Last lesson in a nutshell!

- Proper time defined as

$$d\tau^2 \equiv dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$$

- Minkowski space:

$$x^0 \equiv ct,\ x^1 \equiv x,\ x^2 \equiv y,\ x^3 \equiv z\ [\text{so } \overrightarrow{x}^i \equiv \overrightarrow{X}\ (\text{i=1,2,3})].$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \qquad g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- A 4-vector is an object that conforms to $x'^\alpha = \Lambda^\alpha_\beta x^\beta.$
- Introduced
  - 4-velocity $u^\alpha \equiv \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds}$
  - 4-momentum $p^a \equiv mu^a$

The fundamental dynamical law for particle interactions in SR is that 4-momentum is conserved in any Lorentz frame.

More on Lorentz Transform

- Invariant

$$(ds)^2 = \underbrace{(dx^0)^2 \underbrace{-(dx^1)^2 - (dx^2)^2 - (dx^2)^2}_{\text{rotation}}}_{\text{Lorentz transformations}} = g_{\alpha\beta} dx^\alpha dx^\beta$$

- The Lorentz transform conserve *(ds)²*.

$$g_{\alpha\beta} dx'^\alpha\, dx'^\beta = g_{\gamma\delta} dx^\gamma dx^\delta$$

$$g_{\alpha\beta} \frac{dx'^\alpha}{dx^\gamma} \frac{dx'^\beta}{dx^\delta} = g_{\gamma\delta}$$

$$\boxed{g_{\alpha\beta} \Lambda^\alpha_\gamma \Lambda^\beta_\delta = g_{\gamma\delta}}$$

Determinants: -1 |Λ| |Λ| -1 so $|\Lambda|^2=1$

Particle dynamics in SR: example 1

- Consider $n + n \longrightarrow n + n + n + \bar{n}$ w one incident n at rest

Question: Minimum required energy for the incoming *n* to enable the reaction?

Answer: At threshold the 4 final neutrons are at rest in the lab frame

$$\begin{aligned}
& P_1^\alpha + P_2^\alpha = P_f^\alpha \\
\Rightarrow\ & (P_1^\alpha + P_2^\alpha)(P_{1\alpha} + P_{2\alpha}) = P_f^\alpha P_{f\alpha} = 16(m_n c)^2 \\
& P_1^\alpha P_{1\alpha} + 2P_1^\alpha P_{2\alpha} + P_2^\alpha P_{2\alpha} = 2(m_n c)^2 + 2P_1^\alpha P_{2\alpha} \\
& \Rightarrow P_1^\alpha P_{2\alpha} = 7(m_n c)^2.
\end{aligned}$$

with $P_1^\alpha P_{2\alpha} = g_{\alpha\beta} P_1^\alpha P_2^\beta = g_{00} P_1^0 P_2^0 = m_n c \frac{E}{c}$

we finally get the threshold energy

$$E = 7 m_n c^2.$$

Photon emission & absorption

- Consider

$u^\alpha_{e,a}$ = 4-velocity of emitter, absorber, respectively.

$E_{e,a}$ = photon energy measured by emitter, absorber, respectively.

- Then

Photon 4-momentum

$$\begin{aligned} P_\alpha u^\alpha &= g_{\alpha\beta} P^\beta u^\alpha \\ &= P^0 u^0 - P^i u^i = cP^0 = E. \end{aligned}$$

This also applies to accelerating frames!

- so

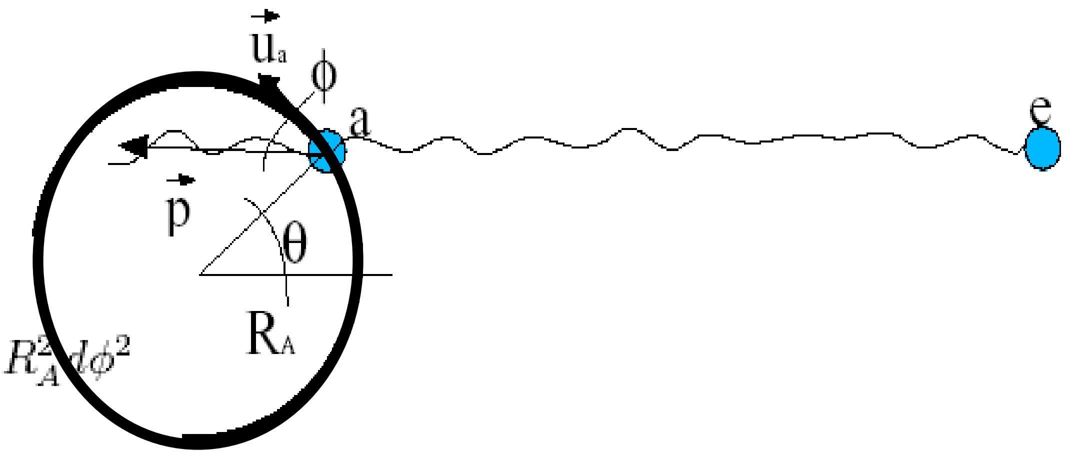
$$E_e = P_\alpha u^\alpha_e \text{ and, } E_a = P_\alpha u^\alpha_a$$

Photon emission & absorption: example 2

- In emitter's frame

$$c^2 d\tau = g_{\alpha\beta} dx^\alpha dx^\beta$$

- In absorber frame



$$\begin{aligned} c^2 (d\tau)^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= c^2 dt^2 - v^2 dt^2 = c^2 dt^2 - R_A^2 d\phi^2 \\ d\tau^2 &= dt^2 - \frac{R_A^2}{c^2} d\phi^2 \end{aligned}$$

- On another hand we have:

$$\begin{aligned} \frac{E_a}{E_e} &= \frac{P_\alpha u_a^\alpha}{P_\alpha u_e^\alpha} = \frac{P_0 u_a^0 - \vec{p}\,\vec{u_a^i}}{P_0 c} \\ &= \frac{P_0 u_\alpha^0 - |\vec{p}||\vec{u_a^i}| \cos\theta}{P_0 c} \end{aligned}$$

for photons ⇒

$$\frac{E_a}{E_e} = \frac{u_a^0 - |\vec{u}_a| \sin\phi}{c}.$$

Photon emission & absorption: example 2

- Expliciting the velocity

$$|\vec{u}_a| = R_A \frac{d\phi}{d\tau} = \frac{R_A \Omega}{\sqrt{1-(R_A\Omega/c)^2}}; \; u_a^0 = \frac{c}{\sqrt{1-(R_A\Omega/c)^2}}$$

$$\frac{E_a}{E_e} = \frac{\lambda_e}{\lambda_a} \Rightarrow \frac{\lambda_e}{\lambda_a} = \frac{1-(R_A\Omega/c)\sin\phi}{\sqrt{1-(R_A\Omega/c)^2}}$$

(Doppler shift)

- For f=90 deg this reduces to

$$\frac{\lambda_e}{\lambda_a} = \frac{1-(R_A\Omega/c)}{\sqrt{1-(R_A\Omega/c)^2}} = \sqrt{\frac{1-(R_A\Omega/c)}{1+(R_A\Omega/c)}}$$

SI (système International) versus CGS (cm Gram Second)

- The prescription to convert from CGS to SI and vice versa is

$$\frac{\overrightarrow{E}^G}{\sqrt{4\pi\epsilon_0}} = \overrightarrow{E}^{SI}; \; \sqrt{\frac{\epsilon_0}{4\pi}}\overrightarrow{D}^G = \overrightarrow{D}^{SI};$$

$$\sqrt{4\pi\epsilon_0}\rho^G(\overrightarrow{J}^G, q^G) = \rho^{SI}(\overrightarrow{J}^{SI}, q^{SI}); \; \sqrt{\frac{\mu_0}{4\pi}}\overrightarrow{B}^G = \overrightarrow{B}^{SI};$$

$$\frac{\overrightarrow{H}^G}{\sqrt{4\pi\mu_0}} = \overrightarrow{H}^{SI}; \; \epsilon_0\epsilon^G = \epsilon^{SI}; \; \mu_0\mu^G = \mu^{SI}; \; c = (\mu_0\epsilon_0)^{-1/2}$$

- Example: Lorentz force

$$\begin{aligned}
\overrightarrow{F}^G &= q^G(\overrightarrow{E}^G + \frac{1}{c}\overrightarrow{v} \times \overrightarrow{B}^G) \\
\Rightarrow \overrightarrow{F}^{SI} &= \frac{q^{SI}}{\sqrt{4\pi\epsilon_0}}\left[\sqrt{4\pi\epsilon_0}\overrightarrow{E}^{SI} + \sqrt{\mu_0\epsilon_0}\overrightarrow{v} \times \sqrt{\frac{4\pi}{\mu_0}}\overrightarrow{B}^{SI}\right] \\
&= q^{SI}(\overrightarrow{E}^{SI} + \overrightarrow{v} \times \overrightarrow{B}^{SI}).
\end{aligned}$$

SI versus CGS (cnt'd)

- Maxwell's equations in SI and CGS are

SI	G
$\overrightarrow{\nabla}.\overrightarrow{D} = \rho$	$\overrightarrow{\nabla}.\overrightarrow{D} = 4\pi\rho$
$\overrightarrow{\nabla} \times \overrightarrow{H} - \partial_t \overrightarrow{D} = \overrightarrow{J}$	$\overrightarrow{\nabla} \times \overrightarrow{H} - \frac{1}{c}\partial_t \overrightarrow{D} = \frac{4\pi}{c}\overrightarrow{J}$
$\overrightarrow{\nabla} \times \overrightarrow{E} + \partial_t \overrightarrow{B} = 0$	$\overrightarrow{\nabla} \times \overrightarrow{E} + \frac{1}{c}\partial_t \overrightarrow{B} = 0$
$\overrightarrow{\nabla}\overrightarrow{B} = 0$	$\overrightarrow{\nabla}\overrightarrow{B} = 0$
$\overrightarrow{F} = q(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}) = 0$	$\overrightarrow{F} = q(\overrightarrow{E} + \frac{\overrightarrow{v}}{c} \times \overrightarrow{B})$

Table 2.1: Maxwell's equations in CGS and SI units.

- The "real pros" also take ***c=1*** (pros because the accept the equivalence between space and time)… but we won't do this!