Field of a moving particle in a dielectric

- The E- and B-field associated to a uniformly moving particle in vacuum are given by

$$\boxed{\vec{E}(\vec{x},\omega) = \sqrt{\frac{2}{\pi}}\frac{q}{v}\left[\frac{\lambda}{\epsilon} K_1(b\lambda)\hat{x} - i\frac{\omega}{v}(\epsilon\beta - 1)K_0(b\lambda)\hat{z}\right].}$$

- and $$\boxed{\vec{B} = \sqrt{\frac{2}{\pi}}\frac{q}{c}\lambda K_1(b\lambda)\hat{y}.}$$

- where $\lambda \equiv \frac{\omega}{v}\sqrt{1-\epsilon\frac{v^2}{c^2}}$

- Previously we considered the case $b\lambda \ll 1$

Field in the limit bλ>>1

- We now consider the extreme case bλ>>1. The modified Bessel functions have the asymptotic expansion:

$$K_0(b\lambda) = K_1(b\lambda) = \sqrt{\frac{\pi}{2}}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}$$

- And the field associated to a uniformly moving charged particle takes the form

$$\begin{aligned}
\overrightarrow{E}(\overrightarrow{x},\omega) &= \frac{q}{v}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}\left(\frac{\lambda}{\epsilon}\hat{x} - i\frac{\omega}{v}(\frac{1}{\epsilon} - \beta^2)\hat{z}\right), \\
\overrightarrow{B}(\overrightarrow{x},\omega) &= \frac{q}{v}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}\hat{y}.
\end{aligned}$$

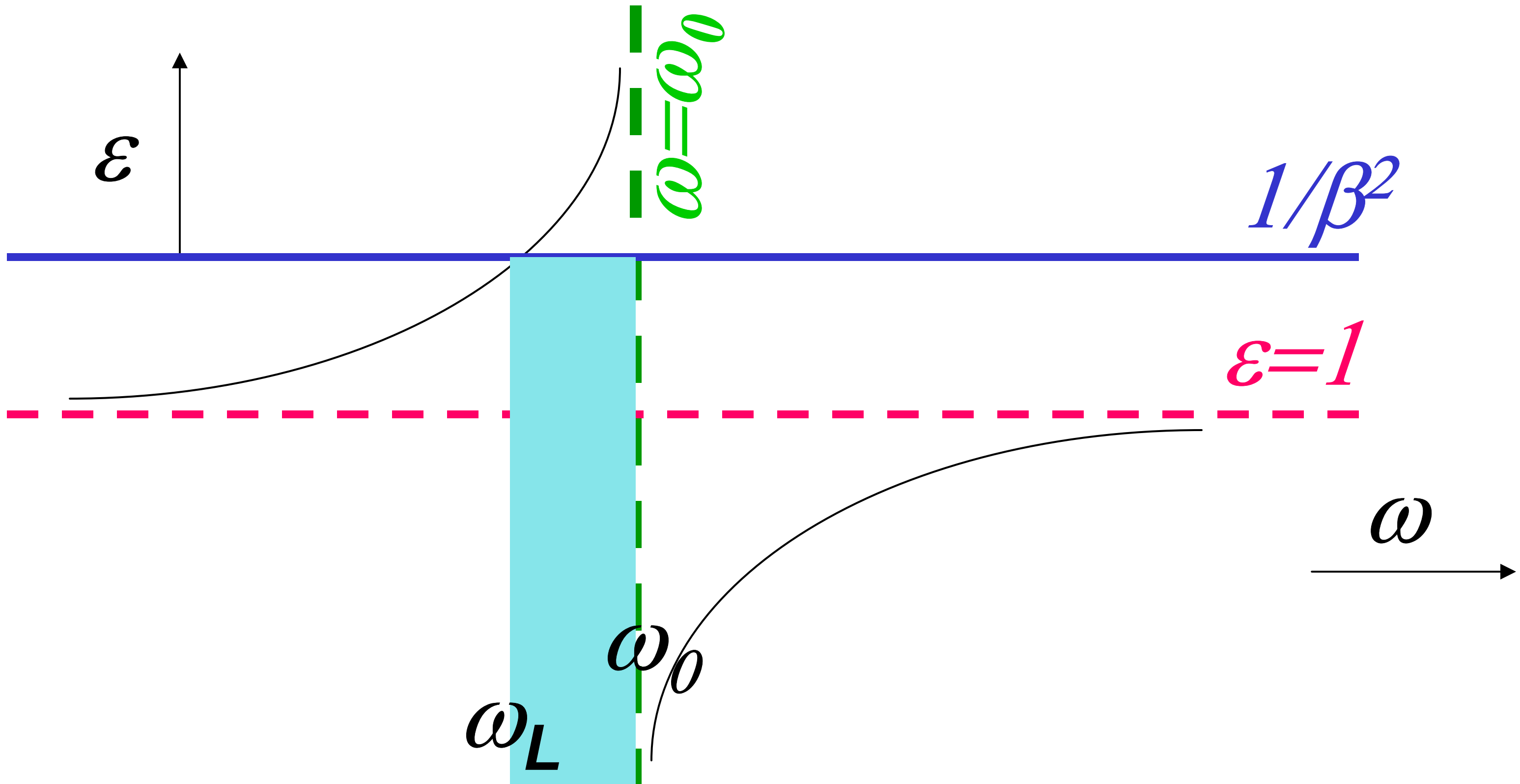
Cerenkov condition

- We saw that to get radiation (or energy loss) we needed either λ or ε to be complex.
- When investigating dielectric screening effects we considered the case $\varepsilon \in$ C
- We now consider the case where λ is a pure imaginary number and $\varepsilon \in$ R.
- from $\lambda = \frac{\omega}{v}\sqrt{1-\epsilon(\omega)\beta^2}$
- $\lambda \in \mathbb{I}$, $1-\epsilon\beta^2 < 0 \Rightarrow \epsilon\beta^2 > 1$, this is the Cerenkov condition.

Cerenkov condition

- Consider the model for permittivity

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - \cancel{i\omega\Gamma}}$$



Energy loss

- Explicit the asymptotic form of the e.m. field in the energy loss equation.

$$
\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= \mathcal{R}e\left\{\int_0^{+\infty} d\omega \frac{2}{\pi}\frac{q^2}{v^2}[i\omega(1/\epsilon-\beta^2)\lambda^* b]K_0(\lambda b)K_1(\lambda^* b)\right\} \\
&= \frac{2}{\pi}\frac{q^2}{v^2}\mathcal{R}e\left\{\int_0^{+\infty} d\omega(i\omega\lambda^* b)(1/\epsilon-\beta^2)K_0(\lambda b)K_1(\lambda^* b)\right\}
\end{aligned}
$$

- gives

$$
\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^{\infty} d\omega(i\omega\lambda^* b)(\frac{1}{\epsilon}-\beta^2)K_0(b\lambda)K_1(b\lambda^*)\right) \\
&= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^{\infty} d\omega(i\omega\lambda^* b)(\frac{1}{\epsilon}-\beta^2)\frac{e^{-b(\lambda+\lambda^*)}}{b\sqrt{\lambda\lambda^*}}\right) \\
&= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^{\infty} i\omega\sqrt{\frac{\lambda^*}{\lambda}}(\frac{1}{\epsilon}-\beta^2)\right)
\end{aligned}
$$

Frank-Tamm energy loss formula

- But $\lambda \in I$ so λ^*/λ=-1 so finally

$$\frac{d\mathcal{E}_f}{dz} = \frac{q^2}{v^2}\int_{\omega_L}^{\omega_0} d\omega\omega\left(1-\frac{1}{\epsilon\beta^2}\right)$$

- This is Frank-Tamm formula derived in 1937.

- History:
  - Cerenkov observed the radiation in Vavilov's labs (1934)
  - Frank and Tamm explained the effect (1937)
  - Cerenkov, Frank and Tamm share Nobel prize (1958)

Direction of propagation

- The direction of the wave is given by ***k***, ***k*** perpendicular to ***E*** and ***B***. Let θ_c be the angle between the velocity of the particle and k then

$$\cos\theta_c = \frac{|E_x|}{|E|} = \frac{E_x}{\sqrt{E_x^2+E_z^2}}$$
$$= \frac{\frac{\lambda}{\epsilon}}{\left[\left(\frac{\lambda}{\epsilon}\right)^2 - \frac{\omega^2}{v^2}\left(\frac{1}{\epsilon}-\beta^2\right)^2\right]^{1/2}}.$$

- From $\lambda^2 = (\omega/v)^2(1-\epsilon\beta^2)$

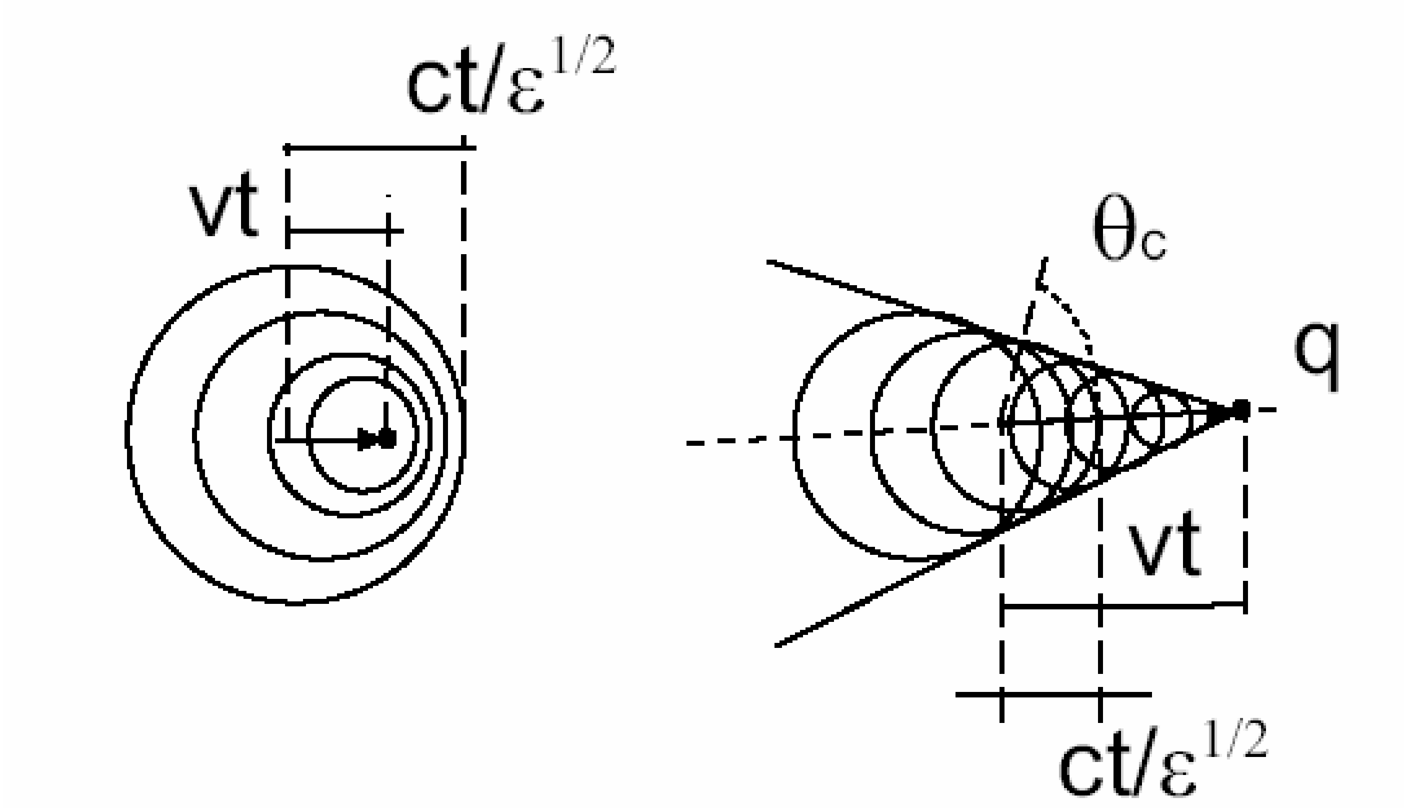
Velocity of light in the considered medium

$$\cos\theta_c = \frac{1}{\sqrt{1-1+\beta^2\epsilon}} = \frac{1}{\beta\sqrt{\epsilon}} = \frac{c_m}{v}$$

$c_m < c$ so $\cos\theta_c < 1$ and $\theta \in \mathbb{R}$.

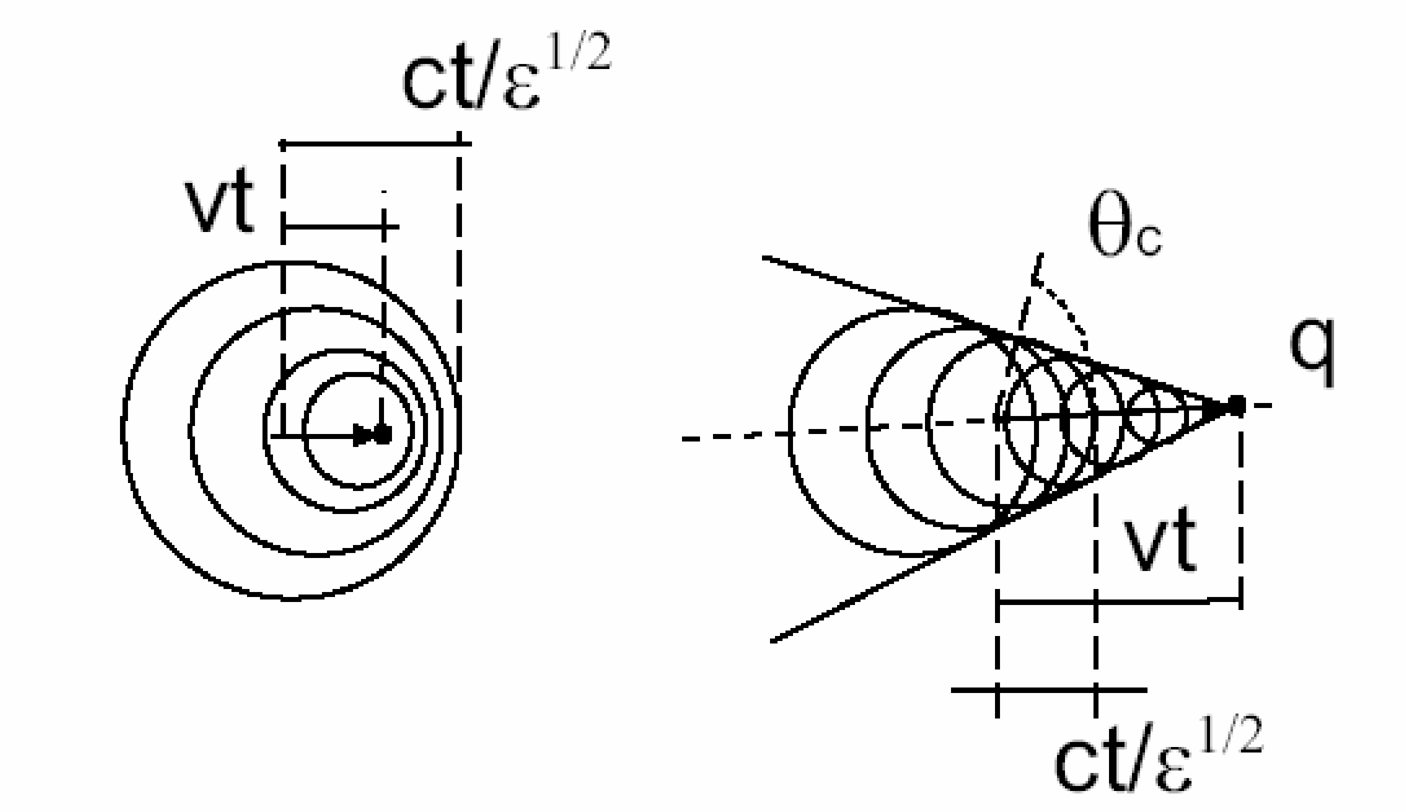
Shock wave feature

- Cerenkov radiation consists of a shock wave
- Effect similar to the Mach effect



Shock wave feature I

- Cerenkov radiation consists of a shock wave
- Effect similar to the Mach effect



Shock wave feature II

- The Shock wave feature inferred geometrically can be derived from the wave equation

$$\left(k^2 - \frac{\omega^2}{c_m^2}\right)\sqrt{\epsilon}\Phi(\overrightarrow{k},\omega) = \frac{4\pi}{\sqrt{\epsilon}}\rho(\overrightarrow{k},\omega)$$
$$\left(k^2 - \frac{\omega^2}{c_m^2}\right)\sqrt{\epsilon}\overrightarrow{A}(\overrightarrow{k},\omega) = \frac{4\pi}{c_m}\overrightarrow{J}(\overrightarrow{k},\omega)$$

- So *A* takes the same form as in vacuum under "renormalization"

$q \rightarrow \; q/\sqrt{\epsilon},\; c \rightarrow c_m.$

- So we can directly write the potentials as

$$\begin{pmatrix} \sqrt{\epsilon}\Phi(\overrightarrow{x},t) \\ \overrightarrow{A}(\overrightarrow{x},t) \end{pmatrix} = \frac{q}{\sqrt{\epsilon}}\frac{1}{[\kappa R]_{ret}}\begin{pmatrix} 1 \\ \frac{\overrightarrow{v}}{c_m} \end{pmatrix}$$

Shock wave feature III

- Consider

$$\left.\begin{aligned}\vec{\zeta} &= \vec{x} - \vec{v}\,t,\\ \vec{R} &= \vec{x} - \vec{x}\,(t') = \vec{x} - \vec{v}\,t'\end{aligned}\right\}\ \vec{R} = \vec{x} - \vec{v}\,t + \vec{v}\,(t - t') = \vec{\zeta} + v(t - t').$$

- On another hand $\underbrace{t - t' = \frac{R(t')}{c_m}}_{\textit{causality}} = \frac{|\vec{\zeta} + \vec{v}\,(t - t')|}{c_m}.$

- So $(t - t')^2 = \frac{1}{c_m^2}[\zeta^2 + 2\,\vec{\zeta}\,.\,\vec{v}\,(t - t') + v^2(t - t')^2].$

$$\Rightarrow (v^2 - c_m^2)(t - t')^2 + 2\,\vec{\zeta}\,.\,\vec{v}\,(t - t') + \zeta^2 = 0;$$

- Solve for *(t-t')* :

$$(t - t')_\pm = \frac{-\vec{\zeta}\,.\,\vec{v} \pm \sqrt{(\vec{\zeta}\,.\,\vec{v})^2 - (v^2 - c_m^2)\zeta^2}}{v^2 - c_m^2}.$$

Shock wave feature IV

$$(t-t')_{\pm} = \frac{-\vec{\zeta}.\vec{v} \pm \sqrt{(\vec{\zeta}.\vec{v})^2 - (v^2 - c_m^2)\zeta^2}}{v^2 - c_m^2}.$$

- For Cerenkov radiation ($v>c_m$), to obtain t-t' real positive we need $\zeta.v>0$ and $(\zeta.v)^2>(v^2-c_m^2)\zeta^2$.
- That is $\zeta v \cos\theta < 0$ or $\theta > \pi/2$, and $\cos^2\theta > 1 - c_m^2/v^2$.
- So $\theta > \arccos(-\sqrt{1-(c_m/v)^2})$,
- Thus potentials and fields exist at time t only within a cone with apex lying at the present time position of the particle and apex angle

$$\pi - \arccos(-\sqrt{1-(c_m/v)^2})$$

Shock wave feature V

- The 4-potential is given by $A^\alpha = A^\alpha_- + A^\alpha_+$

- The denominator is

$$\begin{aligned}
[\kappa R]_{ret} &= |(1 - 1/c_m \vec{v}.\hat{n})\vec{R}| = |\vec{R} - \frac{\hat{n}}{c_m}\vec{v}.[\vec{\zeta} + \vec{v}(t-t')]| \\
&= |\vec{R} - \frac{\hat{n}}{c_m}\vec{\zeta}.\vec{v} - \frac{\hat{n}}{c_m}v^2(t-t')| \\
&= |\hat{n}[c_m(t-t') - \frac{\vec{\zeta}.\vec{v}}{c_m} - \frac{v^2}{c_m}(t-t')]| \\
&= \frac{1}{c_m}|(c_m^2 - v^2)(t-t') - \vec{\zeta}.\vec{v}|.
\end{aligned}$$

- Expliciting (t-t') we gives

$$[\kappa R]_{ret} = \frac{\zeta}{c_m}\sqrt{c_m^2 - v^2\sin^2\theta} = \zeta\sqrt{1 - \frac{v^2}{c_m^2}\sin^2\theta}.$$

Shock wave feature VI

- So finally the potentials are

$$\begin{pmatrix} \sqrt{\epsilon}\Phi(\vec{x},t) \\ \vec{A}(\vec{x},t) \end{pmatrix} = \frac{2q}{\sqrt{\epsilon}} \frac{1}{\zeta\sqrt{1-\frac{v^2}{c_m^2}\sin^2\theta}} \begin{pmatrix} 1 \\ \frac{\vec{v}}{c_m} \end{pmatrix}$$

singularity

- In practice the singularity is smeared by the frequency-dependence of $\varepsilon(\omega)$ which implies $c_m(\omega)$…