"Scattering" of light on charged particle

- A charged particle has no surface, so "scattering" of light is a metaphor.

- Quantum view: collision photon/electron

\[
\begin{align*}
P_\gamma + P_\gamma &= P_{\gamma'} + P_{e-}, \\
P_{e-} &= P_{e-} + P_\gamma - P_{\gamma'}
\end{align*}
\]

- Then

\[
m^2c^2 = (P_{e-} + P_\gamma - P_{\gamma'})(P_{e-},\alpha + P_{\gamma},\alpha - P_{\gamma'},\alpha)
\]

- and

\[
P_{\gamma'}P_{e-},\alpha - P_{\gamma'}P_{e-},\alpha - P_{\gamma'}P_{e-},\alpha = 0
\]
“Scattering” of light on charged particle

- We have

\[ P_{e-,\alpha} = mc(E_\gamma/2) - \vec{p}_{e-} \cdot \vec{p}_\gamma = mE_\gamma \]
\[ P_\gamma^\alpha P_\gamma^\alpha = \frac{E_\gamma E_\gamma'}{c} - \vec{p}_\gamma \cdot \vec{p}_\gamma' = -p_\gamma p_\gamma' \cos \theta + \frac{E_\gamma E_\gamma'}{c^2} \]
\[ P_{e-,\alpha} P_\gamma^\alpha = mE_\gamma. \]

- Taking \( E_\gamma \equiv \frac{hc}{\lambda} \) and similarly for \( \gamma' \), we have

\[ \lambda - \lambda' = \frac{h}{mc} (1 - \cos \theta). \]

This is the usual Compton scattering formula. The non-relativistic limit yields \( \lambda = \lambda' \), which is the regime of Thomson scattering.
Linear Thomson Scattering: cross section I

- Cross section in a figure-of-merit.
  \[ \sigma \equiv \frac{E \text{ radiated/time/solid angle}}{\text{incident flux/unit area/time}}. \]

- Since the electron is at rest:
  \[
  \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c} \left| \hat{n} \times [(\hat{n} - \vec{\beta}) \times \vec{\beta}] \right|^2
  \rightarrow_{\beta \to 0} \frac{e^2}{4\pi c} \left| \hat{n} \times [\hat{n} \times \vec{\beta}] \right| = \frac{e^2}{4\pi c} \hat{\beta}^2 \sin^2 \Theta
  \]
  where \( \Theta \angle (\hat{n}, \vec{\beta}) \)

- So finally
  \[
  \frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3} \alpha^2 \sin^2 \Theta.
  \]
Linear Thomson Scattering: cross section II

• Note that in the non-relativistic limit

\[ \frac{dP(t)}{d\Omega} = \frac{dP(t')}{d\Omega} \]

• Let’s now specialize our problem and consider a plane wave:

\[ \vec{E}(\vec{x}, t) = \hat{e} E_0 e^{i(k \cdot \vec{x} - \omega t)} \]

• The acceleration is therefore given by:

\[ \vec{a}(t) = \frac{e}{m} \hat{e} E_0 e^{i(k \cdot \vec{x} - \omega t)} \]

• We ignore the B-field associated to the plane wave because we assume \( \beta = 0 \)
Linear Thomson Scattering: cross section III

- Given the geometry of the problem we have

\[ \hat{e} = \cos \psi \hat{x} + \sin \psi \hat{y} \]
\[ \hat{n} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \]
\[ \hat{n} \cdot \vec{a} = a s_\theta (c_\psi c_\phi + s_\psi s_\phi) = a s_\theta c_{\psi - \phi} \]
\[ = a \sin \theta \cos (\psi - \phi) = a \cos \Theta. \]

- Thus

\[ \sin^2 \Theta = 1 - \sin^2 \theta \cos^2 (\psi - \phi). \]

- So the time averaging gives

\[ \langle a^2 \sin^2 \Theta \rangle_t = \frac{1}{2} \left( \frac{e E_0}{m} \right)^2 [1 - \sin^2 \theta \cos^2 (\psi - \phi)]. \]
Linear Thomson Scattering: cross section IV

• Assume the incoming wave is unpolarized then

\[ \langle \cos^2(\psi - \phi) \sin^2 \theta \rangle_\psi = \frac{1}{2} \sin^2 \theta. \]

• So

\[ \langle a^2 \sin^2 \Theta \rangle_{t,\psi} = \frac{1}{2} \left( \frac{eE_0}{m} \right)^2 [1 - \frac{1}{2} \sin^2 \theta] \]

• So finally

\[ \langle \frac{dP}{d\Omega} \rangle_{t,\psi} = \frac{cE_0^2}{16\pi} \left( \frac{e^2}{mc^2} \right)^2 [1 + \cos^2 \theta] = \frac{c}{16\pi} r_e (1 + \cos^2 \theta). \]

• The Poynting vector is given by

\[ S = \frac{c}{8\pi} \overrightarrow{E} \times \overrightarrow{H}^* \]
Linear Thomson Scattering: cross section V

- The time averaged power per unit of area is

\[ \frac{dP}{d\sigma} = S = \frac{c}{8\pi} E_0^2. \]

- And so the cross-section is

\[ \frac{d\sigma}{d\Omega} = \frac{d\sigma}{dP} \frac{dP}{d\Omega} = \frac{c r_e^2}{16\pi} E_0^2 \left[1 + \cos^2 \theta\right] \]

\[ = \frac{1}{2} r_e^2 (1 + \cos^2 \theta) \]

this is the scattering Thomson cross section. The integrated cross section is:

\[ \sigma = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin \theta \frac{d\sigma}{d\Omega} = \frac{16\pi}{3} \frac{1}{2} r_e^2 \]

\[ = \frac{8\pi}{3} r_e^2. \]
Notes on Nonlinear Thomson Scattering I

1. Classical Thomson scattering, the scattering of low-intensity light by e-, is a linear process: it does not change the frequency of the radiation;

2. The magnetic-field component of light is not involved.

3. But if the light intensity is extremely high (~10^{18} W/cm^2), the electrons oscillate during the scattering process with velocities approaching c.

4. In this relativistic regime, the effect of the magnetic and electric fields on the electron motion should become comparable
Notes on Nonlinear Thomson Scattering II

- First experimentally observed in 1998
Case of a bounded electron I

- Compton and Thomson scatterings apply to free electrons
- What happen if an electron is bounded (i.e. to an atom)?

- We assume the equation of motion of the bounded electron to be described by:

\[
\ddot{\vec{a}} + \Gamma \dot{\vec{v}} + \omega_0^2 \vec{x} = \frac{q}{m} \vec{E}.
\]

- As before we take

\[
\vec{E} = \epsilon E_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}
\]

and

\[
\vec{x} = \vec{x}_0 e^{-i\omega t}.
\]

then

\[
(-\omega^2 - i\omega\Gamma + \omega_0^2) \vec{x}_0 = \frac{\epsilon q}{m} E_0 e^{i\vec{k} \cdot \vec{x}}
\]
Case of a bounded electron II

• We further assume that \( \vec{k} \cdot \vec{x} = 0 \) i.e. \(|x| \ll \lambda\).

• then

\[
\overrightarrow{x}_0 \simeq \frac{\frac{e}{m} E_0}{\omega_0^2 - \omega^2 - i\omega \Gamma} \hat{\epsilon},
\]

• so

\[
\overrightarrow{a} = -\omega^2 \overrightarrow{x} \Rightarrow |a^2| = \left( \frac{e}{m} E_0 \right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}.
\]

• And finally

\[
|a^2| = \frac{\left( \frac{e}{m} E_0 \right)^2}{\left( \frac{\omega_0}{\omega} \right)^2 + \left[ \left( \frac{1}{\omega} \right)^2 - 1 \right]^2}.
\]

• Same as before but the denominator is different
Case of a bounded electron III

- The radiated power is therefore

\[
\langle dP \rangle = \frac{c}{16\pi r_e^2 E_0^2} \frac{1 + \cos^2 \theta}{\left[ \left( \frac{\omega}{\omega_0} \right)^2 - 1 \right]^2 + \left( \frac{\Gamma}{\omega} \right)^2},
\]

- and the associated cross section is

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \frac{1 + \cos^2 \theta}{\left[ \left( \frac{\omega}{\omega_0} \right)^2 - 1 \right]^2 + \left( \frac{\Gamma}{\omega} \right)^2}.
\]

- \(\omega \ll \omega_0\) and \(\omega \ll \Gamma\) corresponds to Thomson scattering

- \(\omega \ll \omega_0\) and \(\omega \gg \Gamma\) gives the Raleigh scattering cross section

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \left( \frac{\omega}{\omega_0} \right)^4 \left[ 1 + \cos^2 \theta \right] \propto \omega^4
\]

The reason why the sky is blue…