Influence of dielectric screening introduction

- Last time we derive the rate of energy loss for a charge particle *q*

$$\boxed{\frac{dE}{dz} = -4\pi n_e \frac{(qe)^2}{mv^2} \ln \frac{\gamma^2 m v^3}{qe\omega_0} \quad \text{[JDJ Eq (13.9)].}}$$

- Compare to Bohr's result (1915)

$$\boxed{\frac{dE}{dz} = -4\pi n_e \frac{(qe)^2}{mv^2} \left[\ln \frac{1.123\gamma^2 m v^2}{qe\langle\omega\rangle} - \frac{1}{2}\frac{v^2}{c^2} \right]}$$

- Bohr's result accurately describe measurement for non ultra-relativistic particle.

- For ultra-relativistic particle energy loss smaller than Bohr's prediction. This discrepancy is due to density effects

Field of a moving particle in a dielectric I

- In dense medium the particle field is altered by polarization effects

- We need to find the field of a charge particle moving in a medium…i.e. we have to solve

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha$$

$$\Box \equiv \partial_\alpha \partial^\alpha = \tfrac{\epsilon}{c^2} - \partial_t^2 \nabla^2. \qquad A^\alpha = (\Phi, \overrightarrow{A}) \text{ and } J^\alpha = (\rho c/\epsilon, \overrightarrow{J}).$$

- We take $\varepsilon=\varepsilon(\omega)$ and $\mu=1$

- and solve this equation in the Fourier domain

Field of a moving particle in a dielectric II

- Let’s define the Fourier tranforms

$$
\left.\begin{aligned}
F(\vec{x},t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega F(\vec{x},\omega)e^{-i\omega t}\\
F(\vec{x},\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega F(\vec{x},t)e^{+i\omega t}
\end{aligned}\right\}\ \textit{time}
$$

$$
\left.\begin{aligned}
F(\vec{x},\omega) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} d\vec{k}\, F(\vec{k},\omega)e^{i\vec{k}\,.\,\vec{x}}\\
F(\vec{k},\omega) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} d\vec{k}\, F(\vec{x},\omega)e^{-i\vec{k}\,.\,\vec{x}}
\end{aligned}\right\}\ \textit{space}
$$

Typos in the handout!!

Field of a moving particle in a dielectric III

- The sources of the field are of the form

$$\begin{aligned}\rho(\vec{x},t) &= q\delta(\vec{x}-\vec{v}t)\\ J(\vec{x},t) &= \vec{v}\rho(\vec{x},t)\end{aligned}$$

- The time and space Fourier transform is

$$\begin{aligned}\rho(\vec{k},\omega) &= \frac{q}{(2\pi)^2}\int d\vec{x}\int dt[q\delta(\vec{x}-\vec{v}t)]e^{-i(\vec{k}.\vec{x}-\omega t)}\\ &= \frac{q}{(2\pi)^2}\int dt e^{-i(\vec{k}.\vec{v}-\omega)t} = \frac{q}{2\pi}\delta(\omega-\vec{k}.\vec{v})\end{aligned}$$

- So finally

$$\rho(\vec{k},\omega) = \frac{q}{2\pi}\delta(\omega-\vec{k}.\vec{v});\ \text{and}\ \vec{J}(\vec{k},\omega) = \frac{q\vec{v}}{2\pi}\delta(\omega-\vec{k}.\vec{v})$$

Field of a moving particle in a dielectric IV

- Transform the wave equation in the Fourier domain:

$$\Box A^\alpha \rightarrow (k^2 - \epsilon(\omega)\frac{\omega^2}{c^2})A^\alpha = \frac{4\pi}{c}J^\alpha$$

- which give

$$A^\alpha = \frac{4\pi J^\alpha}{c[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]}$$

- So finally

$$\begin{aligned}\overrightarrow{A} &= \frac{4\pi\overrightarrow{J}}{c[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} = \frac{\overrightarrow{v}}{c}\epsilon(\omega)\Phi(\overrightarrow{k},\omega)\\ \Phi &= \frac{4\pi\rho}{\epsilon(\omega)[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} = \frac{2q\delta(\omega - \overrightarrow{k}.\overrightarrow{v})}{\epsilon(\omega)[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]}.\end{aligned}$$

Field of a moving particle in a dielectric V

- The e.m. field are given by

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial A}{\partial t} \text{ or } \vec{E}(\vec{k},\omega) = i(\frac{\omega}{c^2}\epsilon(\omega)\vec{v} - \vec{k})\Phi$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \rightarrow i\vec{k} \times \vec{A} = i\frac{\epsilon(\omega)}{c}\vec{k} \times \vec{v}\Phi.$$

- hence

$$\begin{pmatrix} \vec{E}(\vec{k},\omega) \\ \vec{B}(\vec{k},\omega) \end{pmatrix} = i \begin{pmatrix} \frac{\omega\epsilon(\omega)}{c^2}\vec{v} - \vec{k} \\ \frac{\epsilon(\omega)}{c}\vec{k} \times \vec{v} \end{pmatrix} \Phi(\vec{k},\omega)$$

- We want to find the flow of energy, i.e. the Poynting vector so we need $\vec{E}(\vec{x},\omega)$ and $\vec{B}(\vec{x},\omega)$:

Field of a moving particle in a dielectric VI

- The E-field is

$$\vec{E}(\vec{x},\omega) = \frac{i}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} d\vec{k}\, [\frac{\omega}{c}\epsilon(\omega)\frac{\vec{v}}{c} - \vec{k}]\Phi(\vec{k},\omega)e^{+i\vec{k}.\vec{x}}$$

- Specialize the problem to the case $\vec{x} = b\hat{x}$ and take $\vec{v} = v\hat{z}$.

$$\begin{aligned}
\vec{E}(\vec{x},\omega) &= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int\int dk_x dk_y dk_z \left[\frac{\omega}{c}\epsilon(\omega)\frac{\vec{v}}{c} - \vec{k}\right]\frac{\delta(\omega - k_z v)}{k^2 - \epsilon\frac{\omega^2}{c^2}}e^{ik_x b} \\
&= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int\int dk_x dk_y dk_z \left[-k_x\hat{x} - \cancel{k_y\hat{y}} + \left(\frac{\omega}{c}\frac{v}{c}\epsilon(\omega) - k_z\right)\hat{z}\right] \\
&\quad \times \frac{\delta(\omega - k_z v)}{k^2 - \epsilon\frac{\omega^2}{c^2}}e^{ik_x b}.
\end{aligned}$$

Field of a moving particle in a dielectric VII

- Integrate over k_z

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int dk_x dk_y \left[-k_x\hat{x} + \frac{\omega}{v}\left(\frac{v^2}{c^2}\epsilon - 1\right)\hat{z}\right] \times \frac{e^{ik_x b}}{k_x^2 + k_y^2 + \left(\frac{\omega}{v}\right)^2\left(1 - \epsilon\frac{v^2}{c^2}\right)}$$

- Let $\lambda \equiv \frac{\omega}{v}\sqrt{1-\epsilon\frac{v^2}{c^2}}$ and $\mathcal{I} \equiv \int\int dk^2 \frac{e^{ik_x b}}{k_x^2+k_y^2+\lambda^2}$. Then the E-field takes the form:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{1}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\left[-\frac{d\mathcal{I}}{db}\hat{x} + \frac{i\omega}{v}(\epsilon\beta^2 - 1)\mathcal{I}\hat{z}\right]$$

- Integration over dk_y gives

$$\int_{-\infty}^{+\infty} dk_y \frac{1}{k_x^2+k_y^2+\lambda^2} = \left.\frac{\arctan\left(\frac{k_y}{\sqrt{k_x^2+\lambda^2}}\right)}{\sqrt{k_x^2+\lambda^2}}\right|_{-\infty}^{+\infty} = \frac{\pi}{\sqrt{k_x^2+\lambda^2}}$$

Field of a moving particle in a dielectric VIII

- So finally

$$
\begin{aligned}
\mathcal{I} &= \pi \int_{-\infty}^{\infty} dk_x \frac{e^{ik_x b}}{\sqrt{k_x^2+\lambda^2}} = \pi \int_{0}^{\infty} dk_x \frac{e^{ik_x b}+e^{-ik_x b}}{\sqrt{k_x^2+\lambda^2}} \\
&= 2\pi \int_{0}^{+\infty} dk_x \frac{\cos(k_x b)}{\sqrt{k_x^2+\lambda^2}} = 2\pi K_0(b\lambda).
\end{aligned}
$$

- And $\frac{d\mathcal{I}}{db} = -2\pi\lambda K_1(b\lambda).$

- The E-field finally writes

$$
\boxed{\overrightarrow{E}(\overrightarrow{x},\omega) = \sqrt{\frac{2}{\pi}}\frac{q}{v}\left[\frac{\lambda}{\epsilon}K_1(b\lambda)\hat{x} - i\frac{\omega}{v}(\epsilon\beta-1)K_0(b\lambda)\hat{z}\right].}
$$

$\lambda \equiv \frac{\omega}{v}\sqrt{1-\epsilon\frac{v^2}{c^2}}$

Field of a moving particle in a dielectric IX

- The B-field can be computed in a similar manner:

$$
\begin{aligned}
\vec{B}(\vec{k},\omega) &= i\frac{\epsilon(\omega)}{c}\vec{k}\times\vec{v}\,\Phi(\vec{k},\omega) = i\epsilon\frac{v}{c}(-k_x\hat{y}+k_y\hat{x})\Phi(\vec{k},\omega) \\
&= \frac{i}{(2\pi)^{3/2}}\int\int\int dk_x dk_y dk_z\left[-k_x\hat{y}+\cancel{k_y\hat{x}}\right]\frac{\epsilon v}{c}\frac{2q\delta(\omega-\vec{k}.\vec{v})}{\epsilon[k^2-\epsilon\frac{\omega^2}{c^2}]}e^{i\vec{k}.\vec{x}} \\
&= -\frac{i}{(2\pi)^{3/2}}2q\frac{v}{c}\hat{y}\int\int\int dk_x dk_y dk_z k_x\frac{\delta(\omega-k_z v)}{k_x^2+k_y^2+k_z^2-\epsilon\frac{\omega^2}{c^2}}e^{i\vec{k}.\vec{x}}.
\end{aligned}
$$

- Integrating over dk_z gives

$$
\vec{B} = -\frac{i}{(2\pi)^{3/2}}2q\frac{v}{c}\hat{y}\int\int dk_x dk_y k_x\frac{e^{ik_x b}}{k_x^2+k_y^2+\lambda^2} = -\frac{i}{(2\pi)^{3/2}}2q\frac{1}{c}\hat{y}\frac{dI}{db}.
$$

- This is the same as x-component of the E-field so

$$
\boxed{\vec{B} = -\frac{i}{(2\pi)^{3/2}}2q\frac{v}{c}\hat{y}\int\int dk_x dk_y k_x\frac{e^{ik_x b}}{k_x^2+k_y^2+\lambda^2} = -\frac{i}{(2\pi)^{3/2}}2q\frac{1}{c}\hat{y}\frac{dI}{db}.}
$$

Energy loss in a dielectric I

- The e.m. field energy flowing out of a cylinder surface of radius *b* extending from -∞ to +∞ in *z* is

$$\frac{d\mathcal{E}_f}{dz} = 2\pi b \int_{-\infty}^{\infty} \overrightarrow{S}.\hat{n}dt = 2\pi b \frac{1}{4\pi} \int_{-\infty}^{+\infty} (\overrightarrow{E} \times \overrightarrow{B}).\hat{n}dt$$

- We have
$$\begin{aligned}(\overrightarrow{E} \times \overrightarrow{B}).\hat{n} &= [(E_x\hat{x} + E_z.\hat{z}) \times B_y\hat{y}].\hat{n} \\ &= (E_x B_y \hat{z} - E_z B_y \hat{x}).\hat{n} = -E_z B_y\end{aligned}$$

- So
$$\begin{aligned}\frac{d\mathcal{E}_f}{dz} &= -\frac{b}{2}\int_{-\infty}^{\infty} E_z B_y dt \\ &= -\frac{b}{4\pi}\int_{-\infty}^{\infty} dt \left[\int_{-\infty}^{\infty} d\omega E_z(\omega)e^{-i\omega t}\right]\left[\int_{-\infty}^{\infty} d\omega' B_y(\omega')e^{-i\omega' t}\right] \\ &= -\frac{b}{2}\int_{-\infty}^{+\infty} E_z(\omega)B_y(-\omega)d\omega = -\frac{b}{2}\int_{-\infty}^{+\infty} E_z(\omega)B_y^*(\omega)d\omega \\ &= \mathcal{R}e\left(-b\int_0^{+\infty} E_z(\omega)B_y^*(\omega)d\omega\right)\end{aligned}$$

Energy loss in a dielectric II

- Expliciting the E and B field in the latter equation gives

$$\frac{d\mathcal{E}_f}{dz} = -b\mathcal{R}e\left\{\int_0^{+\infty} d\omega\left[-i\sqrt{\frac{2}{\pi}}\frac{q}{v}\frac{\omega}{v}(1/\epsilon-\beta^2)K_0(\lambda b)\right]\right.$$
$$\left.\times\left[\sqrt{\frac{2}{\pi}}\frac{q}{c}\lambda^* K_1(\lambda^* b)\right]\right\}$$

$$\lambda \equiv \frac{\omega}{v}\sqrt{1-\epsilon\frac{v^2}{c^2}}$$

$$\begin{aligned}\frac{d\mathcal{E}_f}{dz} &= \mathcal{R}e\left\{\int_0^{+\infty} d\omega\frac{2}{\pi}\frac{q^2}{v^2}[i\omega(1/\epsilon-\beta^2)\lambda^* b]K_0(\lambda b)K_1(\lambda^* b)\right\}\\ &= \frac{2}{\pi}\frac{q^2}{v^2}\mathcal{R}e\left\{\int_0^{+\infty} d\omega(i\omega\lambda^* b)(1/\epsilon-\beta^2)K_0(\lambda b)K_1(\lambda^* b)\right\}\end{aligned}$$

- First derived by Enrico Fermi. Energy loss occurs if either λ or ε are complex