ x'_{1n} = 1.841, 5.331, 8.536, \dots \\ x'_{2n} = 3.054, 6.706, 9.970, \dots \end{cases} .$$

$$\begin{aligned}
 E_r(r, \phi, z, t) &= E_0 \left(\frac{R}{x'_{mn}} \sqrt{\mu\epsilon} \omega_{mnp}^{TE} \right) \left(\frac{m}{x'_{mn}} \frac{R}{r} \right) J_m \left(x'_{mn} \frac{r}{R} \right) \sin(m\phi) \sin \left(p\pi \frac{z}{L} \right) \cos(\omega_{mnp}^{TE} t) \\
 E_\phi(r, \phi, z, t) &= E_0 \left(\frac{R}{x'_{mn}} \sqrt{\mu\epsilon} \omega_{mnp}^{TE} \right) \left[\left(\frac{m}{x'_{mn}} \frac{R}{r} \right) J_m \left(x'_{mn} \frac{r}{R} \right) - J_{m+1} \left(x'_{mn} \frac{r}{R} \right) \right] \cos(m\phi) \sin \left(p\pi \frac{z}{L} \right) \cos(\omega_{mnp}^{TE} t) \\
 E_z(r, \phi, z, t) &= 0 \\
 B_r(r, \phi, z, t) &= E_0 \sqrt{\mu\epsilon} \left(\frac{p\pi}{x'_{mn}} \frac{R}{L} \right) \left[\left(\frac{m}{x'_{mn}} \frac{R}{r} \right) J_m \left(x'_{mn} \frac{r}{R} \right) - J_{m+1} \left(x'_{mn} \frac{r}{R} \right) \right] \cos(m\phi) \cos \left(p\pi \frac{z}{L} \right) \sin(\omega_{mnp}^{TE} t) \\
 B_\phi(r, \phi, z, t) &= -E_0 \sqrt{\mu\epsilon} \left(\frac{p\pi}{x'_{mn}} \frac{R}{L} \right) \left[\left(\frac{m}{x'_{mn}} \frac{R}{r} \right) J_m \left(x'_{mn} \frac{r}{R} \right) \sin(m\phi) \cos \left(p\pi \frac{z}{L} \right) \sin(\omega_{mnp}^{TE} t) \right. \\
 B_z(r, \phi, z, t) &= E_0 \sqrt{\mu\epsilon} J_m \left(x'_{mn} \frac{r}{R} \right) \cos(m\phi) \sin \left(p\pi \frac{z}{L} \right) \sin(\omega_{mnp}^{TE} t)
 \end{aligned}$$

Comment on choice of modes and R/L :

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}} \frac{1}{R} \sqrt{(x'_{mn})^2 + (p\pi)^2 (R/L)^2}.$$

Want to choose a frequency such that it is well separated from other resonant frequencies, and such that it is sensitive to both R and L to enable easy tuning. Thus,

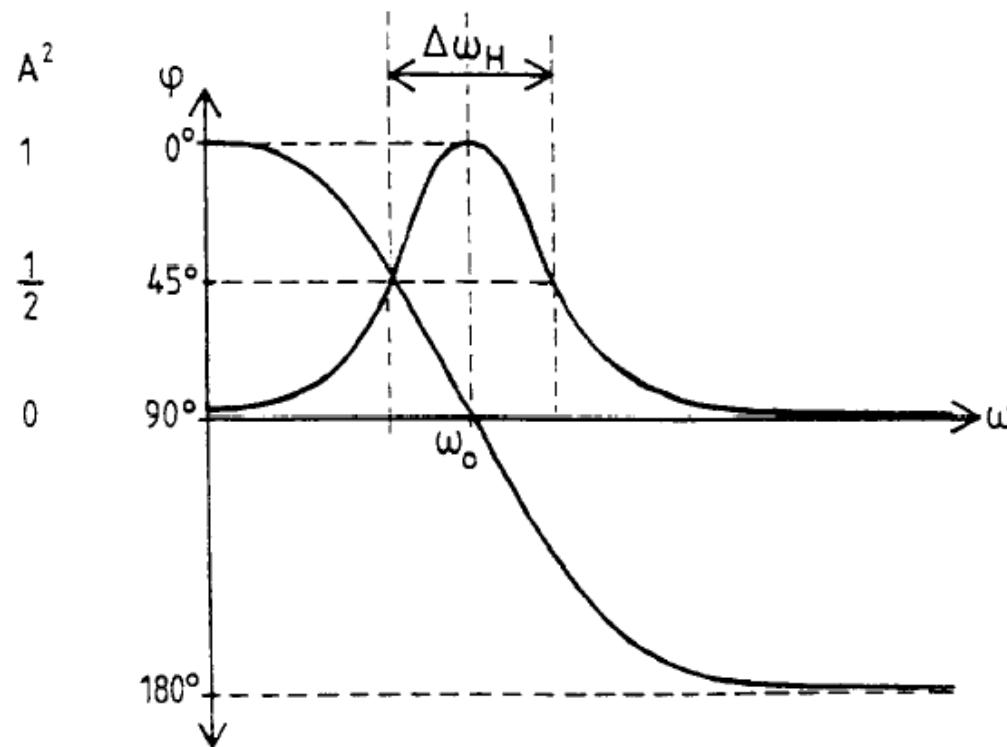
- not interested in large p or R : $\lim_{R \rightarrow \infty, p \rightarrow \infty} \omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}} \frac{p\pi}{L} \Rightarrow R$ -independent.
 - not interested in large m or n (and/or large L): $\omega_{mnp} \rightarrow \frac{1}{\sqrt{\mu\epsilon}} \frac{x_{mn}}{R}$ for large m or n and $R/L \sim 1$ L -independent.
- \Rightarrow The most interesting modes in practical applications should be low order modes with $R/L \sim 1$.

Q-factor:

$$Q \equiv \frac{\omega U}{P} \quad (68)$$

$$(69)$$

U : stored energy, P dissipated power



Note: having cast things into "consistent units", we have in effect inserted into the wave equation:

$$\text{TM: } E_z(r, \phi) = E_0\psi(r, \phi), \quad B_z(r, \phi) = 0;$$

$$\text{TE: } B_z(r, \phi) = -E_0\sqrt{\mu\epsilon}\psi(r, \phi), \quad E_z(r, \phi) = 0;$$

(note: $\Psi = E_0\psi$) so that for both TM and TE-modes:

$$E_z^2(r, \phi) + \frac{1}{\mu\epsilon}B_z^2(r, \phi) = E_0^2\psi^2. \quad (70)$$

And from Eq. 44 and 45:

$$\text{TM: } \begin{bmatrix} \vec{E}_t(r, \phi) \\ \vec{B}_t(r, \phi) \end{bmatrix} = \frac{iE_0}{\gamma^2} \begin{bmatrix} \pm k\vec{\nabla}_t \\ \mu\epsilon\omega\hat{z} \times \vec{\nabla}_t \end{bmatrix} \psi(r, \phi);$$

$$\text{TE: } \begin{bmatrix} \vec{E}_t(r, \phi) \\ \vec{B}_t(r, \phi) \end{bmatrix} = \frac{i\sqrt{\mu\epsilon}E_0}{\gamma^2} \begin{bmatrix} -\omega\hat{z} \times \vec{\nabla}_t \\ \pm k\vec{\nabla}_t \end{bmatrix} \psi(r, \phi);$$

So that, for both TM and TE-modes:

$$E_t^2 + \frac{1}{\mu\epsilon}B_t^2 = \frac{E_0^2}{\gamma^4}(k^2 + \mu\epsilon\omega^2)(\nabla_t\psi)^2. \quad (71)$$

Stored energy in cavity:

$$U = \frac{1}{2} \int_V d\vec{x}^3 (ED + BH) = \frac{\epsilon}{2} \int_V d\vec{x}^3 \left(E^2 + \frac{B^2}{\mu\epsilon} \right) \quad (72)$$

we have $\int dz \sin^2(kz) = L/2$ and $\int dz \cos^2(kz) = L/2(1 + \delta_{0p})$. Time averaging of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ gives a factor 1/2. We also note:

$$\begin{aligned} \frac{k^2 + \mu\epsilon\omega^2}{\gamma^4} &= \frac{2k^2 + (x/R)^2}{(x/R)^4} = \left(\frac{R}{x}\right)^2 \left[1 + 2\left(\frac{kR}{x}\right)^2 \right] \\ &= \left(\frac{R}{x}\right)^2 [1 + 2\xi^2] \text{ where } \xi \equiv \frac{p\pi R}{xL}. \end{aligned} \quad (73)$$

Nota: x denotes either x_{mn} or x'_{mn} .

the energy stored in the cavity is thus

$$U = \frac{1 + \delta_{0p}}{8} \epsilon E_0^2 L \int_A dA \left\{ \left(\frac{R}{x} \right)^2 [1 + 2\xi^2] (\nabla_t \psi)^2 + \psi^2 \right\}, \quad (74)$$

for TE mode $p \neq 0 \Rightarrow \delta_{0p} = 0$.

$$\begin{aligned} \int_A dA (\nabla_t \psi)^2 &= \int_A dA \vec{\nabla}_t \cdot (\psi \vec{\nabla}_t \psi) - \int_A \psi \nabla_t^2 \psi \\ &= \oint_C dl \psi \hat{n} \cdot \vec{\nabla}_t \psi - \int_A dA \psi \nabla_t^2 \psi. \end{aligned} \quad (75)$$

but,

$$\oint_C dl \psi \hat{n} \cdot \vec{\nabla}_t \psi = 0 \left\{ \begin{array}{l} \psi^{TM}(r = R) = 0 \\ \partial_r \psi^{TE}(r = R) = 0 \end{array} \right. \quad (76)$$

Also $\nabla_t^2 \psi = -(k^2 - \mu \epsilon \omega^2) \psi = (x/R)^2 \psi$ (wave eq.)

So

$$\int_A dA (\nabla_t)^2 = \left(\frac{x}{R}\right)^2 \int_A dA \psi^2, \quad (77)$$

and the stored energy is

$$U = \frac{\epsilon L}{4} E_0^2 (1 + \delta_{0p}) [1 + \xi^2] \int_A dA \psi^2 \quad (78)$$

this is JDJ eq, (8.92). Considering the azimuthal dependence $\psi^2 \propto \cos^2(m\phi)$; one has $\int_0^{2\pi} d\phi \psi^2 \rightarrow \pi(1 + \delta_{0m}) J^2$ so that

$$U = \frac{\pi}{4} \epsilon L E_0^2 (1 + \delta_{0m})(1 + \delta_{0p}) [1 + \xi^2] \int_0^R dr r J_m^2 \left(x \frac{r}{R}\right). \quad (79)$$

Case of TM-modes *:

$$\int_0^R dr r J_m^2 \left(\frac{xr}{R}\right) = \frac{1}{2} R^2 J_{m+1}^2(x_{mn}) \quad (80)$$

*the identity $\int_0^1 dx x J_\nu^2(\alpha x) = \frac{1}{2} J_{\nu+1}^2$ if $J_\nu(\alpha) = 0$ was used

Let $V \equiv \pi R^2 L$; the stored energy associated to TM-mode is

$$U_{mnp}^{TM} = \frac{1}{8} V \epsilon E_0^2 (1 + \delta_{0m})(1 + \delta_{0p}) [1 + \xi^2] J_{m+1}^2(x_{mn}). \quad (81)$$

Case of TE-modes:

use the identity:

$$\begin{aligned} \int_0^x d\rho \rho J_m^2(\rho) &= \frac{1}{2} x^2 [J_m^2(x) + J_{m-1}^2(x)] - mx J_m(x) J_{m-1}(x) \\ &\Rightarrow \int_0^R dr r J_m^2\left(r \frac{x}{R}\right) = \frac{R^2}{x^2} \left\{ \frac{1}{2} x^2 [...] - m [...] \right\} \\ &= \frac{1}{2} R^2 [J_m^2(x'_{mn}) + J_{m-1}^2(x'_{mn})] - \frac{m}{x'_{mn}} R^2 J_m(x'_{mn}) J_{m-1}(x'_{mn}) \end{aligned} \quad (82)$$

User recursion relation to get $\frac{m}{x'} J_m(x') = J_{m+1}(x') = J_{m-1}(x')$ *

*Arfken pg. 631 $J_{n-1}(x) = \frac{n}{x} J_n(x) + J'_n(x)$ and $J_{n+1} = \frac{n}{x} J_n(x) - J'_n(x)$.

so finally,

$$\begin{aligned} \int_0^R dr r J_m^2\left(x \frac{r}{R}\right) &= \frac{1}{2} R^2 \left[1 + \left(\frac{m}{x'}\right)^2\right] J_m^2(x') - R^2 \left(\frac{m}{x'}\right)^2 J_m^2(x') \\ &= \frac{1}{2} R^2 \left[1 - \left(\frac{m}{x'}\right)^2\right] J_m^2(x'). \end{aligned} \quad (83)$$

Thus, with $V \equiv \pi R^2 L$ and $\delta_{0p} = 0$ ($p \neq 0$) for TE-modes,

$$U_{mnp}^{TE} = \frac{1}{8} V \epsilon E_0^2 (1 + \delta_{0m}) \left[1 - \left(\frac{m}{x'}\right)^2\right] [1 + \xi'^2] J_m^2(x'_{mn}). \quad (84)$$

with $\xi' = \frac{p\pi R}{x'_{mn} L}$.

Power dissipated in a cavity:

$$\frac{dP}{dA} = \frac{1}{2} R_s H_{\parallel}^2 = \frac{R_s}{2\mu^2} B_{\parallel}^2 \quad (85)$$

with R_s = surface resistance and \parallel means component \parallel to cavity walls. The dissipated power is

$$P = \frac{R_s}{2\mu^2} \left[\int_{side} dAB_{\parallel}^2 + 2 \int_{end} dAB_{\parallel}^2 \right]. \quad (86)$$

Case of TM-modes:

$$\begin{aligned} \text{side: } & \int_{side} dAB_{\parallel}^2 = \int_0^L dz \int_0^{2\pi} d\phi R B_{\phi}^2(r, \phi, z) \\ &= \frac{L}{2} (1 + \delta_{0p}) \pi (1 + \delta_{0m}) \mu \epsilon E_0^2 [1 + \xi^2] R \left[\frac{m}{x} J_m(x) - J_{m+1}(x) \right]^2 \end{aligned}$$

but $J_m(x) = 0$ so

$$\int_{side} dAB_{\parallel}^2 = \frac{1}{2} \mu \epsilon E_0^2 (\pi L R) (1 + \delta_{0p}) (1 + \delta_{0m}) [1 + \xi^2] J_{m+1}^2(x) \quad (87)$$

$$\begin{aligned}
\text{end: } & \int_{end} dAB_{\parallel}^2 = \int_0^{2\pi} d\phi \int_0^R dr r B_t^2 \\
&= \frac{E_0^2 \mu^2 \epsilon^2 \omega^2}{(x/R)^4} \int_A dA (\nabla \psi)^2 = \frac{E_0^2 \mu^2 \epsilon^2 \omega^2}{(x/R)^2} \int_A dA \psi^2 \\
&= \mu \epsilon E_0^2 [1 + \xi^2] \frac{\pi}{2} (1 + \delta_{0m}) R^2 J_{m+1}^2(x) \tag{88}
\end{aligned}$$

$$\text{end: } \int_{end} dAB_{\parallel}^2 = \frac{1}{2} \mu \epsilon E_0^2 (\pi R^2) (1 + \delta_{0m}) [1 + \xi^2] J_{m+1}^2(x_{mn}), \tag{89}$$

where (again) $\xi \equiv \frac{\pi p R}{x_{mn} L}$.

So the total power loss is:

$$P_{mnp}^{TM} = \frac{R_s}{2\mu^2} \mu \epsilon E_0^2 \frac{\pi}{2} R (1 + \delta_{0m}) [1 + \xi^2] [L(1 + \delta_{0p}) + 2R] J_{m+1}^2(x_{mn}),$$

or, with $A_s \equiv 2\pi RL$,

$$P_{mnp}^{TM} = \frac{R_s}{2\mu^2} \mu \epsilon E_0^2 \frac{1}{4} A_s (1 + \delta_{0m}) [1 + \xi^2] \left[1 + \delta_{0p} + 2\frac{R}{L} \right] J_{m+1}^2(x_{mn}) \quad (90)$$

The quality factor is $Q = \frac{\omega U}{P}$ that is:

$$\begin{aligned} Q &= \frac{1}{\sqrt{\mu \epsilon}} \frac{x}{R} \sqrt{1 + \xi^2} \times \\ &\frac{\frac{1}{8} V \epsilon E_0^2 (1 + \delta_{0m})(1 + \delta_{0p})(1 + \xi^2) J_{m+1}^2}{\frac{R_s}{2\mu^2} \mu \epsilon E_0^2 \frac{1}{4} A_s (1 + \delta_{0m})(1 + \delta_{0p} + 2\frac{R}{L})(1 + \xi^2) J_{m+1}^2} \end{aligned} \quad (91)$$

after simplification and using $V/A_s = R/2$,

$$QR_s = \sqrt{\frac{\mu}{\epsilon}}(1 + \delta_{0p}) \frac{x}{R} \frac{R}{2} \frac{\sqrt{1 + \xi^2}}{1 + \delta_{0p} + 2\frac{R}{L}}. \quad (92)$$

Re-arranging

$$Q_{mnp}^{TM} R_s = \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} (1 + \delta_{0p}) \frac{\sqrt{x_{mn}^2 + \left(\frac{p\pi R}{L}\right)^2}}{1 + \delta_{0p} + 2\frac{R}{L}}. \quad (93)$$

TE-mode:

$$\begin{aligned} \text{side: } & \int_{side} dA B_{\parallel}^2 = \int_0^L dz \int_0^{2\pi} d\phi R \left[B_{\phi}^2(R, \phi, z) + B_z^2(R, \phi, z) \right] \\ &= \mu \epsilon E_0^2 \frac{L}{2} \pi (1 + \delta_{0m}) R \left\{ \left(\frac{m}{x'} \right)^2 J_m^2(x') \xi'^2 + J_m^2(x') \right\} \end{aligned} \quad (94)$$

$$\int_{side} dA B_{\parallel}^2 = \mu\epsilon E_0^2 \frac{\pi}{2} LR (1 + \delta_{0m}) \left[1 + \left(\frac{Rmp\pi}{x'^2 L} \right)^2 \right] J_m^2(x') \quad (95)$$

end:

$$\begin{aligned} \int_{end} dA B_{\parallel}^2 &= \int_0^{2\pi} d\phi \int_0^R dr r B_t^2(r, \phi, 0) \\ &= \mu\epsilon E_0^2 \xi'^2 \left(\frac{R}{x'} \right)^2 \int_A dA (\vec{\nabla} \psi)^2 \\ &= \mu\epsilon E_0^2 \xi^2 \int_A dA \psi^2. \end{aligned} \quad (96)$$

$$\begin{aligned} \int_A dA \psi^2 &= \pi (1 + \delta_{0m}) \int_0^R dr r J_m^2 \left(x' \frac{r}{R} \right) \\ &= \frac{\pi}{2} R^2 (1 + \delta_{0m}) \left[1 - \left(\frac{m}{x'} \right)^2 \right] J_m^2(x') \end{aligned} \quad (97)$$

$$\Rightarrow \int_A dA \psi^2 = \mu \epsilon E_0^2 \frac{\pi}{2} R^2 (1 + \delta_{0m}) \xi'^2 \left[1 + \left(\frac{m}{x'_{mn}} \right)^2 \right] J_m^2(x'_{mn}) \quad (98)$$

Thus,

$$P_{mnp}^{TE} = \frac{R_s}{2\mu^2} \mu \epsilon E_0^2 \frac{\pi}{2} R (1 + \delta_{0m}) J_m^2(x'_{mn}) \left\{ L \left[1 + \left(\frac{m}{x'} \right)^2 \xi'^2 \right] + 2R \left[1 - \left(\frac{m}{x'} \right)^2 \right] \xi'^2 \right\}, \quad (99)$$

or,

$$P_{mnp}^{TE} = \frac{R_s}{2\mu^2} \mu \epsilon E_0^2 \frac{1}{4} A_s (1 + \delta_{0m}) \left\{ 1 + \left(\frac{m}{x'} \right)^2 \xi'^2 + 2 \frac{R}{L} \left[1 - \left(\frac{m}{x'} \right)^2 \right] \xi'^2 \right\} J_m^2(x'). \quad (100)$$

and finally,

$$P_{mnp}^{TE} = \frac{R_s}{2\mu^2} \mu \epsilon E_0^2 \frac{1}{4} A_s (1 + \delta_{0m}) \left\{ 1 + \left[2 \frac{R}{L} + (1 - 2 \frac{R}{L}) \left(\frac{m}{x'} \right)^2 \right] \xi'^2 \right\} J_m^2(x'_{mn}). \quad (101)$$

The quality factor is then

$$\begin{aligned} Q &= \frac{x'}{R} \frac{1}{\sqrt{\mu \epsilon}} \sqrt{1 + \xi'^2} \times \\ &\quad \frac{\frac{1}{8} V \epsilon E_0^2 (1 + \delta_{0m}) [1 - (m/x')^2] (1 + \xi'^2) J_m^2}{\frac{R_s}{2\mu^2} \mu \epsilon E_0^2 \frac{1}{4} A_s (1 + \delta_{0m}) \{1 + [2R/L + (1 - 2R/L)(m/x')^2]\xi^2\} J_m^2} \\ &= \frac{1}{2} \sqrt{\frac{\mu}{\epsilon}} \frac{1}{R_s} \underbrace{\frac{x' [1 - (m/x')^2] (1 + \xi'^2)^{3/2}}{1 + [2R/L + 2(1 - 2R/L)(m/x')^2]\xi'^2}}_q. \end{aligned} \quad (102)$$

$$\begin{aligned}
q &= \frac{x'[1 - (m/x')^2](1 + \xi'^2)^{3/2}}{1 + [2R/L + 2(1 - 2R/L)(m/x')^2]\xi'^2} \\
&= \frac{x'[1 - (m/x')^2](1 + \xi'^2)^{3/2}}{1 + 2[1 - (m/x')^2](R/L)\xi'^2 + (m/x')^2\xi'^2} \\
&= \frac{x'[\dots](1 + \xi'^{-2})^{3/2}}{2[\dots]R/L\xi'^{-1} + (\dots)^2\xi'^{-1} + \xi'^{-3}} \\
&= \frac{x'[\dots](1 + \xi'^{-2})^{3/2}}{\frac{x'}{p\pi}\frac{L}{R}\{2[\dots](R/L) + (m/x')^2 + \xi'^{-2}\}}
\end{aligned} \tag{103}$$

Finally one has:

$$Q_{mnp}^{TE} R_s = \frac{p\pi}{2} \sqrt{\frac{\mu}{\epsilon}} \frac{\left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] \left[1 + \left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]^{3/2}}{2 \left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] + \left(\frac{m}{x'_{mn}}\right)^2 \frac{L}{R} + \left(\frac{x'_{mn}}{p\pi}\right)^2 \left(\frac{L}{R}\right)^3} \tag{104}$$

Comment on JDJ's geometry factor for TE_{mnp} mode – Eq. (8.96)

For the power dissipated on the wall we use:

$$\frac{dP}{dA} = \frac{1}{2} R_s H_{\parallel}^2, \quad (105)$$

where the factor $1/2$ comes from time-averaging, and R_s , the surface impedance is defined for normal and super-conductor. JDJ uses:

$$\frac{dP}{dA} = \frac{\mu_c \omega \delta}{4} H_{\parallel}^2 \quad (106)$$

the factor $\frac{\mu_c \omega \delta}{4}$ is valid for normal conductor only. Compare to infer:

$$R_s \Leftrightarrow \frac{\mu_c \omega \delta}{2} \quad (107)$$

JDJ also introduced the geometrical factor G so that:

$$Q = \frac{\mu}{\mu_c} \frac{V}{S \delta} G \quad (108)$$

This means

$$QR_s = \frac{\mu}{\mu_c S \delta} G \frac{\mu_c \omega \delta}{2} = \frac{\mu V}{2 S} \omega G \quad (109)$$

Let $\Gamma \equiv QR_s$ (Γ has units of Ω). One has:

$$G = \frac{2 S \Gamma}{\mu V \omega}. \quad (110)$$

Substitute the earlier results for TE-mode:

$$\begin{aligned} G &= \frac{22\pi RL + 2\pi R^2}{\mu \pi R^2 L} \frac{\Gamma}{\frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{x'_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}} \\ &= \frac{4(1 + \frac{R}{L})}{R} \sqrt{\frac{\epsilon}{\mu p\pi}} \frac{\Gamma}{\sqrt{1 + \left(\frac{x'_{mn}L}{p\pi R}\right)^2}} \end{aligned} \quad (111)$$

$$\begin{aligned}
G &= \frac{4(1+R/L)}{R} \sqrt{\frac{\epsilon}{\mu p\pi}} \frac{1}{\sqrt{1 + \left(\frac{x' L}{p\pi R}\right)^2}} \\
&\quad \times \frac{p\pi}{2} \sqrt{\frac{\mu}{\epsilon}} \frac{\left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] \left[1 + \left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]^{3/2}}{2 \left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] + \left(\frac{m}{x'_{mn}}\right)^2 \frac{L}{R} + \left(\frac{x'_{mn}}{p\pi}\right)^2 \left(\frac{L}{R}\right)^3} \\
&= \left(1 + \frac{L}{R}\right) \frac{2 \left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] \left[1 + \left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]}{2 \left[1 - \left(\frac{m}{x'_{mn}}\right)^2\right] + \left(\frac{m}{x'_{mn}}\right)^2 \frac{L}{R} + \left(\frac{x'_{mn}}{p\pi}\right)^2 \left(\frac{L}{R}\right)^3} \\
G &= \left(1 + \frac{L}{R}\right) \frac{\left[1 + \left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]}{1 + \frac{(m/x'_{mn})^2(L/R) + (x'_{mn}/p\pi)^2(L/R)^3}{2[1-(m/x'_{mn})^2]}} \tag{112}
\end{aligned}$$

Example: Consider TE_{111}

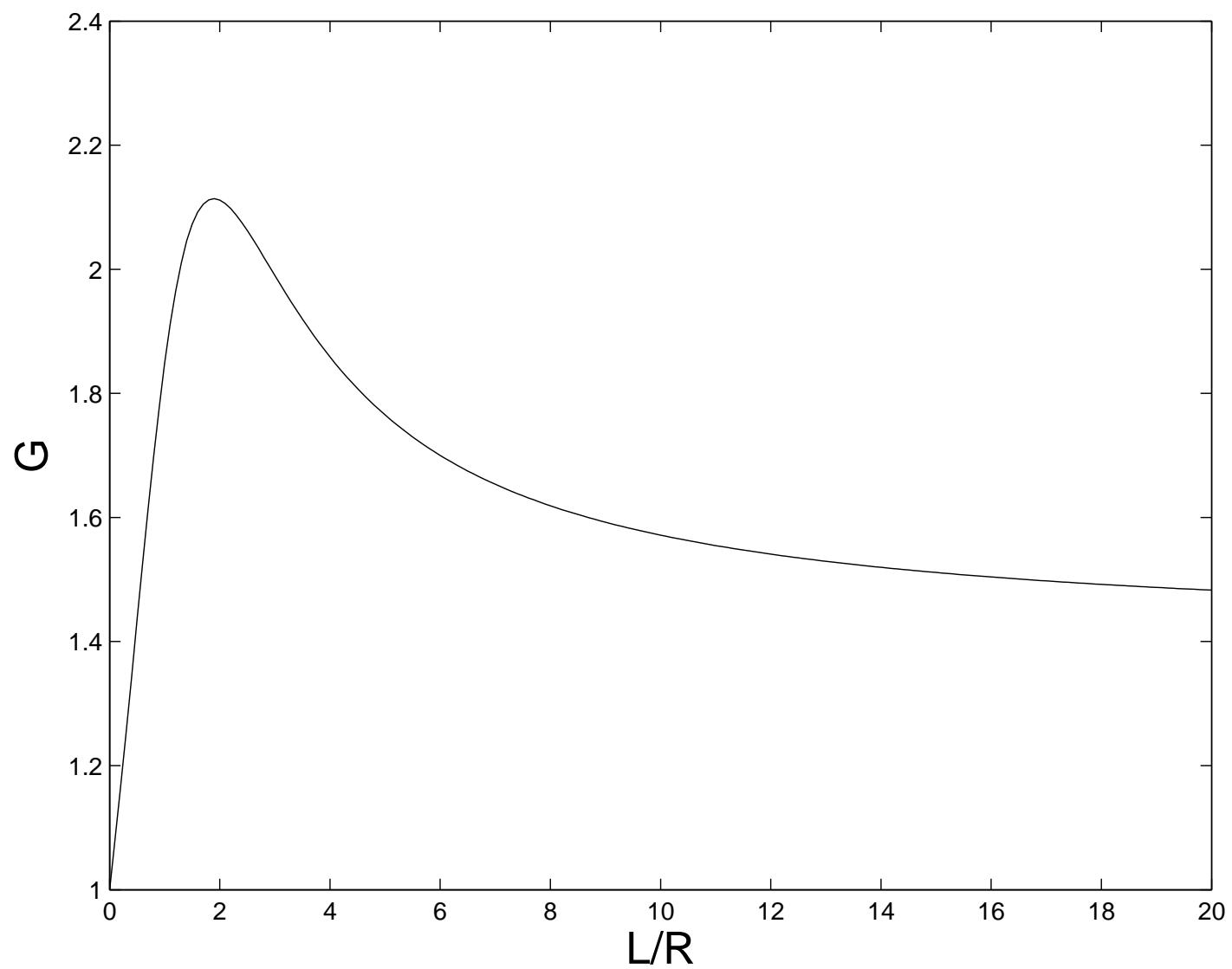
$$x'_{11} = 1.841; \left(\frac{x'_{11}}{\pi}\right)^2 = 0.343; \frac{(1/x'_{11})^2}{2[1-(1/x'_{11})^2]} = 0.209; \frac{(x'_{11}/\pi)^2}{2[1-(1/x'_{11})^2]} = 0.244.$$

$$\Rightarrow G = \left(1 + \frac{L}{R}\right) \frac{1 + 0.343(L/R)^2}{1 + 0.209(L/R) + 0.244(L/R)^3} \quad (113)$$

This is JDJ's Eq. (8.97).

The advantage of using G instead of Γ is that $G = \mathcal{O}(1)$ so that for the right circular cylinder:

$$\begin{aligned} QR_s &\sim \frac{\mu V}{2 S} \omega = \frac{\mu}{22\pi RL + 2\pi R^2} \frac{\pi R^2 L}{2\pi f} \\ &\Rightarrow \Gamma \sim \mu \frac{L}{1 + L/R} f \text{ (unit is } \Omega) \end{aligned} \quad (114)$$



G vs. L/R for the TE_{111} -mode.

Perturbation of cavity wall:

Consider a single resonant mode in a cavity. We perturb the cavity wall and estimate the associated change in resonant frequency. This relates to the cavity tuning and also to removing degeneracies between modes.

Consider a volume \mathcal{V} bounded by a surface \mathcal{S} , then the force associated to e.m. field in the volume is related to the Maxwell stress tensor (see JDJ Chapter 6) via:

$$\vec{F} = \int_{\mathcal{S}} \mathbb{T} \cdot d\vec{A} \quad (115)$$

with $d\vec{A} = dA\hat{n}$ and

$$\vec{\mathbb{T}} = \epsilon \left(\vec{E}\vec{E} + \frac{1}{\mu\epsilon} \vec{B}\vec{B} \right) - \frac{\epsilon}{2} \left(\vec{E}\vec{E} + \frac{1}{\mu\epsilon} \vec{B}\vec{B} \right) \hat{n}\hat{n}. \quad (116)$$

Introducing the displacement $d\vec{\zeta} = d\zeta\hat{n}$, $\Rightarrow dV = dAd\zeta$; δU , the work done by the e.m. field against displacement is

$$\delta U = \int dAd\zeta \hat{n} \cdot \vec{\mathbb{T}} \cdot \hat{n} = \int_{\Delta V} dV \hat{n} \cdot \vec{\mathbb{T}} \cdot \hat{n} \quad (117)$$

Note that $\hat{n} \cdot \vec{T} \cdot \hat{n}$ represents the e.m. pressure on the wall.

$$\hat{n} \cdot \vec{T} \cdot \hat{n} = \epsilon E^2 - \frac{\epsilon}{2} (E^2 + \frac{1}{\mu\epsilon} B^2) \quad (118)$$

because $\hat{n} \cdot \vec{B} = \hat{n} \times \vec{E} = 0$ at the surface of a perfect conductor.

$$\delta U = \int_{\Delta V} \frac{\epsilon}{2} (E^2 - \frac{1}{\mu\epsilon} B^2). \quad (119)$$

or, written in terms of time-averaged fields,

$$\delta U = \frac{\epsilon}{4} \int_{\Delta V} dV (E^2 - \frac{1}{\mu\epsilon} B^2). \quad (120)$$

In the cavity the “photon number” is conserved, which means U/ω is an invariant. So $\delta U/U = \delta\omega/\omega$, from which

$$\frac{\delta\omega}{\omega} = \frac{\epsilon}{4U} \int_{\Delta V} dV (E^2 - \frac{1}{\mu\epsilon} B^2). \quad (121)$$

(E^2 and B^2 are time-averaged).

Example of application: measurement of field profile with a bead pull.

Consider TM_{011} , $r = 0$. The only non zero field component is:

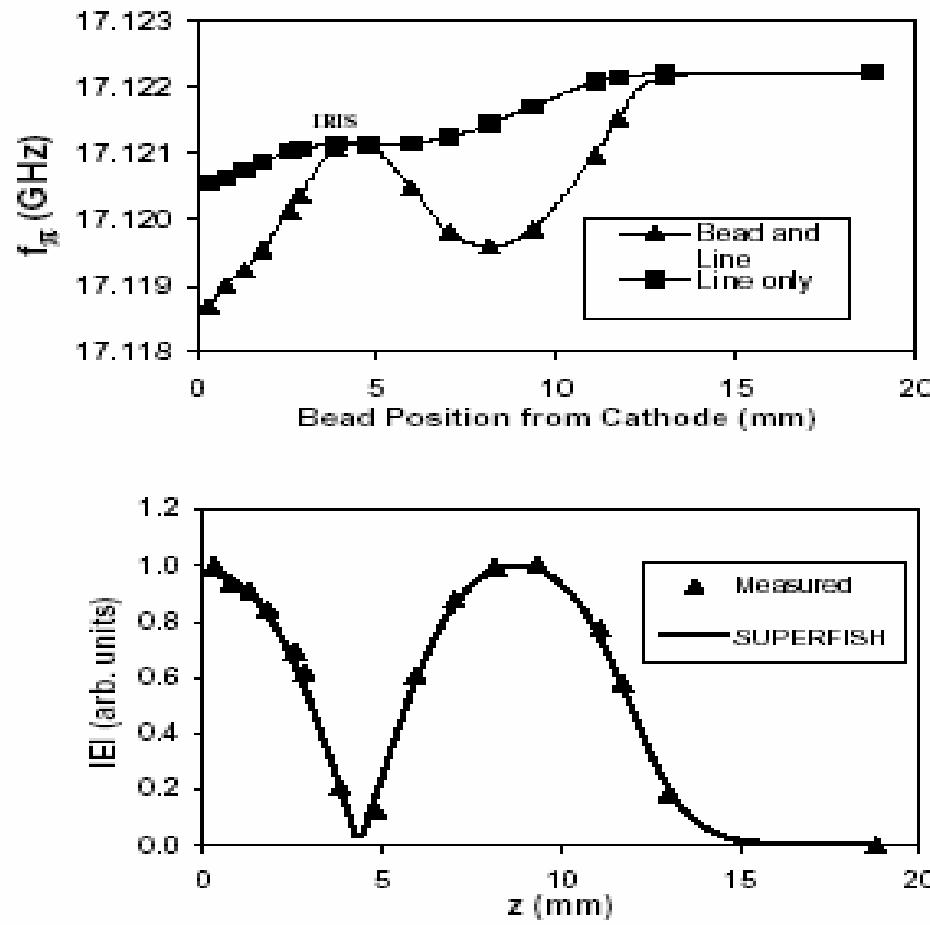
$$E_z = E_0 \cos\left(\frac{\pi z}{L}\right) \cos(\omega_{011}^T t). \quad (122)$$

Imagine pulling a conducting bead along the z -axis. The volume of the bead is ΔV . Then

$$\begin{aligned} \frac{\delta\omega}{\omega} &\simeq \frac{\epsilon}{4U} \Delta V E_z^2 \text{ (time-averaged)} \\ &\simeq \frac{\epsilon E_0^2 \Delta V \cos^2(\pi z/L)}{\frac{\epsilon E_0^2}{8} V 2 \left[1 + \left(\frac{\pi L}{x_{01} R}\right)^2\right] J_1^2(x_{01})}. \end{aligned} \quad (123)$$

$$\Rightarrow \frac{\delta\omega}{\omega} \simeq \frac{\Delta V}{V} \frac{\cos^2(\pi z/l)}{\left[1 + \left(\frac{\pi L}{x_{01} R}\right)^2\right] J_1^2(x_{01})} \quad (124)$$

\Rightarrow can map \cos^2 -dependence. Same principle for any other mode or superimposition of modes.



Example of bead-pull measurement

<http://prst-ab.aps.org/pdf/PRSTAB/v4/i8/e083501>.