4-current density

- Consider a system of particles with positions $\vec{x}_n(t)$ and charges q_n.

$$\vec{J}(\vec{x},t) = \sum_n q_n \delta^3(\vec{x} - \vec{x}_n(t))\dot{\vec{x}}_n(t),$$
$$\rho(\vec{x},t) = \sum_n q_n \delta^3(\vec{x} - \vec{x}_n(t))$$

- Let's suppose J^α is the 4-current density, let $J^\alpha=(c\rho, \boldsymbol{J})$
- So $J^i(\vec{x}) = \sum_n q_n \delta^3(x^i - x^i_n(t)) d_t x^i_n(t)$
- Using the property $\int_{-\infty}^{\infty} f(\vec{x})\delta^3(\vec{x} - \vec{y}) = f(\vec{y})$

Smooth function

we can rewrite J^α as

$$J^\alpha(x) = \int \sum_n q_n \delta^4(x^\alpha - x^\alpha_n(t)) dx^0 \,\frac{dx^\alpha_n(t)}{dt}$$

J^α is a function of $x^\alpha \rightarrow$ it is a Lorentz invariant;

Charge continuity equation

- Consider the divergence of **J**

$$
\begin{aligned}
\vec{\nabla}.\vec{J}(\vec{x},t) &= \sum_n q_n \frac{\partial}{\partial x^i}\delta^3(\vec{x}-\vec{x}_n(t))\frac{dx_n^i(t)}{dt} \\
&= -\sum_n q_n \frac{\partial}{\partial x_n^i}\delta^3(\vec{x}-\vec{x}_n(t))\frac{dx_n^i(t)}{dt} \\
&= -\sum_n q_n \partial_t \delta^3(\vec{x}-\vec{x}_n(t)) \\
&= -\partial_t \rho(\vec{x},t) = -\partial_0[c\rho(\vec{x},t)].
\end{aligned}
$$

- So charge continuity can be written as

$$\boxed{\partial^\alpha J_\alpha = 0}$$

4-gradient

- In previous slide we introduce the 4-gradient operator

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}.$$

- This operator transforms as

$$\partial'_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu_\mu \frac{\partial_\nu}{.}$$

- Note that $\partial_\mu = (\partial_0, \overrightarrow{\nabla}).$

- Can define the covariant form $\partial^\mu = g^{\mu\nu}\partial_\nu = (\partial_0, -\overrightarrow{\nabla})$

- The self-contraction yields the d'Alembertian: $\Box \equiv \partial^\alpha \partial_\alpha.$

4-potential

- Define

$$A^{\alpha} \equiv (\phi, \vec{A})$$

Lorentz Gauge then write $\partial_{\alpha} A^{\alpha} = 0$. We also have

$$\Box A^{\alpha} = \frac{4\pi}{c} J^{\alpha},$$

or in SI units

$$\Box A^{\alpha} = \mu_0 J^{\alpha}, \qquad \text{[SI]}$$

- This is precisely the equation we solved to get the field of a moving charge three lessons ago…

- **In SI unit**: $\phi \to c\phi$

Covariance of Maxwell equations

- Define the tensor of dimension 2

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = g^{\alpha\delta}\partial_\delta A^\beta - g^{\beta\delta}\partial_\delta A^\alpha$$

4 potential

- *F*, is the e.m. field tensor. It is easily found to be

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- In SI units, F is obtained by $E \to E/c$
- The covariant form is

$$F_{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Inhomogeneous Maxwell's eqns

- Consider

$$\partial_\alpha F^{\alpha\beta} = \partial_0 F^{0\beta} + \partial_1 F^{1\beta} + \partial_2 F^{2\beta} + \partial_3 F^{3\beta}:$$

$$\begin{aligned} \partial_\alpha F^{\alpha 0} &= \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} \\ &= \partial_i E^i = \vec{\nabla}.\vec{E} = 4\pi\rho = \frac{4\pi}{c} J^0. \end{aligned}$$

- Similarly

$$\begin{aligned} \partial_\alpha F^{\alpha 1} &= \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} \\ &= \frac{1}{c}\partial_t(-E_x) + \partial_x(0) + \partial_y(-B_z) - \partial_z(B_y) = -\frac{1}{c}\partial_t(E_x) + [\vec{\nabla} \times \vec{B}]_x \\ &= [\vec{\nabla} \times \vec{B}]_x - \frac{1}{c}\partial_t E_x = \frac{4\pi}{c} J^1 \end{aligned}$$

- The inhomogeneous Maxwell's equations can be caster under the equation

$$\boxed{\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta}$$

Homogeneous Maxwell's eqns I

- Consider the Levi-Civita tensor (rank 4)

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if} \alpha, \beta, \gamma, \delta \text{ are } \textbf{even} \text{ permutation of } 0,1,2,3 \\ -1 & \text{if} \alpha, \beta, \gamma, \delta \text{ are } \textbf{odd} \text{ permutation of } 0,1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

- And consider $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\delta\gamma}$:

$$\begin{aligned} \epsilon^{0\beta\gamma\delta}\partial_\beta F_{\gamma\delta} &= \epsilon^{0123}\partial_1 F_{23} + \epsilon^{0132}\partial_1 F_{32} + \\ & \epsilon^{0213}\partial_2 F_{13} + \epsilon^{0231}\partial_2 F_{31} + \epsilon^{0312}\partial_3 F_{12} + \epsilon^{0321}\partial_3 F_{21} \\ &= \partial_1 F_{23} - \partial_1 F_{32} - \partial_2 F_{13} + \partial_2 F_{31} + \partial_3 F_{12} - \partial_3 F_{21} \\ &= \partial_x(-B_x) - \partial_x(B_x) - \partial_y(B_y) + \partial_y(-B_y) + \partial_z(-B_z) - \partial_z(B_z) \\ &= -2\overrightarrow{\nabla}.\overrightarrow{B}(=0) \end{aligned}$$

Homogeneous Maxwell's eqns II

- Consider the Levi-Civita tensor (rank 4)

$$\begin{aligned}
\epsilon^{1\beta\gamma\delta}\partial_\beta F_{\gamma\delta} &= \epsilon^{1023}\partial_0 F_{23} + \epsilon^{1032}\partial_0 F_{32} + \epsilon^{1302}\partial_3 F_{02} + \epsilon^{1320}\partial_3 F_{20} \\
&\quad + \epsilon^{1203}\partial_2 F_{03} + \epsilon^{1230}\partial_2 F_{30} \\
&= -\partial_0 F_{23} + \partial_0 F_{32} - \partial_3 F_{02} + \partial_3 F_{20} + \partial_2 F_{03} - \partial_2 F_{30} \\
&= 2(D_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20}) \\
&= 2\left(\frac{1}{c}\partial_t B_x - \partial_z E_y + \partial_y E_z\right) \\
&= 2\left[(\overrightarrow{\nabla} \times \overrightarrow{E})_x + \frac{1}{c}\partial_t B_x\right] (= 0)
\end{aligned}$$

- The homogeneous Maxwell's equations can be caster under the equation

$$\boxed{\partial_\alpha \mathcal{F}^{\alpha\beta} = 0.}$$

With the Dual field tensor defined as $\mathcal{F}^{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}.$

$$\text{Note: } \mathcal{F}_{\alpha\beta} = F_{\alpha\beta}(\overrightarrow{E} \to \overrightarrow{B}, \overrightarrow{B} \to -\overrightarrow{E}).$$

Covariant form of Maxwell's equation

- To introduce H and D field introduce the rank 2 tensor:

$$G^{\alpha\beta} = F^{\alpha\beta}(\overrightarrow{E} \to \overrightarrow{D}, \overrightarrow{B} \to \overrightarrow{H})$$

- Then Maxwell's equation writes

$$\boxed{\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \text{ and } \partial_\alpha \mathcal{F}^{\alpha\beta} = 0.}$$

- F is a rank 2 tensor that conforms to Lorentz transformation so the (E,B) field can be computed in an other frame by

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} F^{\gamma\delta}$$ or, in matrix notation

$$F' = \tilde{\Lambda} F \Lambda = \Lambda F \Lambda$$

Covariant form of Maxwell's equation

- Example consider the Lorentz boost along ***z-*** axis

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

- Then from

$$F' = \tilde{\Lambda} F \Lambda = \Lambda F \Lambda$$

- We get the same matrix as [JDJ 11.148]

$$F'^{\gamma\delta} = \begin{pmatrix} 0 & \gamma(E_x - \beta B_y) & \gamma(E_y + \beta B_x) & E_z \\ -\gamma(E_x - \beta B_y) & 0 & B_z & -\gamma(B_y - \beta E_x) \\ -\gamma(E_y + \beta B_x) & -B_z & 0 & \gamma(B_x + \beta E_y) \\ -E_z & \gamma(B_y - \beta E_x) & -\gamma(B_x + \beta E_y) & 0 \end{pmatrix}$$

Invariant of the e.m. field tensor

- Consider the following invariant quantities

$$F^{\mu\nu}F_{\mu\nu} = 2(E^2 - B^2), \text{ and } F^{\mu\nu}\mathcal{F}_{\mu\nu} = 4\vec{E}.\vec{B}$$

- Usually one redefine these invariants as

$$\mathcal{I}_1 \equiv -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(B^2 - E^2), \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4}F^{\mu\nu}\mathcal{F}_{\mu\nu} = -\vec{E}.\vec{B}.$$

- Which can be rewritten as $\mathcal{I}_1 \equiv -\frac{1}{4}\mathrm{tr}(F^2) \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4}\mathrm{tr}(F\mathcal{F})$

where $F \equiv F^{\nu}_{\mu} = F^{\mu\alpha}g_{\alpha\nu} \text{ and } \mathcal{F} \equiv \mathcal{F}^{\nu}_{\mu} = \mathcal{F}^{\mu\alpha}g_{\alpha\nu}.$

- Finally note the identities

$$F\mathcal{F} = \mathcal{F}F = -\mathcal{I}_2 I, \text{ and } F^2 - \mathcal{F}^2 = -2\mathcal{I}_1 I$$

Eigenvalues of the e.m. field tensor

- The eigenvalues are given by

$$F\Psi = \lambda\Psi \Rightarrow \mathcal{F}F\Psi = \lambda\mathcal{F}\Psi \Rightarrow \mathcal{F}\Psi = -\frac{\mathcal{I}_2}{\lambda}\Psi.$$

$$(F^2 - \mathcal{F}^2)\Psi = -2I\mathcal{I}_1\Psi = [\lambda^2 - (\mathcal{I}_2/\lambda)^2]\Psi,$$

- Characteristic polynomial $\lambda^4 + 2\mathcal{I}_1\lambda^2 - \mathcal{I}_2^2 = 0.$
- With solutions

$$\lambda_\pm = \sqrt{\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2} \pm \mathcal{I}_1}$$

$$\lambda_1 = -\lambda_2 = \lambda_-,\ \lambda_3 = -\lambda_4 = i\lambda_+.$$

Equation of motion

- The equation of motion (EOM) can be written

$$\boxed{\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^\alpha_\beta u^\beta.}$$

where $u^\alpha = (\gamma c, \gamma \vec{v}).$

- This is equivalent to defining a 4-force:

$$f^\mu = F^{\mu\nu} u_\nu.$$

- We need to solve EOM once we have specified the external e.m. tensor (assuming no other fields)