# e.m. field of charge in rectilinear motion I

- Start with Maxwell's equations

$$\overrightarrow{\nabla}.\overrightarrow{D} = \rho,\ \overrightarrow{\nabla}.\overrightarrow{H} = 0,$$
$$\overrightarrow{\nabla} \times \overrightarrow{E} + \partial_t \overrightarrow{B} = 0,\ \text{and}\ \overrightarrow{\nabla} \times \overrightarrow{H} - \partial_t \overrightarrow{D} = \overrightarrow{J}.$$

- Let's work out our way to an equation for ***A*** and $F$

$$\begin{aligned} \overrightarrow{B} &= \overrightarrow{\nabla} \times \overrightarrow{A} \Rightarrow \overrightarrow{\nabla} \times (\overrightarrow{E} + \partial_t \overrightarrow{A}) = 0 \Rightarrow \overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \partial_t \overrightarrow{A} \\ \frac{1}{\mu}\overrightarrow{\nabla} \times \overrightarrow{B} - \epsilon\partial_t \overrightarrow{E} &= \overrightarrow{J} \Rightarrow \overrightarrow{\nabla} \times \overrightarrow{B} - \mu\epsilon\partial_t \overrightarrow{E} = \mu \overrightarrow{J} \\ &\Rightarrow \overrightarrow{\nabla} \times (\overrightarrow{\nabla} \times \overrightarrow{A}) + \mu\epsilon(\overrightarrow{\nabla}\partial_t\Phi + \partial_t^2 \overrightarrow{A}) = \mu \overrightarrow{J} \end{aligned}$$

# e.m. field of charge in rectilinear motion II

- Using $\overrightarrow{\nabla} \times (\overrightarrow{\nabla} \times \overrightarrow{A}) = \overrightarrow{\nabla}(\overrightarrow{\nabla}.\overrightarrow{A}) - \nabla^2 \overrightarrow{A},$

  we get:

$$-\nabla^2 \overrightarrow{A} + \overrightarrow{\nabla}(\overrightarrow{\nabla}.\overrightarrow{A} + \mu\epsilon\partial_t \Phi) + \mu\epsilon\partial_t^2 \overrightarrow{A} = \mu \overrightarrow{J}$$

- In the Lorenz' Gauge: $\overrightarrow{\nabla}.\overrightarrow{A} + \mu\epsilon\partial_t \Phi = 0$
- Thus $\boxed{\nabla^2 \overrightarrow{A} - \mu\epsilon\partial_t^2 \overrightarrow{A} = -\mu \overrightarrow{J}}$ **[JDJ 6.16]**
- Using $\overrightarrow{\nabla}.\overrightarrow{D} = \rho \Rightarrow -\nabla^2 \Phi - \partial_t \overrightarrow{\nabla}.\overrightarrow{A} = \frac{\rho}{\epsilon}$ gives

$$\boxed{\nabla^2 \overrightarrow{\Phi} - \mu\epsilon\partial_t^2 \Phi = -\frac{\rho}{\epsilon}}$$ **[JDJ 6.15]**

**Inhomogeneous wave equations**

# e.m. field of charge in rectilinear motion III

- So the problem is to solve the equation for ***A*** and $F$ given the form of $r$ and ***J***

$$\left[\nabla^2 - me\frac{\partial^2}{\partial \boldsymbol{t}^2}\right]\begin{bmatrix}\Phi \\ \vec{\mathrm{A}}\end{bmatrix} = \begin{bmatrix}-\frac{r}{e} \\ -m\vec{\boldsymbol{J}}\end{bmatrix}$$

- For a moving charge distribution with velocity ***v***

$$\rho = \rho(\overrightarrow{x} - \overrightarrow{v}t)$$
$$\overrightarrow{J} = \overrightarrow{v}\rho(\overrightarrow{x} - \overrightarrow{v}t)$$

# e.m. field of charge in rectilinear motion IV

- For both ***A*** and $F$ we have to solve an inhomogeneous d'Alembert equation of the form

$$\Box f = g(\vec{x} - \vec{v}t).$$

- Consider $\vec{v} = v\hat{z} \Rightarrow f(\vec{x} - \vec{v}t) = (x, y, z - vt) = f(x, y, \zeta)$

with $\zeta \equiv z - vt$

- Then

$$\begin{aligned} \partial_z f &\rightarrow \frac{\partial \zeta}{\partial z}\partial_\zeta f = \partial_\zeta f \\ \partial_t f &\rightarrow \frac{\partial \zeta}{\partial t}\partial_\zeta f = -v\partial_\zeta f \end{aligned}$$

- So $\Box f \rightarrow \left(\partial_x^2 + \partial_y^2 + \partial_\zeta^2 - \mu\epsilon v^2 \partial_\zeta^2\right) f = \left(\partial_x^2 + \partial_y^2 + \gamma^{-2}\partial_\zeta^2\right) f.$

with $\gamma \equiv \frac{1}{\sqrt{1-\mu\epsilon v^2}}.$

# e.m. field of charge in rectilinear motion V

- Let $z' = \gamma\zeta \Rightarrow \partial_\zeta = \frac{\partial z'}{\partial \zeta}\partial_{z'} = \gamma\partial_{z'}$
- Then our d'Alembert equation has the form

$$\left(\partial_x^2 + \partial_y^2 + \partial_{z'}^2\right) f(x, y, \gamma^{-1}z') = g(x, y, \gamma^{-1}z')$$

- Consider a point charge

$$\rho(\overrightarrow{x} - \overrightarrow{v}t) \rightarrow \delta(x)\delta(y)\delta(\gamma^{-1}z') = \gamma\delta(x)\delta(y)\delta(z') = \gamma\delta(\overrightarrow{x}')$$

- Vector potential is along *z* $\overrightarrow{A} \rightarrow A\hat{z}\ (A_x = A_y = 0);$
- And we have to solve

$$\nabla_{x'}^2 A = -\gamma\mu q v \delta(\overrightarrow{x'}),\ \nabla_{x'}^2 \Phi = -\gamma\frac{q}{\epsilon}\delta(\overrightarrow{x'}).$$

# e.m. field of charge in rectilinear motion VI

- The equation $\nabla^2_{x'} A = -\gamma\mu q v \delta(\overrightarrow{x'}),\ \nabla^2_{x'}\Phi = -\gamma\frac{q}{\epsilon}\delta(\overrightarrow{x'}).$

  is solve by inspection from

$$\nabla^2_{x'}\left(\frac{1}{|\overrightarrow{x'}|}\right) = -4\pi\delta(\overrightarrow{x'})$$

- The results are

$$\boxed{\left\{\begin{array}{l} A = \frac{\gamma\mu}{4\pi}\frac{qv}{R}, \\ \Phi = \frac{\gamma}{4\pi\epsilon}\frac{q}{R}, \end{array}\right.} \quad \text{where } R \equiv \sqrt{x^2+y^2+\gamma^2(z-vt)^2}.$$

# e.m. field of charge in rectilinear motion VII

- The E-field can be calculated from

$$\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \partial_t \overrightarrow{A}$$

- Which gives

$$\begin{aligned}\overrightarrow{E} &= -\frac{\gamma q}{4\pi\epsilon}(\overrightarrow{\nabla} + \mu\epsilon v \partial_t \hat{z})\frac{1}{R} \\ &= \frac{\gamma q}{4\pi\epsilon R^3}\left[x\hat{x} + y\hat{y} + \gamma^2(z - vt)(1 - \mu\epsilon v^2)\hat{z}\right]\end{aligned}$$

$$\boxed{\overrightarrow{E} = \frac{\gamma q}{4\pi\epsilon R^3}\left[x\hat{x} + y\hat{y} + (z - vt)\hat{z}\right]}$$

# e.m. field of charge in rectilinear motion VIII

- In spherical coordinate

$$x^2 + y^2 = r^2 \sin^2\theta, \; z - vt = r\cos\theta.$$

- So

$$\begin{aligned} \Rightarrow R^2 &= r^2(\sin^2\theta + \gamma^2\cos^2\theta) \\ &= \gamma^2 r^2 \left(1 + \frac{1-\gamma^2}{\gamma^2}\sin^2\theta\right) = \gamma^2 r^2 (1 - \mu\epsilon v^2 \sin^2\theta), \end{aligned}$$

- and

$$\begin{aligned} E &= \frac{\gamma q}{4\pi\epsilon} \frac{r}{\gamma^3 r^3 (1-\mu\epsilon v^2\sin^2\theta)^{3/2}} \\ &= \frac{q}{4\pi\epsilon r^2} \frac{1-\mu\epsilon v^2}{(1-\mu\epsilon v^2\sin^2\theta)^{3/2}}. \end{aligned}$$

- In vacuum

$$\vec{E} = \frac{q}{4\pi\epsilon r^2} \frac{\vec{r}}{\gamma^2(1-\beta^2\sin^2\theta)^{3/2}}. \quad \text{[JDJ, Eq. (11.154)]}$$

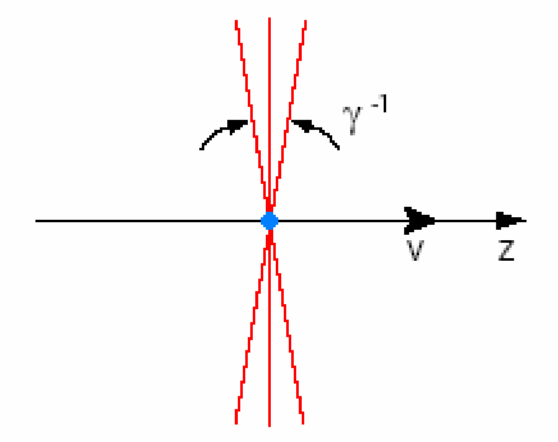
# e.m. field of charge in rectilinear motion IX

- Consider

$$\vec{E} = \frac{q}{4\pi\epsilon r^2}\frac{\vec{r}}{\gamma^2(1-\beta^2\sin^2\theta)^{3/2}}.$$

- Note that $E(\pi/2)/E(0) = \gamma^3$

the field line are squashed long the direction of motion.




**E-field line associated to a moving charge with Lorentz factor $g$**

# e.m. field of charge in rectilinear motion X

- B- field can also be computed $\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$

$$\overrightarrow{A} = \mu\epsilon\Phi\overrightarrow{v} \;\Rightarrow\; \overrightarrow{B} = \mu\epsilon\overrightarrow{\nabla} \times (\Phi\overrightarrow{v}) = \mu\epsilon[\overrightarrow{\nabla}\Phi \times \overrightarrow{v} + \Phi\overrightarrow{\nabla} \times \overrightarrow{v}]$$

$$\Rightarrow\; \overrightarrow{B} = \mu\epsilon\overrightarrow{\nabla}\Phi \times \overrightarrow{v}.$$

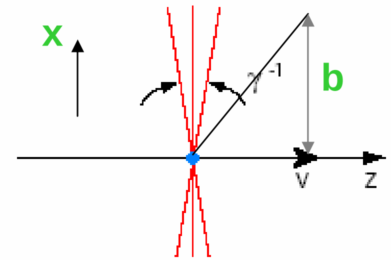
$$\overrightarrow{v} \times \overrightarrow{E} = -\overrightarrow{v} \times (\overrightarrow{\nabla}\Phi + \partial_t\overrightarrow{A}) = \overrightarrow{\nabla}\Phi \times \overrightarrow{v}.$$

$$\boxed{\overrightarrow{B} = \mu\epsilon\overrightarrow{v} \times \overrightarrow{E},} \text{ or } \overrightarrow{B} = \frac{\mu}{4\pi}\frac{\gamma q}{R^3}\overrightarrow{v}(x\hat{y} - y\hat{x}).$$

# e.m. field of charge in rectilinear motion XI

- Further reduction (try to introduce an impact parameter *b*)

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\frac{\hat{r}}{\gamma^2(1-\beta^2\sin^2\theta)^{3/2}}$$




- introducing $\sin\theta = \frac{b}{r} = \frac{b}{\sqrt{b^2+v^2t^2}}.$

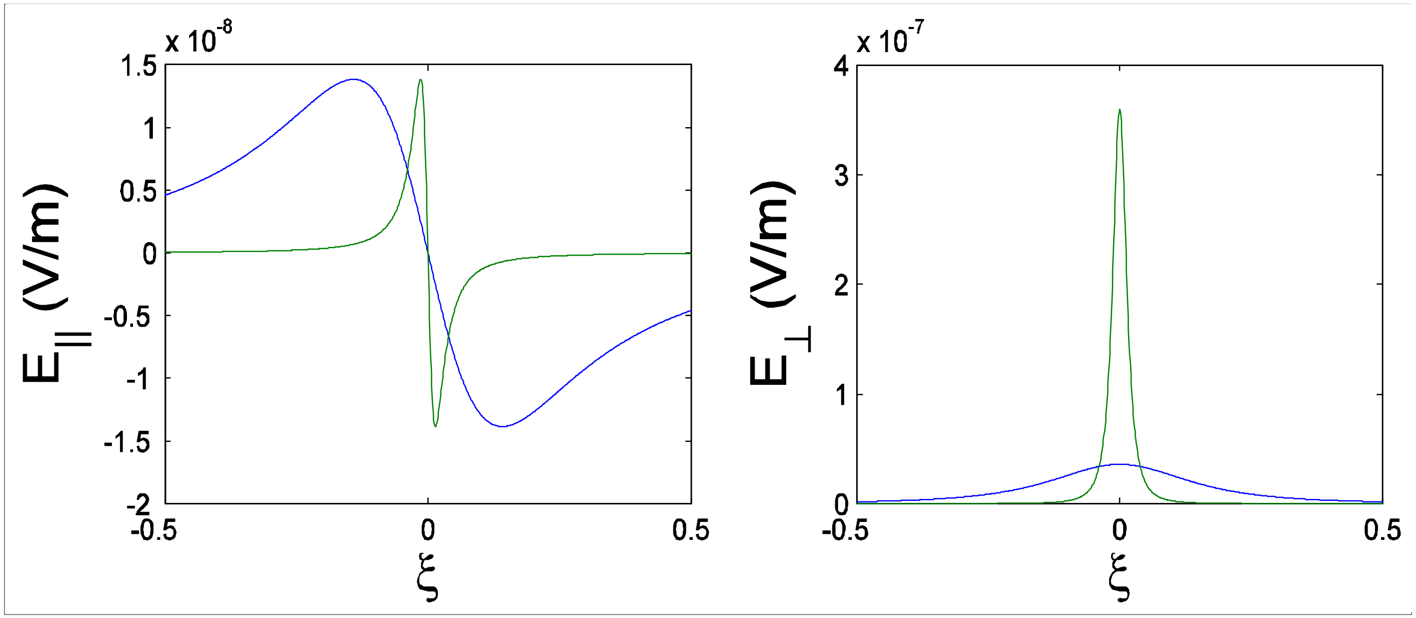
$$1-\beta^2\sin^2\theta = 1-\frac{\beta^2b^2}{b^2+(vt)^2} = \frac{b^2+v^2t^2-\beta^2b^2}{b^2+v^2t^2} = \frac{(1-\beta^2)b^2+v^2t^2}{b^2+v^2t^2}$$

$$1-\beta^2\sin^2\theta = \frac{b^2+\gamma^2v^2t^2}{\gamma^2r^2} \Rightarrow \gamma r\sqrt{1-\beta^2\sin^2\theta} = \sqrt{b^2+\gamma^2v^2t^2}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\vec{r}}{(b^2+\gamma^2v^2t^2)^{3/2}} \Rightarrow \vec{E}_\perp = \frac{q}{4\pi\epsilon_0}\frac{\gamma b\hat{x}}{(b^2+\gamma^2v^2t^2)^{3/2}}.$$

# e.m. field of charge in rectilinear motion XII

- Example of E-field associated to a charge at rest ($g$=1)and moving with $g$=10.



# Space charge effects I

- Let's consider the interaction of two particle moving at parallel to each other and consider the force experience by the particle of charge $q_0$ from the other particle

$$\begin{aligned} F &= q_0(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}) \\ &= q_0[\overrightarrow{E} + \mu\epsilon\overrightarrow{v} \times (\overrightarrow{v} \times \overrightarrow{E})] \end{aligned}$$

$$\begin{aligned} \Rightarrow \overrightarrow{F} &= q_0\left[(1-\mu\epsilon v^2)\overrightarrow{E} + \mu\epsilon v^2 E_z\hat{z}\right] \\ &= q_0\left(\frac{1}{\gamma^2}\overrightarrow{E} + \frac{\gamma^2-1}{\gamma^2}E_z\hat{z}\right) = q_0\left[\frac{1}{\gamma^2}(\overrightarrow{E} - E_z\hat{z}) + E_z\hat{z}\right] \end{aligned}$$

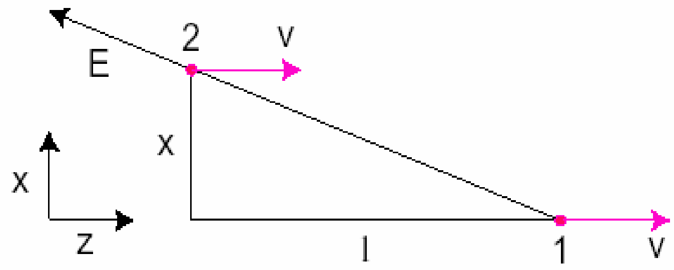
$$\Rightarrow \overrightarrow{F} = q_0\left[\frac{1}{\gamma^2}\overrightarrow{E}_\perp + \overrightarrow{E}_\parallel\right]$$

# Space charge effects II

- Let's consider the fields are generated by a source particle of unit charge then

$$F_l = E_z = -\frac{1}{4\pi\epsilon_0}\frac{ql}{\gamma^2(l^2+x^2/\gamma^2)^{3/2}},$$

$$F_t = E_x - vB_y = \frac{1}{4\pi\epsilon_0}\frac{qx}{\gamma^4(l^2+x^2/\gamma^2)^{3/2}}.$$




In accelerator physics, the force $\boldsymbol{F}$ is often called *the space charge force.*

The longitudinal force decreases with the growth of $\gamma$ as $\gamma^{-2}$ (for $l \gtrsim x/\gamma$). For the transverse force, if $l \gg x/\gamma$, $F_t \sim \gamma^{-4}$, and for $l = 0$, $F_t \sim \gamma^{-1}$. Hence, in the limit $\gamma \to \infty$, the electromagnetic interaction in free space between two particles on parallel paths vanishes.