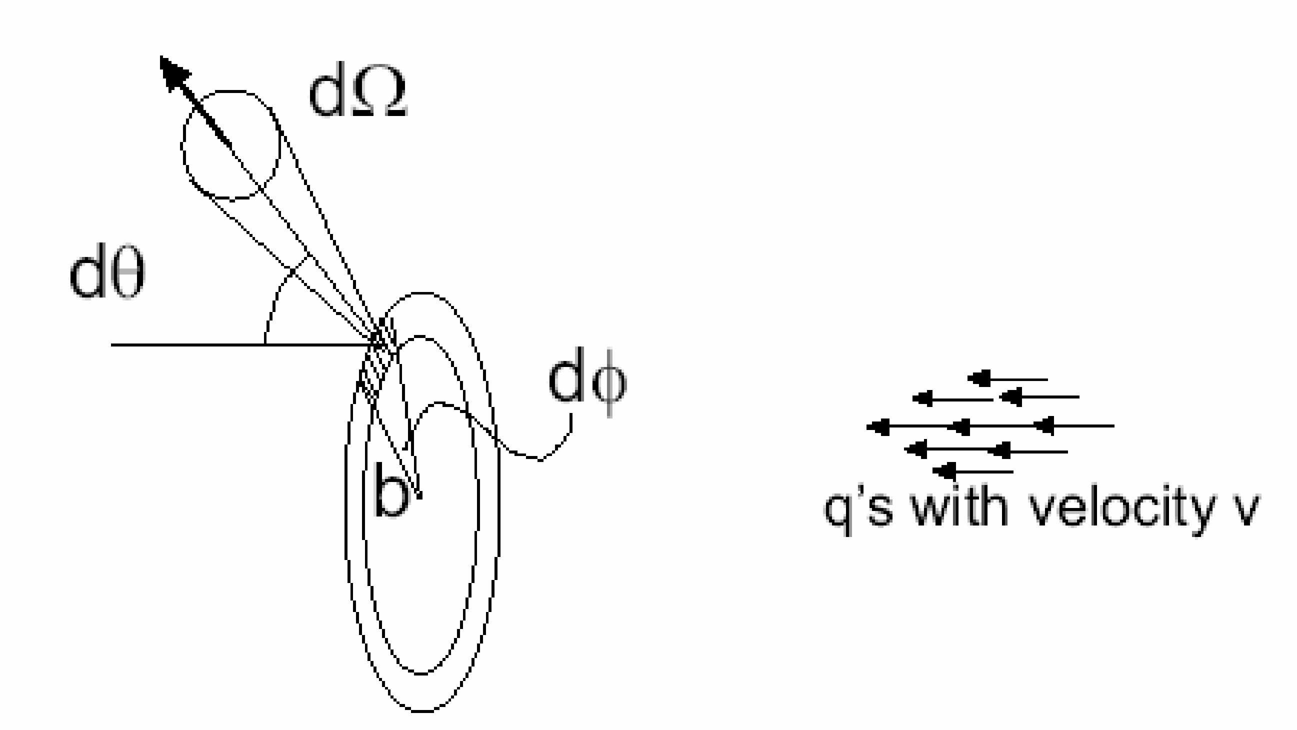
Scattering: introduction

- Thus far we concentrated on energy loss
- We now wish to quantify scattering, i.e. momentum transfer



Differential cross-section I

- Let N' be the number of particles scattered from $bdbd\phi$ into the solid angle $d\Omega$ per unit of time.

- Then

$$d^2N = nvbdbd\phi = N'd\Omega \Rightarrow bdbd\phi = \frac{N'}{nv}d\Omega = \frac{d\sigma}{d\Omega}d\Omega; \Rightarrow \frac{d\sigma}{d\Omega} = \frac{N'}{nv}.$$

$$bdb = \frac{d\sigma}{d\Omega}\sin\theta d\theta \Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta} = \frac{\boldsymbol{bdb}}{\boldsymbol{d}(\cos\theta)}$$

- Where $$\frac{d\sigma}{d\Omega} \equiv \frac{bdbd\varphi}{d\Omega}$$

is the differential cross-section

Differential cross-section II

- Under the small-angle approximation we have:

$$\sin\theta \sim \theta \Rightarrow \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{d\sigma}{d\Omega}$$

- Then

$$|\theta| = \frac{\Delta p}{p} = \frac{2qe}{\gamma b M v^2}$$

- For e- we saw that $b = \frac{2qe}{\gamma\theta M v^2} \Rightarrow \left|\frac{db}{d\theta}\right| = \frac{2qe}{\gamma\theta^2 M v^2}.$

- So $\frac{d\sigma}{d\Omega} = \frac{b\theta}{\theta^2}\left|\frac{db}{d\theta}\right| = \left(\frac{2qe}{\gamma M v^2}\right)^2 \frac{1}{\theta^4}$

- This is a small-angle approximation of the Rutherford differential cross section

Differential cross-section III

- For electrons targets:

$$\frac{d\sigma}{d\Omega} = \frac{b\theta}{\theta^2}\left|\frac{db}{d\theta}\right| = \left(\frac{2qe}{\gamma M v^2}\right)^2 \frac{1}{\theta^4}$$

- For nuclei targets

$$\frac{d\sigma}{d\Omega} = \frac{1}{\theta^4}\left(\frac{2qZe}{\gamma M v^2}\right)^2$$

- Therefore scattering by nuclei is Z^2 more probable that scattering by electron target.

- But there are Z times more nuclei than e- $\Rightarrow$ scattering by nuclei is *Z* times more probable that by e-

- So scattering in a block of matter is dominated by nuclei (while energy loss is dominated by the electrons)

Remarks on Rutherford Cross section

- The Rutherford cross section is actually:

$$\frac{d\sigma}{d\Omega} \equiv \left(\frac{Z_1 Z_2 e^2}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

- Which we could have inferred from a detailed analysis of the scattering geometry, wherein the impact parameter is actually related to the angle via

$$b = \frac{Z_1 Z_2 e^2}{2E} \frac{1}{\tan(\theta/2)}$$

- While we only had a 1/θ dependence

- This is an historical conincidence that the Rutherford cross section which was derived in the framework of classical mechanics keep the same form in Quantum mechanics (when spin effects are introduced).

rms scattering angle I

- When we have an incoming cloud of particle, e.g. a charged particle beam, how do we quantify the scattering?
- Use statistical concept: the variance: if a variable *x* is distributed with a probability function *F(x)* then the variance of the function which is an indication of the spread of the variable *x* is

$$\langle \boldsymbol{x}^2 \rangle_F \equiv \frac{\int \boldsymbol{F}(\boldsymbol{x})\boldsymbol{x}^2 \boldsymbol{dx} - \int \boldsymbol{F}(\boldsymbol{x})\boldsymbol{x}\boldsymbol{dx}}{\int \boldsymbol{F}(\boldsymbol{x})\boldsymbol{dx}}$$

- Correspondingly the rms value is defined as

$$\sigma_x \equiv \langle \boldsymbol{x}^2 \rangle^{1/2}$$

rms scattering angle II

- Evaluating the integral for the dσ/dΩ we have

$$\langle\theta^2\rangle = \frac{\int d\Omega\theta^2 d\sigma/d\Omega}{\int d\Omega d\sigma/d\Omega} \simeq \frac{\int d\theta\theta^3 1/\theta^4}{\int d\theta\theta 1/\theta^4}$$
$$= \frac{\int_{\theta_{min}}^{\theta_{max}} d\theta 1/\theta}{\int d\theta 1/\theta^3} = \frac{\ln\frac{\theta_{max}}{\theta_{min}}}{\frac{1}{2}(1/\theta_{min}^2 - 1/\theta_{max}^2)}$$

- Thus for single event scattering

$$\langle\theta^2\rangle \simeq 2\theta_{min}^2 \ln\frac{\theta_{max}}{\theta_{min}}$$

- Where we can estimate

$$\theta_{min} = \frac{2qZe}{\gamma b_{max}Mv^2} \simeq \frac{2qZe}{\gamma a M v^2}$$
$$\sim \frac{e^2}{am_pc^2} \sim \frac{e^2/(m_ec^2)}{am_pc^2} \sim \frac{r_e}{1836a} \ll 1$$

a: atomic radius

Case of many small angle scattering

- The charge q performs a random walk through the material and the rms deflection angle is given by

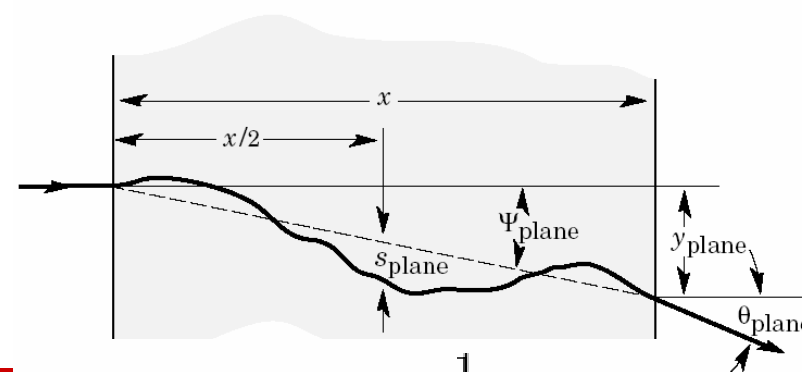
$$\langle\Theta^2\rangle = N\langle\theta^2\rangle$$

- So that

$$\frac{d\langle\Theta^2\rangle}{dz} = n\sigma\langle\theta^2\rangle \simeq 2n\sigma\theta^2_{min}\ln(\theta_{max}/\theta_{min})$$

- And from central-limit theorem we infer the probability distribution to be

$$P_{RW}(\theta_p) \propto e^{\frac{-\theta_p^2}{2\langle\Theta_p^2\rangle}}.$$



$$\theta_0 = \theta^{\rm rms}_{\rm plane} = \frac{1}{\sqrt{2}}\,\theta^{\rm rms}_{\rm space}\,.$$

Case of single scattering events

- Consider the single scattering event cross-section:

$$\frac{d\sigma}{d\Omega}d\Omega = \left(\frac{2qZe}{\gamma M v^2}\right)\frac{1}{\theta^4}d\phi\theta d\theta$$

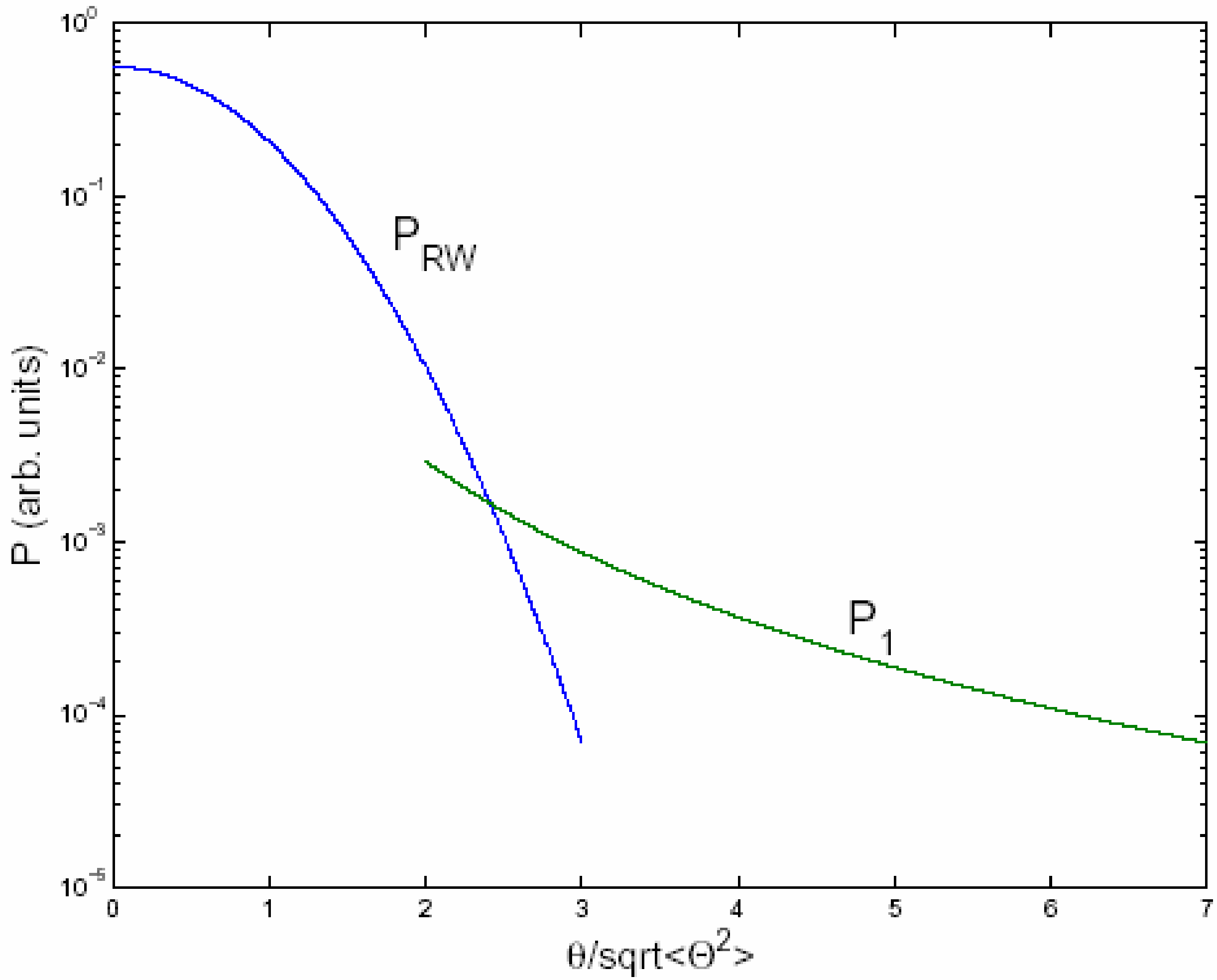
- Introducing the projected angle $\theta_p = \theta \sin\phi$

$$\frac{d\sigma}{d\Omega}d\Omega = \left(\frac{2qZe}{\gamma M v^2}\right)\frac{1}{\theta_p^3}d\theta_p \sin^2\phi d\phi$$

- Upon integration over ϕ we get

$$P_1(\theta_p)d\theta_p \propto \frac{d\theta_p}{\theta_p^3} \leftrightarrow P_1(\theta_p) \propto \frac{1}{\theta_p^3}$$

Scattering probability



Summary (extract from the PDG)

A charged particle traversing a medium is deflected by many small-angle scatters. Most of this deflection is due to Coulomb scattering from nuclei, and hence the effect is called multiple Coulomb scattering. (However, for hadronic projectiles, the strong interactions also contribute to multiple scattering.) The Coulomb scattering distribution is well represented by the theory of Molière [31]. It is roughly Gaussian for small deflection angles, but at larger angles (greater than a few θ_0, defined below) it behaves like Rutherford scattering, with larger tails than does a Gaussian distribution.

If we define

$$\theta_0 = \theta_{\rm plane}^{\rm rms} = \frac{1}{\sqrt{2}}\,\theta_{\rm space}^{\rm rms}\,. \qquad (27.11)$$

then it is sufficient for many applications to use a Gaussian approximation for the central 98% of the projected angular distribution, with a width given by [32,33]

$$\boxed{\theta_0 = \frac{13.6\ \mathrm{MeV}}{\beta c p}\, z\, \sqrt{x/X_0}\Big[1 + 0.038 \ln(x/X_0)\Big]\,.} \qquad (27.12)$$

Here p, βc, and z are the momentum, velocity, and charge number of the incident particle, and x/X_0 is the thickness of the scattering medium in radiation lengths (defined below). This value of θ_0 is from a fit to Molière distribution [31] for singly charged particles with $\beta = 1$ for all Z, and is accurate to 11% or better for $10^{-3} < x/X_0 < 100$.

http://pdg.lbl.gov/