Special relativity

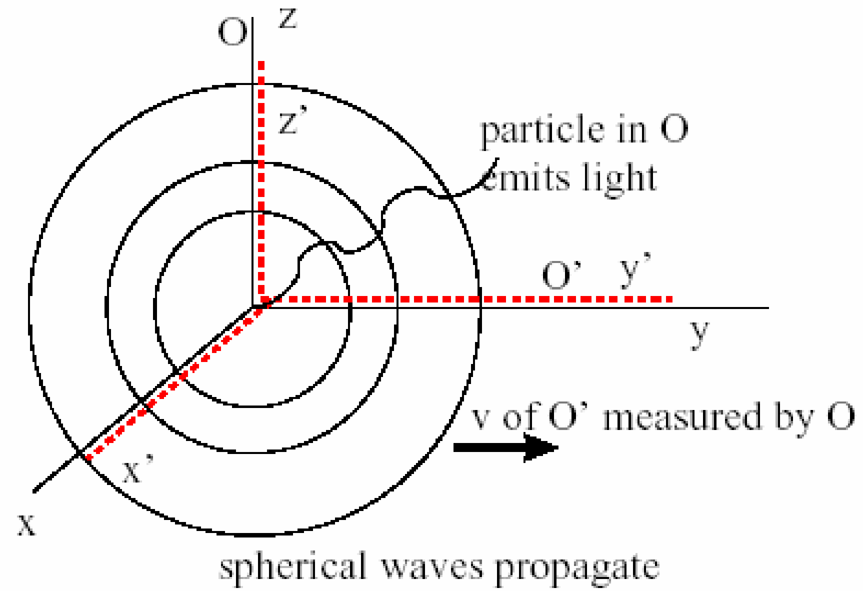
- Squashing of the E-field line associated to a moving charge is suggestive of Lorentz contraction
- e.m. law and eqn of motion should be invariant with respect to Lorentz transformations
- Let's refresh our memory with some basic concepts of special relativity (SR in short)

Proper time

- Consider two spherical waves

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2$$

$$\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2 = c^2$$



- So for **photons** we can write

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0$$

- This holds true for any inertial frames, and one generally define the proper time which is an invariant (it is a scalar)

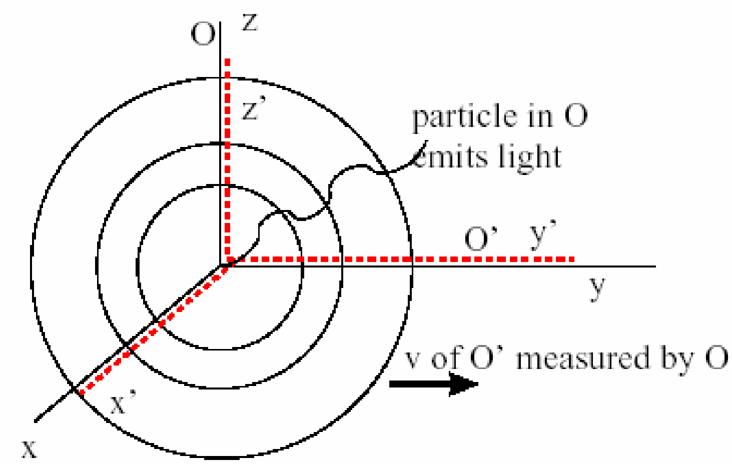
$$d\tau^2 \equiv dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$$

Proper time

- In SR proper time is an invariant.
- Note that

$$\boxed{d\tau^2 = dt^2(1-\beta^2) = \frac{1}{\gamma^2}dt^2}$$

all inertial observers measure the same $d\tau$.



$\vec{\beta} \equiv \frac{1}{c}\vec{v}$; $\vec{v}$ = velocity measured in lab frame ($\mathcal{O}$), dt = period between "ticks" of clock in lab frame.

When $\vec{v} = 0$, $d\tau = dt \Rightarrow d\tau$ = period between "ticks" of clock comoving with $\mathcal{O}'$. Every inertial observer measure the same value for this time interval: it is a scalar – a fixed physical quantity!

If δt represents the period between ticks of $\mathcal{O}'$'s clock, then $\mathcal{O}$ sees it ticks with period

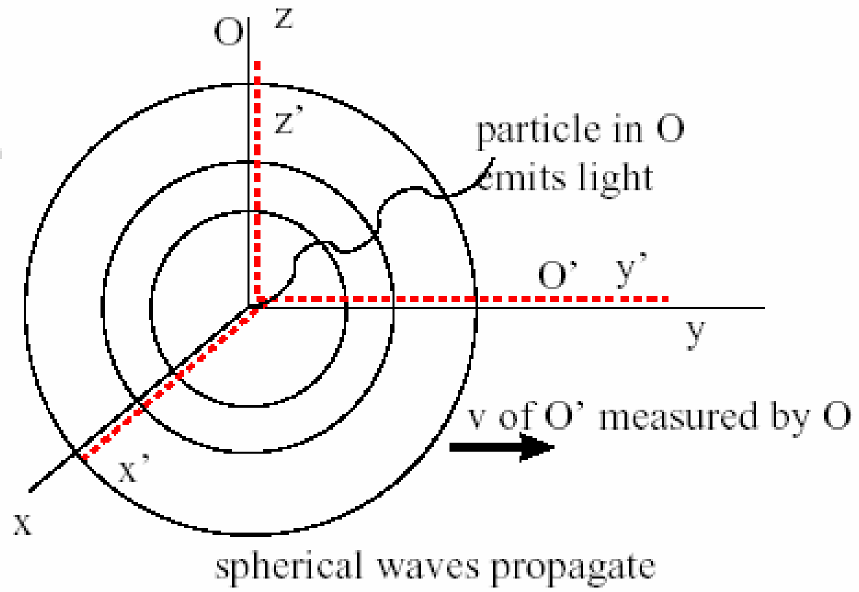
$dt = \gamma\delta t$ This is "time dilatation": $\mathcal{O}$ thinks $\mathcal{O}'$'s clock runs slow.

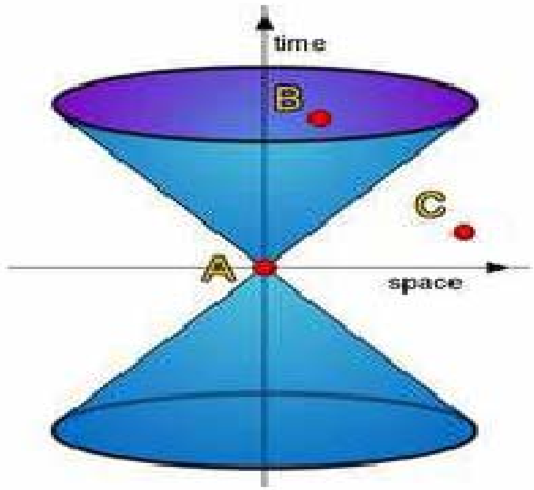
Proper time

- Proper time defined as

$$d\tau^2 \equiv dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$$

- This holds true for any inertial frames, and one generally define the proper time which is an invariant (it is a scalar)





$[AB]$ is time-like

$[AC]$ is space-like.

3+1 dimension space & Minkowski's metric

- Let

$$x^0 \equiv ct,\ x^1 \equiv x,\ x^2 \equiv y,\ x^3 \equiv z\ [\text{so } \overrightarrow{x}^i \equiv \overrightarrow{X}\ (\text{i=1,2,3})].$$

contravariant

- Then we can write

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

with $\alpha, \beta = 0, 1, 2, 3$ and

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is the Minkowski's metric.

3+1 dimension space -- some properties

- Some useful properties

$$x^{\alpha} = g^{\alpha\beta} x_{\beta}$$

contravariant **covariant**

- The scalar product is defined as

$$x^{\alpha} . x_{\alpha} = g_{\alpha\beta} x^{\alpha} x^{\beta} = g^{\alpha\beta} x_{\alpha} x_{\beta}$$

- Contravariant and covariant form of the metric are equal

$$g_{\gamma\delta} = g_{\gamma\alpha} g_{\delta\beta} g^{\alpha\beta}$$

mixed form is the Kroenecker delta function

$$g_{\gamma}^{\beta} = g_{\gamma\alpha} g^{\alpha\beta} = \delta_{\gamma}^{\beta}$$

Lorentz transformation (LT) I

- Derived to in sure Laws of Physics have the same form in inertial frames
- Many proofs …
- The transformation must let dτ invariant. A possible transformation is

$$\begin{cases} x = x'\cosh\varphi + ct'\sinh\varphi \\ ct = x'\sinh\varphi - ct'\cosh\varphi \end{cases}$$

Consider the coordinate in *O* corresponding to the origin (*x'*=*0*) in *O'*

$$\begin{cases} x = ct'\sinh\varphi \\ ct = ct'\sinh\varphi \end{cases} \Rightarrow \tanh\varphi = \frac{x}{Vt} = \beta \Rightarrow \begin{cases} \sinh\varphi = \gamma\beta \\ \cosh\varphi = \gamma \end{cases}$$

Rapidity (additive for LT compositions)

- Which gives (in 1+1 dim) the usual Lorentz transform (LT).
- If e.m. only is considered other transformation can let Maxwell's equation invariant (e.g. just dilations) but LT are universal.

Lorentz transformation II

- The Lorentz transform from $\mathcal{O}$ to $\mathcal{O}$' (two aligned inertial frames) is given by the boost matrix [see JDJ eqn (11.98)]

$$\Lambda^{\nu}_{\mu} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\left(\frac{\beta_x}{\beta}\right)^2(\gamma-1) & \frac{\beta_x\beta_y}{\beta^2}(\gamma-1) & \frac{\beta_x\beta_z}{\beta^2}(\gamma-1) \\ -\gamma\beta_y & \frac{\beta_x\beta_y}{\beta^2}(\gamma-1) & 1+\left(\frac{\beta_y}{\beta}\right)^2(\gamma-1) & \frac{\beta_y\beta_z}{\beta^2}(\gamma-1) \\ -\gamma\beta_z & \frac{\beta_x\beta_z}{\beta^2}(\gamma-1) & \frac{\beta_y\beta_z}{\beta^2}(\gamma-1) & 1+\left(\frac{\beta_z}{\beta}\right)^2(\gamma-1) \end{pmatrix}$$

- Note that $\Lambda^{\alpha}_{\gamma}\Lambda^{\beta}_{\delta}g_{\alpha\beta} = g_{\gamma\delta}$
- The Lorentz transformation is $x'^{\alpha} = \Lambda^{\alpha}_{\beta}x^{\beta}$.
- Formally $\Lambda^{\alpha}_{\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}}$
- If $\mathcal{O}$ and $\mathcal{O}$' not aligned Lorentz transformation would be $\Lambda^{\alpha}{}_{\beta}$ multiplied by a rotation matrix

Particle dynamics in SR

- **The principle of SR is :**
  - All laws of physics must be **invariant** under Lorentz transformations.
  - "**Invariant**" $\Leftrightarrow$ Physics laws retain the same mathematical forms and numerical constants (scalars) keep the same value.

Particle dynamics in SR: 4- velocity

- define

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds}$$

- Then $u^0 = c\dfrac{dt}{d\tau} = \gamma c \text{ and } u^i = \dfrac{1}{c}\dfrac{dx^i}{d\tau} = c\dfrac{dt}{d\tau}\dfrac{dx^i}{dt} = c\gamma\beta^i$

- An invariant can be form via the scalar product

$$u_\alpha u^\alpha = g_{\alpha\beta} u^\beta u^\alpha = \gamma^2 - \gamma^2\beta^2 = c^2$$

- Moreover since dτ is an invariant and $x'^\alpha = \Lambda^\alpha_\beta x^\beta$. then

$$u'^\alpha = \Lambda^\alpha_\beta u^\beta$$

u conforms to Lorentz transformation i.e. satisfies the principle of SR!

Particle dynamics in SR: 4- momentum

- Define $P_\alpha \equiv m u^\alpha$

$E =$ total energy

$$\Rightarrow P^0 = \gamma m c = E/c, \; P^i = p^i;$$

rest mass

$p^i =$ ordinary 3-momentum.

- Then

$$P_\alpha P^\alpha = m^2 u_\alpha u^\alpha = m^2 c^2 = E/c^2$$ and $$P'^\alpha = \Lambda^\alpha_\beta P^\beta$$

- The fundamental dynamical law for particle interactions in SR is that 4-momentum is conserved in any Lorentz frame.
- So $P_\alpha P^\alpha = g_{\alpha\beta} P^\beta P^\alpha = E^2/c^2 - p^2$

Kinetic energy

$$E^2/c^2 - p^2 = (mc)^2$$
$$\Rightarrow \quad E = \sqrt{(pc)^2 + (mc^2)^2}. \quad \Rightarrow \quad T = \sqrt{(pc)^2 + (mc^2)^2} - mc^2$$

Particle dynamics in SR: example

- Consider $n + n \longrightarrow n + n + n + \bar{n}$ w one incident n at rest

Question: Minimum required energy for the incoming *n* to enable the reaction?

Answer: At threshold the 4 final neutrons are at rest in the lab frame

$$
\begin{aligned}
& P_1^\alpha + P_2^\alpha = P_f^\alpha \\
\Rightarrow \quad & (P_1^\alpha + P_2^\alpha)(P_{1\alpha} + P_{2\alpha}) = P_f^\alpha P_{f\alpha} = 16(m_n c)^2 \\
& P_1^\alpha P_{1\alpha} + 2P_1^\alpha P_{2\alpha} + P_2^\alpha P_{2\alpha} = 2(m_n c)^2 + 2P_1^\alpha P_{2\alpha} \\
& \Rightarrow P_1^\alpha P_{2\alpha} = 7(m_n c)^2.
\end{aligned}
$$

with $P_1^\alpha P_{2\alpha} = g_{\alpha\beta} P_1^\alpha P_2^\beta = g_{00} P_1^0 P_2^0 = m_n c \frac{E}{c}$

we finally get the threshold energy

$$E = 7 m_n c^2.$$