Energy loss in a dielectric

- Expliciting the E and B field in the latter equation gives

$$\frac{d\mathcal{E}_f}{dz} = -b\mathcal{R}e\left\{\int_0^{+\infty} d\omega\left[-i\sqrt{\frac{2}{\pi}}\frac{q}{v}\frac{\omega}{v}(1/\epsilon-\beta^2)K_0(\lambda b)\right]\right.$$
$$\left.\times\left[\sqrt{\frac{2}{\pi}}\frac{q}{c}\lambda^* K_1(\lambda^* b)\right]\right\}$$

$$\boxed{\begin{aligned}\frac{d\mathcal{E}_f}{dz} &= \mathcal{R}e\left\{\int_0^{+\infty} d\omega\frac{2}{\pi}\frac{q^2}{v^2}[i\omega(1/\epsilon-\beta^2)\lambda^* b]K_0(\lambda b)K_1(\lambda^* b)\right\}\\ &= \frac{2}{\pi}\frac{q^2}{v^2}\mathcal{R}e\left\{\int_0^{+\infty} d\omega(i\omega\lambda^* b)(1/\epsilon-\beta^2)K_0(\lambda b)K_1(\lambda^* b)\right\}\end{aligned}}$$

- First derived by Enrico Fermi. Energy loss occurs if either λ or ε are complex

Energy loss in a dielectric

- We now introduce a simple model for the dielectric permittivity

- Consider the electron to be bounded to the nuclei via a damped harmonic oscillator type force

External field

$$\vec{x}(\omega) = \frac{-\frac{e}{m}\vec{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

Damping term

"Natural oscillation" frequency

- Then the polarization is defined as $-n_e e\vec{x}$:

$$\begin{aligned}\vec{P}(\omega) &= \frac{n_e e^2}{m}\frac{\vec{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\Gamma} \\ &= \frac{\epsilon(\omega) - 1}{4\pi}\vec{E}(\omega).\end{aligned}$$

Energy loss in a dielectric

- So the electric permittivity can be written as

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

- Where $\omega_p \equiv \sqrt{4\pi n_e e^2/m}$ is the plasma frequency

- If we explicit this form of $\varepsilon(\omega)$ in the energy loss equation and perform the integral…

- Not trivial, need make a "narrow band resonance" approximation

$$\omega \simeq \omega_0 \Rightarrow b\lambda = b\frac{\omega}{v}\sqrt{1-\epsilon\beta^2} \sim b\frac{\omega_0}{v}\sqrt{1-\epsilon\beta^2}$$

Energy loss in a dielectric

- Which leads to

$$b\frac{\omega_0}{v} \simeq 2\pi\frac{b}{\lambda_e}$$

- Also assume *bλ<<1* that is *b< atomic radius*

- Using the small argument approximation for the modified Bessel functions gives

$$\begin{aligned} b\lambda^* K_1(b\lambda^*) &\sim b\lambda^* \frac{1}{b\lambda^*} \sim 1 \\ K_0(b\lambda) &\sim \ln 2 - \ln(b\lambda) - \gamma = \ln\left(\frac{2e^{-\gamma}}{b\lambda}\right) = \ln\left(\frac{1.123}{b\lambda}\right) \end{aligned}$$

- where $\gamma = 0.577$ Euler constant.

Energy loss in a dielectric

- The energy loss for our model for $\varepsilon(\omega)$ is

$$\begin{aligned}\frac{d\mathcal{E}_f}{dz} &= \frac{2}{\pi}q^2v^2\mathcal{R}e\left\{\int_0^{+\infty} d\omega i\omega(1/\epsilon-\beta^2)\ln\left(\frac{1.123}{b\lambda}\right)\right\} \\ &\equiv \frac{2}{\pi}q^2v^2\mathcal{R}e(\mathcal{I})\end{aligned}$$

- where $\mathcal{I} \equiv \int_0^{+\infty} d\omega i\omega(\frac{\epsilon-1}{\epsilon})\ln\left(\frac{1.123}{b\lambda}\right)$ (we took $\beta = 1$).

- Explicit $\varepsilon(\omega)$ gives

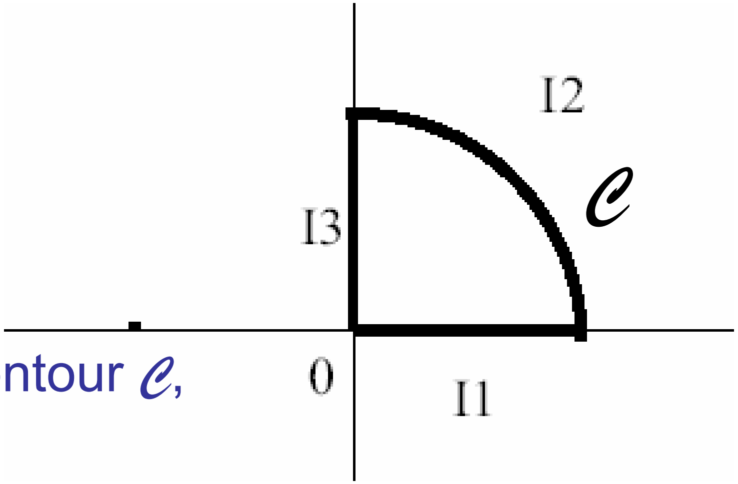
$$\begin{aligned}\mathcal{I} &= i\int_0^{+\infty} d\omega\omega\left(\frac{\omega_p^2}{\omega_p^2+\omega_0^2-\omega^2-i\omega\Gamma}\right)\left[\ln\left(\frac{1.123c}{b\omega_p}\right)-\ln\omega+\right. \\ &\left.+\frac{1}{2}\ln(\omega^2-\omega_0^2+i\omega\Gamma)\right]\end{aligned}$$

Energy loss in a dielectric

- We need to perform the integral. This is done in the Complex plane

- Two sources of poles

$$-\omega_0^2 + \omega^2 + i\omega\Gamma = 0$$
$$\omega_p^2 + \omega_0^2 - \omega^2 - i\omega\Gamma = 0$$



- Consider the path integral along the contour $\mathcal{C}$, we have:

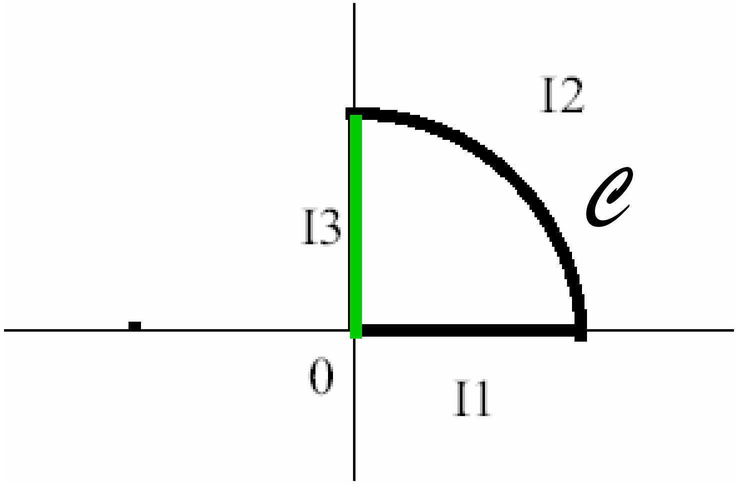
$$I_1+I_2+I_3=0$$

- Note that $\mathcal{I} = iI_1 = i(-I_2 - I_3).$

Energy loss in a dielectric

- Start with evaluating the integral

$$I_3 = \int_{+i\infty}^{0} d\omega\omega(\dots)\ln(\dots)$$



- Introduce $\omega \equiv i\Omega \text{ with } \Omega \in \mathbb{R}$

- then

$$\begin{aligned} I_3 &= -\int_{\infty}^{0} d\Omega\Omega(\dots)\ln(\dots) = \int_{0}^{\infty} d\Omega\Omega\frac{\omega_0^2+\Omega^2+\Omega\Gamma}{\omega_p^2+\omega_0^2+\Omega^2+\Omega\Gamma} \\ &\times\left(\ln\frac{1.123c}{b\omega_p} - \ln i\Omega + \frac{1}{2}\ln(-\Omega^2-\omega_0^2-\Omega\Gamma)\right) \end{aligned}$$

Energy loss in a dielectric

- The brackets simplifies to

$$(...) = \ln \frac{1.123c}{b\omega_p} - \cancel{\ln i} - \ln \Omega + \cancel{\frac{1}{2}\ln(-1)} + \frac{1}{2}\ln(\Omega^2 + \omega_0^2 + \Omega\Gamma)$$

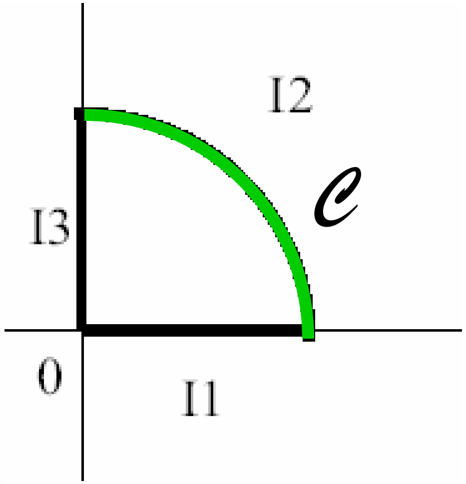
- And finally

$$\begin{aligned} I_3 &= \int_0^{\infty} d\Omega \Omega \frac{\omega_0^2 + \Omega^2 + \Omega\Gamma}{\omega_p^2 + \omega_0^2 + \Omega^2 + \Omega\Gamma} \\ &\times \left(\ln \frac{1.123c}{b\omega_p} - \ln \Omega + \frac{1}{2}\ln(\Omega^2 + \omega_0^2 + \Omega\Gamma) \right) \end{aligned}$$

- I_3 is real so iI_3 is imaginary so this integral has NO contribution to the energy loss

Energy loss in a dielectric

- Start with evaluating the integral I_2
- introduce $\omega \equiv Re^{i\theta}$
- Then



$$I_2 = \lim_{R\to\infty} \int_0^{\pi/2} i d\theta Re^{i\theta} Re^{i\theta} \frac{\omega_p^2}{\omega_p^2 + \omega_0^2 - R^2 e^{2i\theta} - iRe^{i\theta}\Gamma} \times \left(\ln \frac{1.123c}{b\omega_p} - \ln Re^{i\theta} + \frac{1}{2} \ln(-\omega_0^2 + R^2 e^{2i\theta} + iRe^{i\theta}\Gamma) \right)$$

- Taking the limit R→∞ gives

$$I_2 = \int_0^{\pi/2} i d\theta \omega_p^2 \ln \frac{1.123c}{b\omega_p} = i\frac{\pi\omega_p^2}{2} \ln \frac{1.123c}{b\omega_p}$$

Energy loss in a dielectric

- So finally the energy loss is

$$\boxed{\frac{d\mathcal{E}_f}{dz} = \frac{2}{\pi} q^2 v^2 \mathcal{R}e(\mathcal{I}) = -\frac{q^2\omega_p^2}{c^2} \ln \frac{1.123c}{b\omega_p}.}$$

- Compare with our initial derivation without dielectric screening and use the impulse approximation

$$\boxed{\frac{d\mathcal{E}_f}{dz} = -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\gamma v^2}{\omega_0 b} = -\frac{q^2\omega_p^2}{c^2} \ln \frac{\gamma c}{b\omega_0}.}$$

- **Influence of dielectric screening is two-folds**:
  - It removes the energy loss dependence on atomic structure ω_0 replaced by ω_p
  - It reduces the dependence on γ (γ in the *ln* argument is gone)