

Case of Circular motion: angular spectral fluence

- Finally the *angular spectral fluence* takes the form

$$\begin{aligned}\frac{d^2I}{d\Omega d\omega} &= |A_{\parallel}(\omega)|^2 + |A_{\perp}(\omega)|^2 \\ &= \frac{q^2}{3\pi^2 c} \left(\frac{\omega}{\omega_0} \right)^2 (\gamma^{-2} + \theta^2)^2 \left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2} + \theta^2} K_{1/3}^2(\xi) \right]\end{aligned}$$

- or, introducing $\xi = \frac{1}{3} \frac{\omega}{\omega_0} [\gamma^{-2} + \theta^2]^{3/2} \equiv \frac{1}{2} \frac{\omega}{\omega_e} [1 + \gamma^2 \theta^2]^{3/2}$:

$$\frac{d^2I}{d\Omega d\omega} = \frac{3q^2}{\pi^2 c} \xi^2 \frac{1}{\gamma^{-2} + \theta^2} \left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2} + \theta^2} K_{1/3}^2(\xi) \right]$$

Angle-integrated spectrum I

- Last Lesson we noted
 - High frequency radiation occupies angles $\theta < \gamma^{-1}$ ($\ll \gamma^{-1}$ for $\omega \ll \omega_c$)
 - Low frequency ($\omega \ll \omega_c$) we have

$$\begin{aligned}\frac{dI}{d\omega} &\simeq 2\pi\theta_c \left[\frac{d^2I}{d\omega d\Omega} \right]_{\theta=0} \\ &= \frac{2\pi}{\gamma} \left(\frac{2\omega_c}{\omega} \right)^{1/3} \frac{3}{\pi^2} \frac{q^2}{c} \gamma^2 [\xi^2 K_{2/3}^2(\xi)]_{\theta=0}.\end{aligned}$$

where the critical angle was defined as $\theta_c = \frac{1}{\gamma} \left(\frac{2\omega_c}{\omega} \right)^{1/3}$

Angle-integrated spectrum II

- But $\xi(0) = \frac{\omega}{2\omega_c} \ll 1$ so

$$[\xi K_{2/3}(\xi)]_{\theta=0}^2 \simeq \left[\frac{\Gamma(2/3)}{2^{1/3}} \right]^2 [\xi(0)]^{2/3} \simeq \left(\frac{\omega}{2\omega_c} \right)^{2/3}.$$

- And

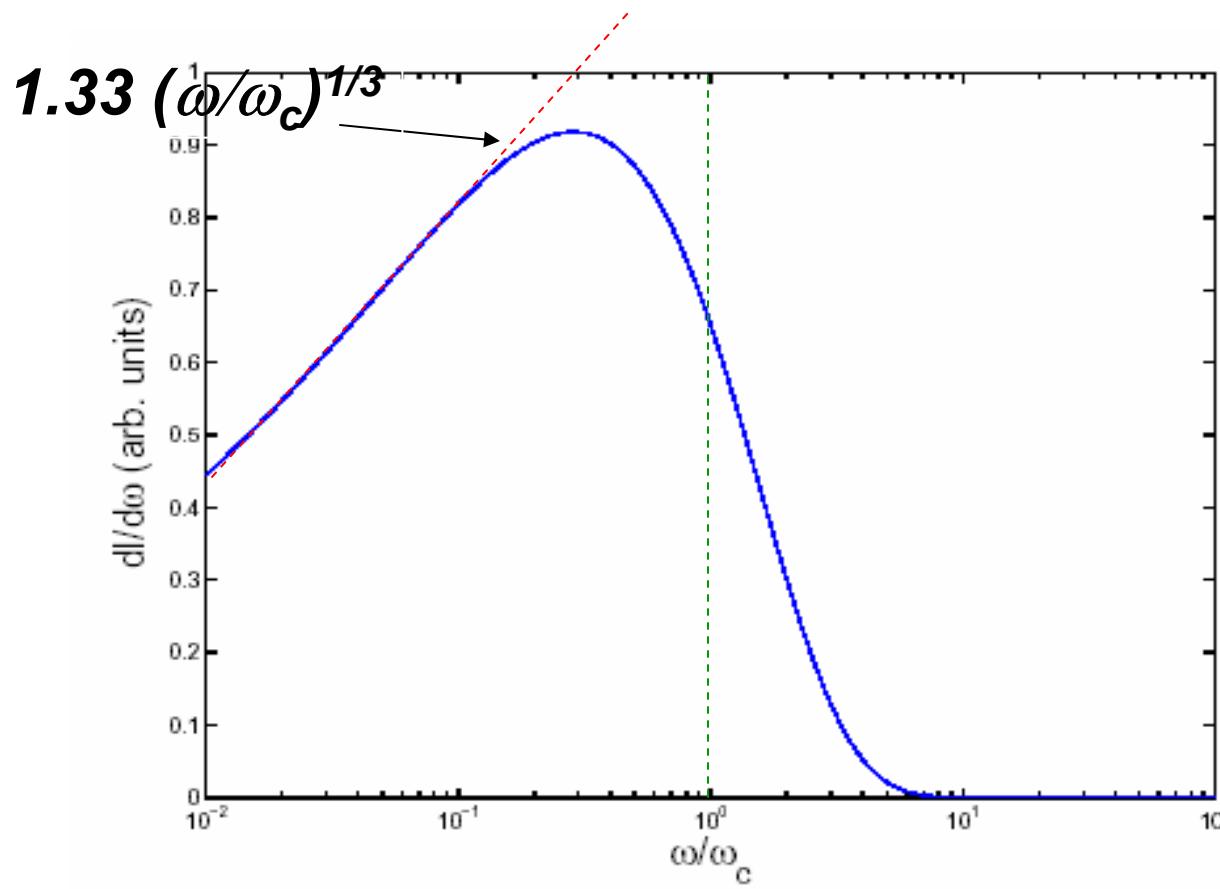
$$\frac{dI}{d\omega} \simeq \frac{6}{\pi} \frac{q^2}{c} \gamma \left(\frac{\omega}{2\omega_c} \right)^{1/3} = \frac{6}{\pi} \frac{q^2}{c} \gamma \left(\frac{\omega}{3\gamma^3 \omega_0} \right)^{1/3} \propto \omega^{1/3}$$

- Broad spectrum γ -independent
- Can do a similar asymptotic expansion for the high frequency region of the angle-integrated spectrum – let as an exercise...

Angle-integrated spectrum III

- Derived by *Schwinger* to be

$$\frac{dI}{d\omega} = \sqrt{3} \frac{q^2}{c} \gamma \frac{\omega}{\omega_c} \int_{\omega/\omega_c}^{+\infty} dx K_{5/3}(x).$$



Angular distribution (frequency integrated) I

- Need to evaluate $\int_0^\infty d\omega \frac{d^2I}{d\omega d\Omega}$
- Change of variable $\xi = \frac{1}{3} \frac{\omega}{\omega_0} [\gamma^{-2} + \theta^2]^{3/2}$ gives:

$$\begin{aligned}\frac{dI}{d\Omega} &= \frac{3q^2}{\pi^2 c} \frac{3\omega_0}{[\gamma^{-2} + \theta^2]^{5/2}} \int_0^\infty \xi^2 \left\{ K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2} + \theta^2} K_{1/3}^2(\xi) \right\} d\xi \\ &= \frac{9}{\pi^2} \frac{q^2}{c} \frac{\gamma^5 \omega_0}{[1 + (\gamma\theta)^2]^{5/2}} \left[\frac{7\pi^2}{144} + \frac{5\pi^2}{144} \frac{\gamma^2 \theta^2}{(1 + \gamma^2 \theta^2)} \right]\end{aligned}$$

- Where the identity

$$\int_0^\infty \omega^2 K_\mu^2(a\omega) d\omega = \frac{\pi^2}{32a^3} \frac{1 - 4\mu^2}{\cos(\pi\mu)}$$

Angular distribution II

- So finally we have

$$\frac{dI}{d\Omega} = \frac{7}{16} \frac{q^2}{c} \frac{\gamma^5 \omega_0}{[\gamma^{-2} + \theta^2]^{5/2}} \left[1 + \frac{5}{7} \frac{\gamma^2 \theta^2}{(1 + \gamma^2 \theta^2)} \right]$$

- Let's do a consistency check and consider the total radiated energy

$$\Delta W = \int d\Omega \frac{dI}{d\Omega} = 2\pi \int d\theta \frac{dI}{d\theta}$$

!!!!

- then

$$\begin{aligned}\Delta W &= 2\pi \int_0^\infty d\theta \frac{dI}{d\theta} \\ &= \frac{7\pi q^2}{8c} \gamma^5 \omega_0 \int_{-\infty}^{+\infty} \left[\frac{1}{(1 + \gamma^2 \theta^2)^{5/2}} + \frac{5}{7} \frac{\gamma^2 \theta^2}{(1 + \gamma^2 \theta^2)^{7/2}} \right] \\ &= \frac{7\pi q^2}{8c} \gamma^5 \omega_0 \left[\frac{4}{3\gamma} + \frac{4}{15\gamma} \right] = \frac{7\pi q^2}{6c} \gamma^4 \omega_0 \left[1 + \frac{1}{7} \right]\end{aligned}$$

σ

π

Total power

- So finally we have $\Delta W = \frac{4\pi}{3} \frac{q^2}{c} \gamma^4 \omega_0$
with $\omega_0 = c/r$.
- In agreement with the P_{circ} we derived at the beginning of chapter 4:

$$\begin{aligned}\Delta W_{circ} &= P_{circ} \frac{2\pi}{\omega_0} = \left(\frac{2}{3} \frac{q^2}{m^2 c^3} \gamma^2 \dot{p}^2 \right) \frac{2\pi}{\omega_0} \\ &= \frac{2}{3} \frac{q^2}{m^2 c^3} \gamma^2 (\gamma m r \omega_0^2)^2 \frac{2\pi}{\omega_0} = \frac{4\pi}{3} \frac{q^2}{c^3} \gamma^4 r^2 \omega_0^3 \\ &= \frac{4\pi}{3} \frac{q^2}{r} \gamma^4.\end{aligned}$$

Case of periodic circular motion I

- Up to now we considered the steady case circular motion (no transient) and computed instantaneous spectra.
- If the motion is periodic

$$\vec{A}(t) = \sqrt{\frac{c}{4\pi}} [E \vec{E}]_{ext} = \sum_{n=-\infty}^{n=+\infty} \vec{A}_n e^{-in\omega_0 t},$$

$$\vec{A}_n = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \vec{A}(t) e^{in\omega_0 t}$$

Case of periodic circular motion II

- And we can show (following the steps we did for the instantaneous case) that

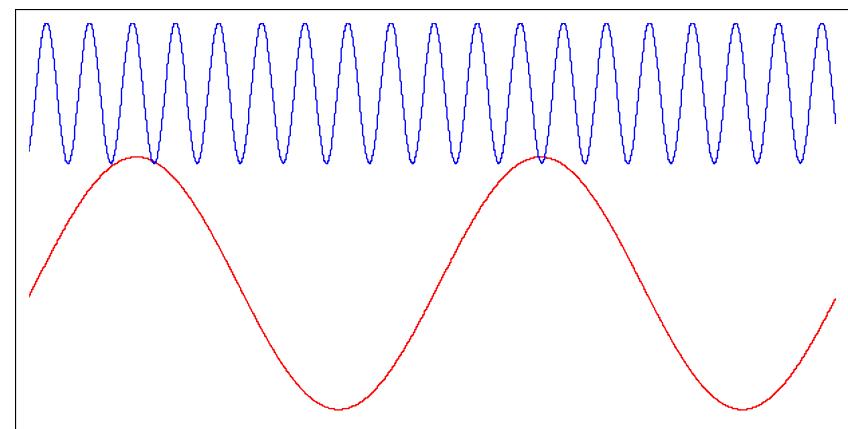
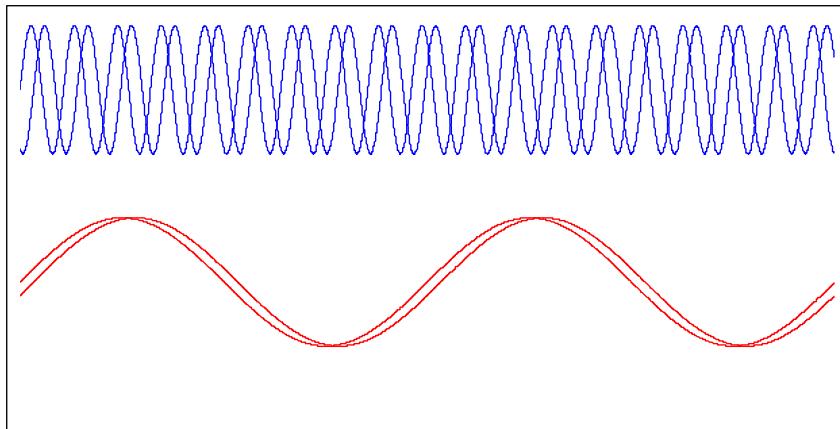
$$\vec{A}_n = \sqrt{2\pi} \frac{q}{2\pi\sqrt{2c}} \frac{\omega_0}{2\pi} (-in\omega_0) \int_0^{2\pi/\omega_0} dt \hat{n} \times (\hat{n} \times \vec{\beta}) e^{in\omega_0(t-\hat{n} \cdot \vec{r})}$$

**Same general form as for instantaneous motion
a factor $\sqrt{2\pi}$ come from the difference in
normalization between Fourier transforms and
Fourier series...**

- The spectrum is now discrete at $\omega = n \omega_0$

Multiparticle Coherence I

- In real life a bunch consists of many particle so one may wonder how does this affect all the results previously derived
- It depends on the frequency (wavelength) of observation!



Electric field radiated by two particle at “small” (right) and “long” wavelength (compared to the particle spacing)

Multiparticle Coherence II

- Let's compute the total field generated by an ensemble of N electrons.

$$E_N(\mathbf{P}) = \sum_k E_k(\mathbf{P}) e^{-i\omega\delta t_k}$$

- Let's assume the single particle field have the same value at the observation P. Then spectral angular fluence is

$$\left| \frac{d^2W}{d\omega d\Omega} \right|_N \propto |E_N(\mathbf{P})|^2 \equiv \left| \frac{d^2W}{d\omega d\Omega} \right|_1 \left| \sum_k e^{-i\omega\delta t_k} \right|^2$$

- Let's evaluate the multiplicative factor

Multiparticle Coherence III

- We have.

$$\left| \sum_j e^{-i\omega\delta t} \right|^2 = \left(\sum_j e^{-i\omega\delta t_j} \right) \left(\sum_k e^{+i\omega\delta t_k} \right) = N + \left(\sum_j \sum_{k \neq j} e^{+i\omega\delta t_k} e^{-i\omega\delta t_j} \right)$$

- Introducing the line charge density $\Lambda(t)$ we can write

$$\left| \sum_j e^{-i\omega\delta t} \right|^2 = N + N(N-1)\tilde{\Lambda}(\omega)$$

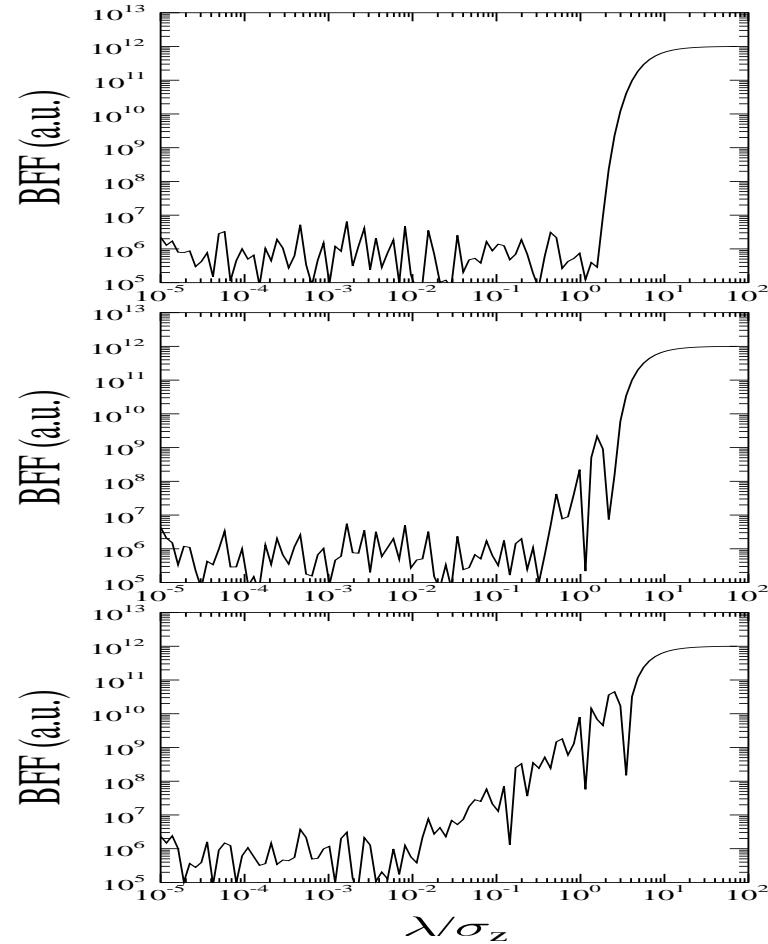
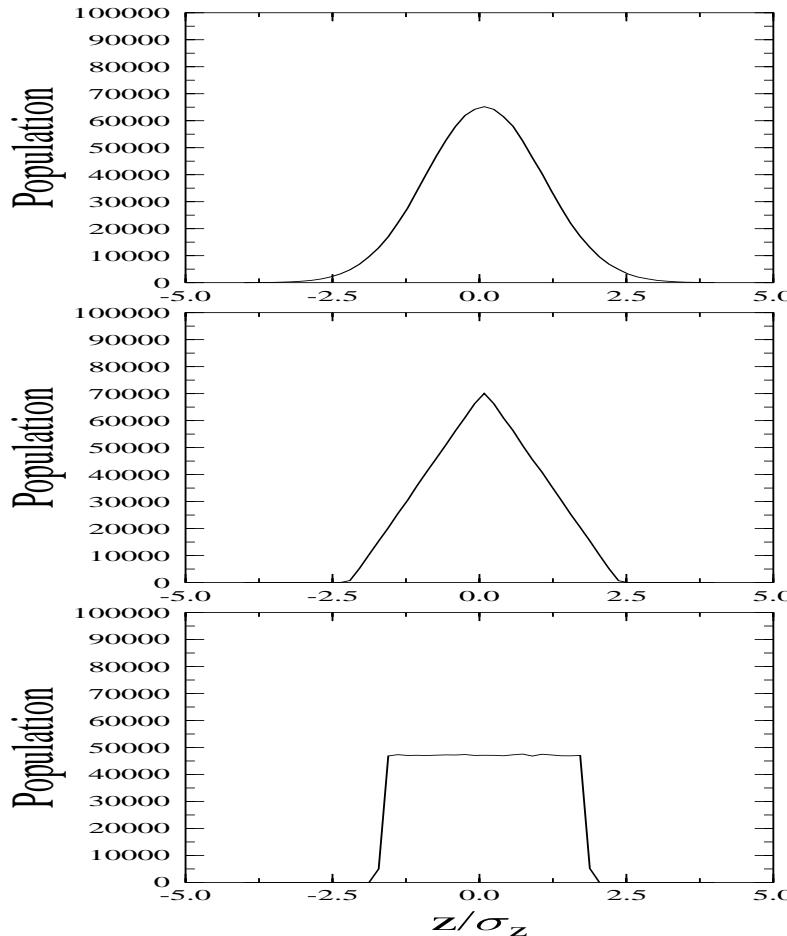
$$\approx N + N^2 \tilde{\Lambda}(\omega)$$

Typically $N \gg 1$

*Fourier transform of
the line charge density*

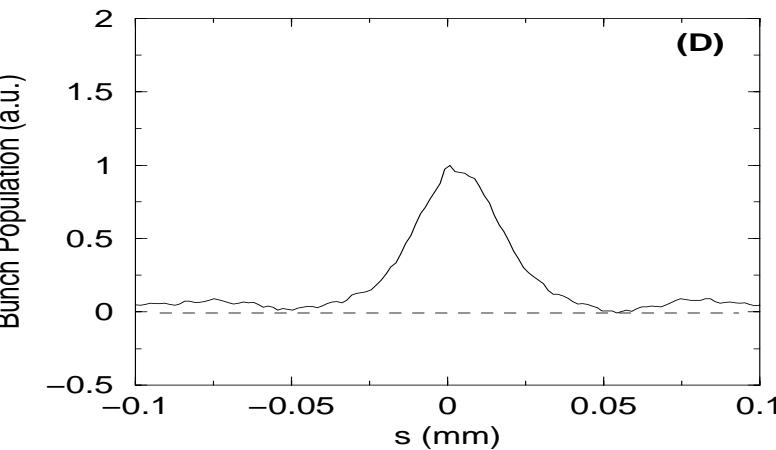
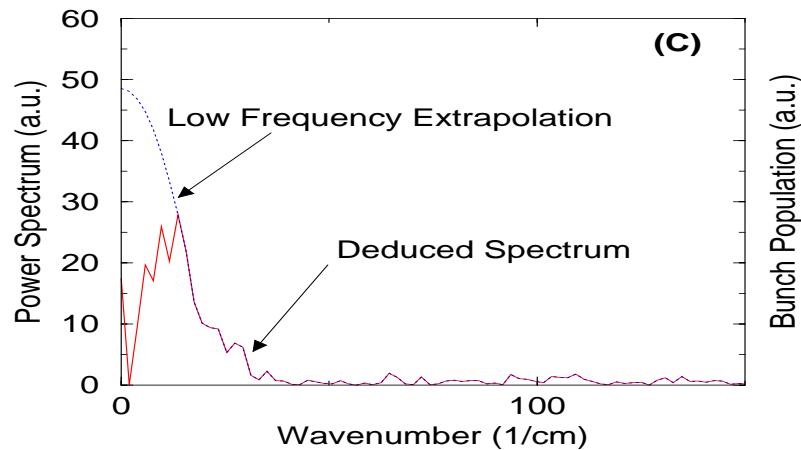
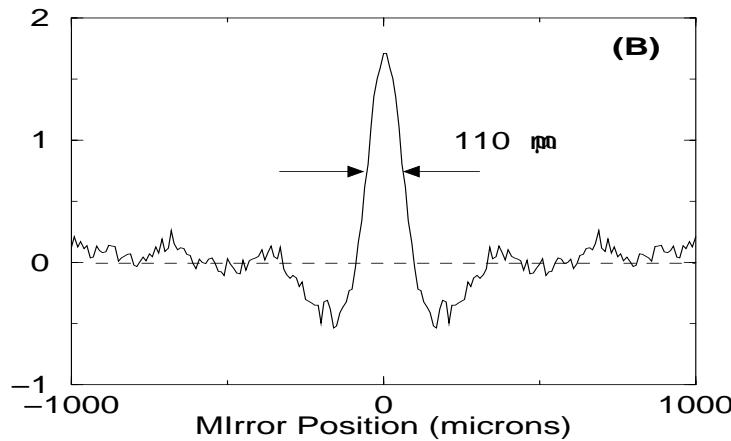
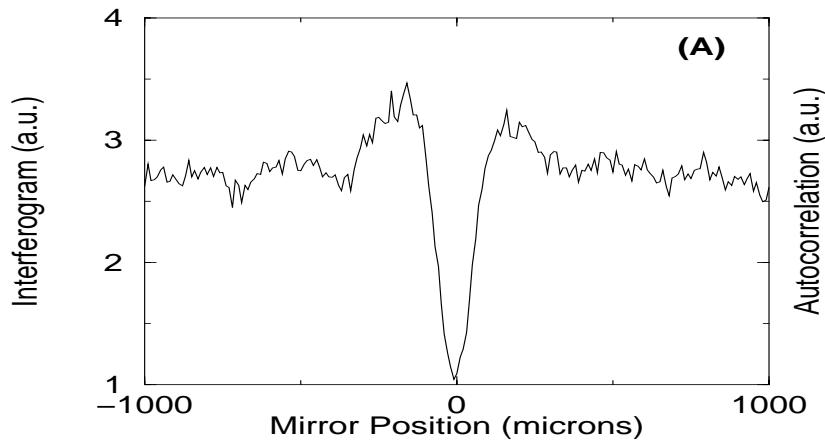
Multiparticle coherence IV

BBF measurement (easy!) can provide information on the bunch longitudinal charge distribution



Multiparticle coherence V

Example of real measurement...



Multi-particle coherence: example of CSR

Beam pipe induced frequency cut-off

Coherent Synchrotron Radiation

CSR enhancement

