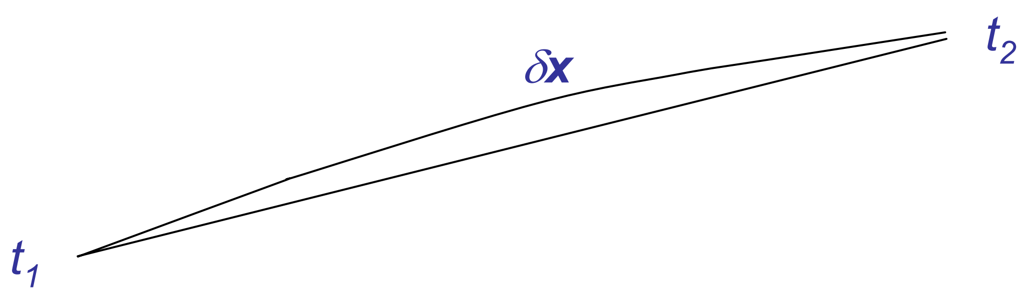
Particle Dynamics in e.m. fields

- **Lagrangian**
  given $\vec{x}(x^1, x^2, x^3)$ and d**x**/dt, a system can be described by a Lagrangian $\mathcal{L}(x^i, \dot{x}^i, t)$. The **action**

$$\mathcal{A} \equiv \int_{t_1}^{t_2} \mathcal{L}(x^i, \dot{x}^i, t)dt$$

is a functional of $\vec{x}(t)$, $\forall \vec{x}(t)$ defined for $t \in [t_1, t_2]$.

- **Least action principle:**
  $\mathcal{A}$ is a stationary function for any small variation δ**x** (*t*) verifying δ**x** (*t*1) = δ**x** (*t*2) = 0.



Lagrangian: equation of motion

- Once the Lagragian of a system is know the equation of motion are found from Euler-Lagrange equations:

$$
\begin{aligned}
P^i &= \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \\
\frac{dP^i}{dt} &= \frac{\partial \mathcal{L}}{\partial x^i}
\end{aligned}
$$

Lagrangian: case of a free relativistic particle

- The equation of motion must be the same in any inertial frame
  $\Rightarrow$ $\mathcal{A}$ is a scalar invariant

- $\mathcal{A}$ is a sum of infinitesimal displacement along a universe line $x^i(t)$
  $\Rightarrow$ $\mathcal{L}$dt associated to an infinitesimal displacement must be a scalar invariant, that is:

$$\mathcal{L}dt = \alpha ds = \alpha\sqrt{1-\frac{V^2}{c^2}}dt$$

- We also must have the NR limit

$$\lim_{V \ll c} \mathcal{L} = \frac{1}{2}mV^2 + \text{const} = \alpha\left(1 - \frac{V^2}{2c^2} + \mathcal{O}((V/c)^4)\right)$$

- So $\alpha = -mc$, and the relativistic Lagrangian is

$$\boxed{\mathcal{L}_{free} = -mc^2\sqrt{1-\frac{V^2}{c^2}} = -\frac{mc}{\gamma}\sqrt{u^\alpha u_\alpha},} \qquad u^\alpha = (\gamma c, \gamma\vec{v})$$

Case of a relativistic particle in an e.m. field

- Now we have $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$

Previous slide

From interaction potential

- The NR limit is: $\mathcal{L}_{int}^{NR} = -e\Phi = -eA^0$

- Let's try

$$\begin{aligned}\mathcal{L}_{int} &= -\frac{e}{\gamma c} u_\alpha A^\alpha \\ &= -\frac{e}{\gamma c} g_{\alpha\beta} u^\beta A^\alpha \\ &= -\frac{e}{\gamma c}\left(\gamma c\Phi - \gamma \overrightarrow{V}.\overrightarrow{A}\right) \\ \mathcal{L}_{int} &= -e\Phi + e\overrightarrow{\beta}\,\overrightarrow{A}.\end{aligned}$$

- So the total Lagrangian is

$$\mathcal{L} = -mc^2\sqrt{1-\frac{V^2}{c^2}} + \frac{e}{c}\overrightarrow{V}.\overrightarrow{A}(\overrightarrow{x}) - e\Phi(\overrightarrow{x}).$$

Checking equation of motion

- Let's check the Lagrangian derive gives the equation of motions: let's compute:

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = 0$$

- Let'go…

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \vec{V}} &= \frac{d}{dt}\left(\gamma m \vec{V} + \frac{e}{c}\vec{A}\right) \\ &= \frac{d(\gamma m \vec{V})}{dt} + \frac{e}{c}\left(\frac{\partial \vec{A}}{\partial t} + \frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i}\vec{A}\right) = \frac{d(\gamma m \vec{V})}{dt} + \frac{e}{c}\left(\frac{\partial \vec{A}}{\partial t} + (\vec{V}.\vec{\nabla})\vec{A}\right)\end{aligned}$$

$$\frac{\partial}{\partial \vec{x}}\mathcal{L} = \frac{e}{c}\vec{\nabla}(\vec{V}.\vec{A}) - e\vec{\nabla}\Phi = \frac{e}{c}\left[(\vec{V}.\vec{\nabla})\vec{A} + \vec{V}\times(\vec{\nabla}\times\vec{A})\right] - e\vec{\nabla}\Phi$$

With $\vec{B} = \vec{\nabla}\times\vec{A}$, one finally has:

Equation of motion in CGS units!

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \vec{V}} - \frac{\partial \mathcal{L}}{\partial \vec{x}} = \boxed{\frac{d}{dt}(\gamma m \vec{V}) + \frac{e}{c}\frac{\partial \vec{A}}{\partial t} + e\vec{\nabla}\Phi - \frac{e}{c}\vec{V}\times\vec{B} = 0}$$

Checking Least action principle

- The Lagrangian is $\mathcal{L} = -\frac{mc}{\gamma}\sqrt{u^\alpha u_\alpha} - \frac{q}{\gamma c} u_\alpha A^\alpha(x^\beta).$
- define $\tilde{\mathcal{L}} \equiv \gamma \mathcal{L}$. The action is $\mathcal{A} = \int_{\tau_1}^{\tau_2} d\tau \tilde{\mathcal{L}}$.
  least action principle $\delta \mathcal{A} = 0$.

$$\delta \mathcal{A} = \delta \left[\int_{\tau_1}^{\tau_2} d\tau \tilde{\mathcal{L}} \right] = \int_{\tau_1}^{\tau_2} d\tau \delta \tilde{\mathcal{L}}$$

$$-\delta \tilde{\mathcal{L}} = mc \frac{1}{2} \frac{1}{\sqrt{u^\alpha u_\alpha}} \left[\frac{\partial (u^\alpha u_\alpha)}{\partial u^\beta} \right] \delta u^\beta + q A_\alpha \delta u^\alpha + q u^\alpha \frac{\partial A_\alpha}{\partial x^\beta} \delta x^\beta .$$

One has $\delta u^\alpha = \delta(\frac{\partial x^\alpha}{\partial \tau}) = \frac{\partial}{\partial \tau}(\delta x^\alpha)$, and

$$\frac{\partial (u^\alpha u_\alpha)}{\partial u^\beta} = g_{\alpha\gamma} \frac{\partial (u^\alpha u^\gamma)}{\partial u^\beta} = g_{\alpha\gamma} \left(\delta^\alpha_\beta u^\gamma + \delta^\gamma_\beta u^\alpha \right) = 2 u_\beta$$

Checking Least action principle

- using $\delta \frac{dx^\alpha}{d\tau} = \frac{d(\delta x^\alpha)}{d\tau}$ (commutation of δ and d operators), one gets:

$$-c\delta\tilde{\mathcal{L}} = (mcu_\beta + qA_\beta)\frac{d(\delta x^\beta)}{d\tau} + qu^\alpha \partial_\beta A_\alpha \delta x^\beta.$$

Evaluating the integral by part and noting that $\delta x^\beta(\tau_1) = \delta x^\beta(\tau_2) = 0$ gives:

$$\delta\mathcal{A} = -\int_{\tau_1}^{\tau_2} d\tau \left[-mc\frac{du_\beta}{d\tau} - q(\partial_\alpha A_\beta)u^\alpha + qu^\alpha \partial_\beta A_\alpha\right] \delta x^\beta,$$

- So $\delta\mathcal{A} = 0 \Rightarrow [...] = 0$ (linear independence argument)

- And we recover the equation of motion

$$m\frac{d}{d\tau}u_\beta = \frac{q}{c}F_{\alpha\beta}u^\alpha$$

Canonical Conjugate and Hamiltonian

- The canonical momentum $\overrightarrow{P}$ conjugate to $\overrightarrow{x}$ is, by definition,

$$\overrightarrow{P} = \frac{\partial \mathcal{L}}{\partial \overrightarrow{V}} = \gamma m \overrightarrow{\dot{x}} + \frac{e}{c}\overrightarrow{A}$$
$$\overrightarrow{P} = \overrightarrow{p} + \frac{e}{c}\overrightarrow{A}$$

and the hamiltonian is defined as:

$$\mathcal{H} \equiv \overrightarrow{P}.\overrightarrow{V} - \mathcal{L}$$

- And Hamilton's equations are

$$\begin{cases} \dfrac{dx^\alpha}{d\tau} = \dfrac{\partial H}{dp^\alpha} \\ \dfrac{dp^\alpha}{d\tau} = \dfrac{\partial H}{dx^\alpha} \end{cases}$$

Relativistic Hamiltonian

We use $\overrightarrow{P} = \gamma m \overrightarrow{v} + \frac{e}{c}\overrightarrow{A}$ and calculate $\mathcal{H}$ then express $\mathcal{H}$ only as a function of $\overrightarrow{P}$ and $\overrightarrow{x}$. On can do the algebra (namely explicit $\overrightarrow{v}$ as a function $\overrightarrow{P}$ and replace in the expression of $\mathcal{H}$.

$$\begin{aligned}\mathcal{H} &= \overrightarrow{v}.\left(\gamma m \overrightarrow{v} + \frac{e}{c}\overrightarrow{A}\right) + \gamma mc^2\frac{1}{\gamma} + e\Phi - \frac{e}{c}\overrightarrow{A}\overrightarrow{v} & (3.16)\\ &= \gamma m v^2 + \frac{mc^2}{\gamma} + e\Phi = \gamma mc^2 + e\Phi. & (3.17)\end{aligned}$$

We note that the relation between $\overrightarrow{P} - \frac{e}{c}\overrightarrow{A}$ and $\mathcal{H} - e\Phi$ is the same as between $\mathcal{H}$ and $\overrightarrow{p}$ for the case of zero-field so we have:

$$(\mathcal{H} - e\Phi)^2 = \left(\overrightarrow{P} - \frac{e}{c}\overrightarrow{A}\right)^2 c^2 + m^2c^4 \tag{3.18}$$

So finally,

$$\mathcal{H} = \sqrt{\left(\overrightarrow{P}c - e\overrightarrow{A}\right)^2 + m^2c^4} + e\Phi \tag{3.19}$$

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