Summary of TM mode

$$\omega_{mnp}^{TM} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}; \left\{ \begin{array}{c} m = 0,1,2,... \\ n = 1,2,3,... \\ p = 0,1,2,... \end{array} \right. \rightarrow \left\{ \begin{array}{l} x_{0n} = 2.405, 5.520, 8.564, ... \\ x_{1n} = 3.832, 7.016, 10.714, ... \\ x_{2n} = 5.136, 8.417, 11.620, ... \end{array} \right. .$$

$$\begin{aligned}
E_r(r,\phi,z,t) &= -E_0\left(\frac{p\pi}{x_{mn}}\frac{R}{L}\right)\left[\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right) - J_{m+1}\left(x_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TM}t) \\
E_\phi(r,\phi,z,t) &= E_0\left(\frac{p\pi}{x_{mn}}\frac{R}{L}\right)\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right)\sin(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TM}t) \\
E_z(r,\phi,z,t) &= E_0 J_m\left(x_{mn}\frac{r}{R}\right)\cos(m\phi)\cos\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TM}t) \\
B_r(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}\left(\frac{R}{x_{mn}}\sqrt{\mu\epsilon}\omega_{mnp}^{TM}\right)\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right)\sin(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TM}t) \\
B_\phi(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}\left(\frac{R}{x_{mn}}\sqrt{\mu\epsilon}\omega_{mnp}^{TM}\right)\left[\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right) - J_{m+1}\left(x_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TM}t) \\
B_z(r,\phi,z,t) &= 0
\end{aligned}$$

Summary of TE mode

$$\omega_{mnp}^{TE} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x'_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}; \begin{cases} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \\ p = 1, 2, 3, \ldots \end{cases} \rightarrow \begin{cases} x'_{0n} = 3.832, 7.016, 10.714, \ldots \\ x'_{1n} = 1.841, 5.331, 8.536, \ldots \\ x'_{2n} = 3.054, 6.706, 9.970, \ldots \end{cases}.$$

$$\begin{aligned}
E_r(r,\phi,z,t) &= E_0\left(\frac{R}{x'_{mn}}\sqrt{\mu\epsilon}\,\omega_{mnp}^{TE}\right)\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right)\sin(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TE}t) \\
E_\phi(r,\phi,z,t) &= E_0\left(\frac{R}{x'_{mn}}\sqrt{\mu\epsilon}\,\omega_{mnp}^{TE}\right)\left[\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right) - J_{m+1}\left(x'_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TE}t) \\
E_z(r,\phi,z,t) &= 0 \\
B_r(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}\left(\frac{p\pi}{x'_{mn}}\frac{R}{L}\right)\left[\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right) - J_{m+1}\left(x'_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TE}t) \\
B_\phi(r,\phi,z,t) &= -E_0\sqrt{\mu\epsilon}\left(\frac{p\pi}{x'_{mn}}\frac{R}{L}\right)[[\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right)\sin(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TE}t) \\
B_z(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}J_m\left(x'_{mn}\frac{r}{R}\right)\cos(m\phi)\sin\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TE}t)
\end{aligned}$$

Comment on cavity geometry

- Resonant frequency

$$\boxed{\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\frac{1}{R}\sqrt{(x'_{mn})^2 + (p\pi)^2(R/L)^2}.}$$

- not interested in large p or R: $\lim_{R\to\infty, p\to\infty} \omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\frac{p\pi}{L} \Rightarrow R$-independent.
- not interested in large m or n (and/or large L): $\omega_{mnp} \to \frac{1}{\sqrt{\mu\epsilon}}\frac{x_{mn}}{R}$ for large m or n and $R/L \sim 1$ L-independent.

$\Rightarrow$ The most interesting modes in practical applications should be low order modes with $R/L \sim 1$.

Some convenient Formulae

- Let's write the axial field for TM and TE mode as

$$
\begin{aligned}
\text{TM:} \qquad & E_z(r,\phi) = E_0\psi(r,\phi),\ B_z(r,\phi) = 0; \\
\text{TE:} \qquad & B_z(r,\phi) = -E_0\sqrt{\mu\epsilon}\,\psi(r,\phi),\ E_z(r,\phi) = 0;
\end{aligned}
$$

[note: here we introduce y not to be confused with the Y previously used]

- For both mode with therefore have

$$E_z^2(r,\phi) + \frac{1}{\mu\epsilon}B_z^2(r,\phi) = E_0^2\psi^2$$

Some convenient Formulae

- From last lesson we have, for TM and TE mode

$$\begin{bmatrix} \overrightarrow{E}_t(r,\phi) \\ \overrightarrow{B}_t(r,\phi) \end{bmatrix} = \frac{iE_0}{\gamma^2} \begin{bmatrix} \pm k\overrightarrow{\nabla}_t \\ \mu\epsilon\omega\hat{z}\times\overrightarrow{\nabla}_t \end{bmatrix} \psi(r,\phi);$$

$$\begin{bmatrix} \overrightarrow{E}_t(r,\phi) \\ \overrightarrow{B}_t(r,\phi) \end{bmatrix} = \frac{i\sqrt{\mu\epsilon}E_0}{\gamma^2} \begin{bmatrix} -\omega\hat{z}\times\overrightarrow{\nabla}_t \\ \pm k\overrightarrow{\nabla}_t \end{bmatrix} \psi(r,\phi);$$

- So for both mode we

$$E_t^2 + \frac{1}{\mu\epsilon}B_t^2 = \frac{E_0^2}{\gamma^4}(k^2+\mu\epsilon\omega^2)(\nabla_t\psi)^2.$$

Computing *U*

- The stored energy in the resonant cavity is

$$U = \frac{1}{2}\int_V d\overrightarrow{x}^3 (ED + BH) = \frac{\epsilon}{2}\int_V d\overrightarrow{x}^3 \left(E^2 + \frac{B^2}{\mu\epsilon}\right)$$

- Note the convenient formulae
  - $\int dz \sin^2(kz) = L/2$
  - $\int dz \cos^2(kz) = L/2(1 + \delta_{0p})$
  - Time averaging $\sin^2(\omega t)$ or $\cos^2(\omega t)$ gives 1/2

Computing *U*

- We also have

$$\frac{k^2+\mu\epsilon\omega^2}{\gamma^4} = \frac{2k^2+(x/R)^2}{(x/R)^4} = \left(\frac{R}{x}\right)^2\left[1+2\left(\frac{kR}{x}\right)^2\right]$$
$$= \left(\frac{R}{x}\right)^2\left[1+2\xi^2\right] \text{ where } \xi \equiv \frac{p\pi R}{xL}.$$

- So integrating (by head) over z and t-averaging gives

$$U = \frac{1+\delta_{0p}}{8}\epsilon E_0^2 L\int_A dA\left\{\left(\frac{R}{x}\right)^2\left[1+2\xi^2\right](\nabla_t\psi)^2+\psi^2\right\}.$$

Computing *U*

- Consider $\int_A dA(\nabla_t \psi)^2$
- We have

$$\begin{aligned} \int_A dA(\nabla_t \psi)^2 &= \int_A dA \overrightarrow{\nabla}_t.(\psi \overrightarrow{\nabla}_t \psi) - \int_A \psi \nabla_t^2 \psi \\ &= \oint_C dl \psi \hat{n}.\overrightarrow{\nabla}_t \psi - \int_A dA \psi \nabla_t^2 \psi. \end{aligned}$$

- But the boundary conditions gives

$$\oint_C dl \psi \hat{n}.\overrightarrow{\nabla}_t \psi = 0 \left\{ \begin{array}{c} \psi^{TM}(r=R)=0 \\ \partial_r \psi^{TE}(r=R)=0 \end{array} \right.$$

- And from wave eqn. $\nabla_t^2 \psi = -(k^2 - \mu\epsilon\omega^2)\psi = (x/R)^2 \psi$
- So finally

$$\boxed{\int_A dA(\nabla_t)^2 = \left(\frac{x}{R}\right)^2 \int_A dA \psi^2,}$$

Computing *U*

- So we end-up with JDJ Eq. 8.92

$$U = \frac{\epsilon L}{4} E_0^2 (1 + \delta_{0p}) \left[1 + \xi^2\right] \int_A dA \psi^2$$

- To perform the integral consider $\psi^2 \propto \cos^2(m\phi)$ so that $\int_0^{2\pi} d\phi \psi^2 \to \pi(1 + \delta_{0m}) J^2$ and

$$U = \frac{\pi}{4} \epsilon L E_0^2 (1 + \delta_{0m})(1 + \delta_{0p}) \left[1 + \xi^2\right] \int_0^R dr r J_m^2 \left(x \frac{r}{R}\right).$$

- To continue we now need to specialize to either TM or TE modes…

U for TM mode

- The integral is straightforward

$$\int_0^R dr r J_m^2\left(\frac{xr}{R}\right) = \frac{1}{2}R^2 J_{m+1}^2(x_{mn})$$

the identity $\int_0^1 dx x J_\nu^2(\alpha x) = \frac{1}{2}J_{\nu+1}^2$ if $J_\nu(\alpha) = 0$ was used

- And finally the stored energy associated to the TM mode is

$$\boxed{U_{mnp}^{TM} = \frac{1}{8}V\epsilon E_0^2(1+\delta_{0m})(1+\delta_{0p})\left[1+\xi^2\right]J_{m+1}^2(x_{mn}).}$$

wherein $V \equiv \pi R^2 L$

U for TE mode

- Consider the identity

$$\int_0^x d\rho \rho J_m^2(\rho) = \frac{1}{2}x^2 \left[J_m^2(x) + J_{m-1}^2(x)\right] - mxJ_m(x)J_{m-1}(x)$$

- Applied to our radial integral this gives

$$\Rightarrow \int_0^R dr r J_m^2\left(x\frac{r}{R}\right) = \frac{R^2}{x^2}\left\{\frac{1}{2}x^2\left[...\right] - m...\right\}$$

$$= \frac{1}{2}R^2\left[J_m^2(x'_{mn}) + J_{m-1}^2(x'_{mn})\right] - \frac{m}{x'_{mn}}R^2 J_m(x'_{mn})J_{m-1}(x'_{mn}),$$

the identity $\frac{m}{x'}J_m(x') = J_{m+1}(x') = J_{m-1}(x')$ is then used

$$\begin{aligned}\int_0^R dr r J_m^2\left(x\frac{r}{R}\right) &= \frac{1}{2}R^2\left[1 + \left(\frac{m}{x'}\right)^2\right] J_m^2(x') - R^2\left(\frac{m}{x'}\right)^2 J_m^2(x') \\ &= \frac{1}{2}R^2\left[1 - \left(\frac{m}{x'}\right)^2\right] J_m^2(x').\end{aligned}$$

U for TE mode

- So finally the store energy associated to TE mode is ($\delta_{0p} = 0 \ (p \neq 0)$)

$$\boxed{U_{mnp}^{TE} = \frac{1}{8} V \epsilon E_0^2 (1 + \delta_{0m}) \left[1 - \left(\frac{m}{x'} \right)^2 \right] \left[1 + \xi'^2 \right] J_m^2 (x'_{mn}).}$$

$$\delta_{0p} = 0 \ (p \neq 0) \text{ for TE-modes.}$$

where in
$$V \equiv \pi R^2 L$$
$$\xi' = \frac{p \pi R}{x'_{mn} L}$$

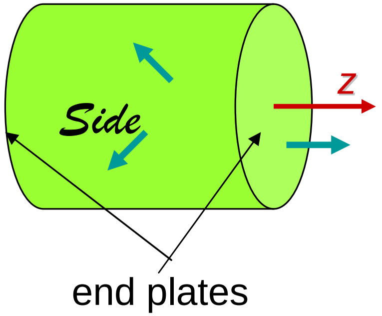
Dissipated Power

- Let’s define the dissipated power per surface area

Field component || to cavity walls

$$\frac{dP}{dA} = \frac{1}{2} R_s H_{\parallel}^2 = \frac{R_s}{2\mu^2} B_{\parallel}^2$$

Shunt impedance



- For a pillbox cavity

$$P = \frac{R_s}{2\mu^2} \left[\int_{side} dA B_{\parallel}^2 + 2 \int_{end} dA B_{\parallel}^2 \right].$$

P for TM mode

- For the sides of the cavity

$$\int_{side} dAB_{\|}^2 = \int_0^L dz \int_0^{2\pi} d\phi R B_\phi^2(r,\phi,z)$$
$$= \frac{L}{2}(1+\delta_{0p})\pi(1+\delta_{0m})\mu\epsilon E_0^2\left[1+\xi^2\right] R\left[\frac{m}{x}J_m(x) - J_{m+1}(x)\right]^2$$

but $J_m(x) = 0$ so

$$\int_{side} dAB_{\|}^2 = \frac{1}{2}\mu\epsilon E_0^2(\pi L R)(1+\delta_{0p})(1+\delta_{0m})\left[1+\xi^2\right] J_{m+1}^2(x)$$

and again $\xi \equiv \frac{\pi p R}{x_{mn} L}$.

P for TM mode

- For the ends plate

$$\begin{aligned}\int_{end} dA B_{\parallel}^2 &= \int_0^{2\pi} d\phi \int_0^R dr r B_t^2 \\ &= \frac{E_0^2 \mu^2 \epsilon^2 \omega^2}{(x/R)^4} \int_A dA (\nabla \psi)^2 = \frac{E_0^2 \mu^2 \epsilon^2 \omega^2}{(x/R)^2} \int_A dA \psi^2 \\ &= \mu \epsilon E_0^2 \left[1 + \xi^2\right] \frac{\pi}{2} (1 + \delta_{0m}) R^2 J_{m+1}^2(x)\end{aligned}$$

So

$$\int_{end} dA B_{\parallel}^2 = \frac{1}{2} \mu \epsilon E_0^2 (\pi R^2)(1 + \delta_{0m}) \left[1 + \xi^2\right] J_{m+1}^2(x_{mn}).$$

P for TM mode

- In summary

$$\int_{side} dAB_{\|}^2 = \frac{1}{2}\mu\epsilon E_0^2(\pi LR)(1+\delta_{0p})(1+\delta_{0m})\left[1+\xi^2\right] J_{m+1}^2(x)$$

$$\int_{end} dAB_{\|}^2 = \frac{1}{2}\mu\epsilon E_0^2(\pi R^2)(1+\delta_{0m})\left[1+\xi^2\right] J_{m+1}^2(x_{mn}).$$

- So the total power loss is (with $A_s \equiv 2\pi RL$)

$$\boxed{P_{mnp}^{TM} = \frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{1}{4}A_S(1+\delta_{0m})\left[1+\xi^2\right]\left[1+\delta_{0p}+2\frac{R}{L}\right] J_{m+1}^2(x_{mn}).}$$

P for TE mode

- Consider the side wall:

$$\int_{side} dAB_{\parallel}^2 = \int_0^L dz \int_0^{2\pi} d\phi R \left[B_\phi^2(R,\phi,z) + B_z^2(R,\phi,z)\right]$$
$$= \mu\epsilon E_0^2 \frac{L}{2}\pi(1+\delta_{0m}) R \left\{ \left(\frac{m}{x'}\right)^2 J_m^2(x')\xi'^2 + J_m^2(x') \right\}$$

$$\Rightarrow \int_{side} dAB_{\parallel}^2 = \mu\epsilon E_0^2 \frac{\pi}{2} LR(1+\delta_{0m}) \left[1 + \left(\frac{Rmp\pi}{x'^2 L}\right)^2\right] J_m^2(x')$$

- Consider the end plates

$$\int_{end} dAB_{\parallel}^2 = \int_0^{2\pi} d\phi \int_0^R dr r B_t^2(r,\phi,0)$$
$$= \mu\epsilon E_0^2 \xi'^2 \left(\frac{R}{x'}\right)^2 \int_A dA \left(\vec{\nabla}\psi\right)^2$$
$$= \mu\epsilon E_0^2 \xi^2 \int_A dA \psi^2.$$

P for TE mode

$$\begin{aligned}\int_A dA\psi^2 &= \pi(1+\delta_{0m})\int_0^R dr r J_m^2\left(x'\frac{r}{R}\right)\\ &= \frac{\pi}{2}R^2(1+\delta_{0m})\left[1-\left(\frac{m}{x'}\right)^2\right]J_m^2(x')\end{aligned}$$

$$\Rightarrow \int_A dA\psi^2 = \mu\epsilon E_0^2\frac{\pi}{2}R^2(1+\delta_{0m})\xi'^2\left[1+\left(\frac{m}{x'_{mn}}\right)^2\right]J_m^2(x'_{mn}).$$

- So finally

$$\boxed{\begin{aligned}P_{mnp}^{TE} &= \frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{1}{4}A_s(1+\delta_{0m})\left\{1+\left[2\frac{R}{L}+\right.\right.\\ &\left.\left.+(1-2\frac{R}{L})\left(\frac{m}{x'}\right)^2\right]\xi'^2\right\}J_m^2(x'_{mn}).\end{aligned}}$$