Radiation Spectrum I

- Starting from the radiation field, we have

$$\frac{dP(t)}{d\Omega} = \frac{1}{\kappa(t')}\frac{dP(t')}{d\Omega}$$
$$= \frac{q^2}{4\pi c}\left[\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2}{\kappa^6}\right]_{ret} \equiv |\vec{A}(t)|^2$$

- Where *A* is defined as

$$\vec{A}(t) = \sqrt{\frac{c}{4\pi}}[R\vec{E}]_{ret}$$

- To obtain the power spectrum we need to work in the frequency domain

Radiation Spectrum II

- Let's define the "symmetrized Fourier transform"

$$\overrightarrow{A}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt \overrightarrow{A}(t) e^{i\omega t},$$

$$\overrightarrow{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \overrightarrow{A}(\omega) e^{-i\omega t},$$

- Parseval's theorem states that

$$\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} dt |\overrightarrow{A}(t)|^2 = \int_{-\infty}^{+\infty} d\omega |\overrightarrow{A}(\omega)|^2$$

- Since A is a real function

$$\frac{dW}{d\Omega} = 2 \int_{0}^{\infty} d\omega |\overrightarrow{A}(\omega)|^2$$

Radiation Spectrum III

- The radiation spectrum is therefore

$$\frac{d^2 I(\hat{n},\omega)}{d\Omega d\omega} = 2|A(\omega)|^2$$

- Starting from *A(t)*

$$\vec{A}(t) = \frac{q}{\sqrt{4\pi c}}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{\kappa^3}\right]_{ret}$$

- *A(ω)* is

$$\vec{A}(\omega) = \frac{q}{2\pi\sqrt{2c}}\int_{-\infty}^{+\infty} dt \left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{\kappa^3}\right]_{ret} e^{i\omega t}$$

Radiation Spectrum IV

- This must be evaluated at the retarded time t'

$$\overrightarrow{A}(\omega) = \frac{q}{2\pi\sqrt{2c}} \int_{-\infty}^{+\infty} dt' \frac{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \dot{\overrightarrow{\beta}}]}{\kappa^2} e^{i\omega(t' + \frac{R(t')}{c})}$$

- Note that in the far-field regime

$\hat{n} = \frac{\overrightarrow{x} - \overrightarrow{r}(t')}{|\overrightarrow{x} - \overrightarrow{r}(t')|} \simeq \hat{x}$ is constant in time

and $R = x - \overrightarrow{r}.\hat{n} + \mathcal{O}(1/x).$

argument of the exponential in the far-field is

ignore

$$\Xi = i\omega[t' + \frac{R(t)}{c}] = \cancel{i\omega x} + i\omega[t' - \frac{\hat{n}.\overrightarrow{r}(t')}{c}] \Rightarrow \boxed{\Xi(t') = i\omega[t' - \frac{\hat{n}.\overrightarrow{r}(t')}{c}],}$$

Radiation Spectrum V

- We finally have

$$\vec{A}(\omega) = \frac{q}{2\pi\sqrt{2c}} \int_{-\infty}^{+\infty} dt \frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{\kappa^2} e^{\Xi(t)},$$

- And the corresponding *angular spectral fluence* distribution

$$\boxed{\frac{d^2 I(\hat{n},\omega)}{d\Omega d\omega} = 2A^2(\omega) = \frac{q^2}{4\pi^2 c} \left| \int_{-\infty}^{+\infty} dt \frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{\kappa^2} e^{\Xi(t)} \right|^2.}$$

- This is the most general formula for computing the angular spectral fluence.

- JDJ re-write the vector part of the integrand as a total time derivative

Radiation Spectrum VI

- Consider

$$\frac{\hat{n} \times (\hat{n} \times \vec{\beta})}{\kappa},$$

- Then the time derivative is

$$\frac{d}{dt}\left[\frac{\hat{n} \times (\hat{n} \times \vec{\beta})}{\kappa}\right] = \frac{(-\dot{\kappa}\hat{n} + (1-\kappa)\dot{\hat{n}} - \dot{\vec{\beta}})\kappa - \dot{\kappa}[(1-\kappa)\hat{n} - \vec{\beta}]}{\kappa^2}$$

- We have

$$\begin{aligned} \frac{d}{dt}[...] &= \frac{1}{\kappa^2}\left\{[(\dot{\vec{\beta}}.\hat{n})\hat{n} - 0 - \dot{\vec{\beta}}]\kappa + (\dot{\vec{\beta}}.\hat{n})[(1-\kappa)\hat{n} - \vec{\beta}]\right\} \\ &= \frac{1}{\kappa^2}\left\{-\dot{\vec{\beta}}\kappa + (\dot{\vec{\beta}}.\hat{n})(\hat{n} - \vec{\beta})\right\} + \mathcal{O}(1/R) = \underbrace{\frac{1}{\kappa^2}\left\{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]\right\}}. \end{aligned}$$

Radiation Spectrum VII

- So we can do an integration-by-part

$$
\begin{aligned}
\overrightarrow{A}(\omega) &= \frac{q}{2\pi\sqrt{2c}} \int_{-\infty}^{+\infty} dt \frac{d}{dt} \left[\frac{\hat{n} \times (\hat{n} \times \overrightarrow{\beta})}{\kappa} \right] e^{\Xi(t)} \\
&= \frac{q}{2\pi\sqrt{2c}} \left\{ \underbrace{\left| \left[\frac{\hat{n} \times (\hat{n} \times \overrightarrow{\beta})}{\kappa} \right] e^{\Xi(t)} \right|_{-\infty}^{+\infty}}_{\substack{=0 \text{ in principle} \\ \text{but care must be taken} \\ \text{to verify this in practice}}} - i\omega \int_{-\infty}^{+\infty} dt \left[\hat{n} \times (\hat{n} \times \overrightarrow{\beta}) \right] e^{\Xi(t)} \right\}
\end{aligned}
$$

=0 in principle
but care must be taken
to verify this in practice

Radiation Spectrum VIII

- So finally the *angular spectral fluence* is

$$\boxed{\frac{d^2 I(\hat{n},\omega)}{d\Omega d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} dt[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})]e^{i\omega[t'-\frac{\hat{n}.\overrightarrow{r}(t)}{c}]}\right|^2}$$

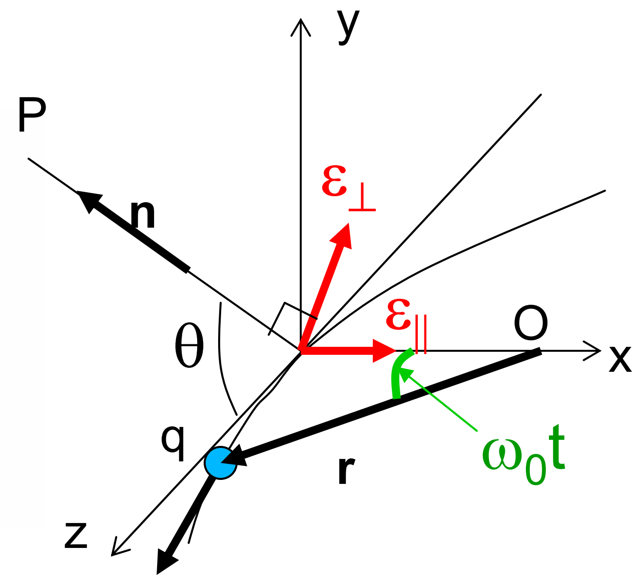
note that $[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})] = \beta\sin\theta = |\hat{n}\times\overrightarrow{\beta}|$ where $\theta = \angle(\hat{n},\overrightarrow{\beta})$.

- Here we could also start introducing the polarization, but we will do this for the special case of circular motion

Case of Circular motion I

- Introduce the polarization unit vectors ε's

$$\begin{aligned}
\hat{n} &= \sin\theta\hat{y} + \cos\theta\hat{z}, \\
\overrightarrow{\beta} &= \beta[\sin(\omega_0 t')\hat{x} + \cos(\omega_0 t')\hat{z}], \\
\hat{\epsilon}_{||} &= \hat{x}, \\
\hat{\epsilon}_{\perp} &= \hat{n}\times\hat{x} = -\sin\theta\hat{z} + \cos\theta\hat{y}.
\end{aligned}$$



- Then

$$\begin{aligned}
\hat{n}\times(\hat{n}\times\overrightarrow{\beta}) &= (\hat{n}.\overrightarrow{\beta})\hat{n} - \overrightarrow{\beta} \\
&= \beta[c_{\omega_0 t}c_\theta\hat{y} + c_{\omega_0 t}(c_\theta^2 - 1)\hat{z} - c_{\omega_0 t}\hat{x} \\
&= \beta[-s_{\omega_0 t}\hat{\epsilon}_{||} + c_{\omega_0 t}s_\theta\hat{\epsilon}_{\perp}]
\end{aligned}$$

Case of Circular motion II

- The argument of the exponential writes

$$\hat{n}.\overrightarrow{r} = r\cos\theta\cos(\pi/2 - \omega_0 t') = r\sin(\omega_0 t')\cos\theta$$

$$\Xi = i\omega(t' - \frac{\hat{n}.\overrightarrow{r}}{c}) = \omega[t' - \frac{r}{c}\sin(\omega_0 t')\cos\theta]$$

- If an observer catches an impulse from the charge *q*: θ is small and the pulse originated close to *t = 0*, so under these approximations

$$\lim_{\theta \ll 1, \omega_0 t \ll 1} \hat{n} \times (\hat{n} \times \overrightarrow{\beta}) = \beta(-\omega_0 t \hat{\epsilon}_{||} + \theta \hat{\epsilon}_{\perp})$$

and

$$\begin{aligned} \lim_{\theta \ll 1, \omega_0 t \ll 1} \frac{1}{i}\Xi &= \omega\left\{t' - \frac{r}{c}[\omega_0 t' - \frac{1}{6}(\omega_0 t')^3](1 - \frac{\theta^2}{2})\right\} \\ &= \omega\left\{(1-\beta)t' + \frac{\beta t'}{2}\theta^2 + \frac{1}{6}\frac{r}{c}(\omega_0 t')^3\right\} \\ &= \frac{\omega t'}{2}(\gamma^{-2} + \beta\theta^2) + \frac{\omega\beta}{6\omega_0}(\omega_0 t')^3. \end{aligned}$$

Case of Circular motion III

- The angular spectral fluence is

$$\frac{d^2 I}{d\Omega d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} dt \beta(-\omega_0 t \hat{\epsilon}_{||} + \theta \hat{\epsilon}_{\perp}) e^{\Xi}\right|^2$$
$$= \left| \underbrace{- A_{||}(\omega)\hat{\epsilon}_{||}}_{\sigma} + \underbrace{A_{\perp}(\omega)\hat{\epsilon}_{\perp}}_{\pi}\right|^2$$

- This displays the two polarizations.

$$\begin{pmatrix} A_{||} \\ A_{\perp} \end{pmatrix} = \frac{q\omega}{2\pi\sqrt{c}}\int_{-\infty}^{+\infty} dt \begin{pmatrix} \omega_0 t \\ \theta \end{pmatrix} e^{i\frac{\omega}{2}[(\gamma^{-2}+\theta^2)t + \frac{1}{3\omega_0}(\omega_0 t')^3]}.$$

$x = \frac{\omega_0 t}{\sqrt{\gamma^{-2}+\theta^2}}$, $dt = \frac{1}{\omega_0}\sqrt{\gamma^{-2}+\theta^2}dx$; and let $\xi \equiv \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2}$,

Case of Circular motion IV

- We have to compute the integrals

$$\begin{pmatrix} A_{||}(\omega) \\ A_{\perp}(\omega) \end{pmatrix} = \frac{q\omega}{2\pi\sqrt{c}} \int_{-\infty}^{+\infty} dx \begin{pmatrix} (\gamma^{-2}+\theta^2)x\frac{1}{\omega_0} \\ (\gamma^{-2}+\theta^2)^{1/2}\theta\frac{1}{\omega_0} \end{pmatrix} e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]}.$$

- We have

$$\int_{-\infty}^{+\infty} dt e^{i(xt+at^3)} = \frac{2\pi}{(2a)^{1/3}} A_i\left(\frac{x}{(3a)^{1/3}}\right),$$

$$A_i(x) = \frac{1}{\pi}\sqrt{\frac{1}{3}x}K_{1/3}\left(\frac{2}{3}x^{3/2}\right).$$

$$\boxed{\int_{-\infty}^{+\infty} dx e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]} = \frac{2\pi}{(3\xi/2)^{1/3}} A_i\left[\left(\frac{3\xi}{2}\right)^{2/3}\right] = \frac{2}{\sqrt{3}}K_{1/3}(\xi).}$$

$$\int_{-\infty}^{+\infty} dt t e^{i(xt+at^3)} = \frac{1}{i}\frac{d}{dx}\int_{-\infty}^{+\infty} t e^{i(xt+at^3)} dt = \frac{2\pi}{(2a)^{1/3}} A_i'\left(\frac{x}{(3a)^{1/3}}\right)$$

$$A_i'(x) = \frac{-1}{\pi}\sqrt{\frac{1}{3}x}K_{2/3}\left(\frac{2}{3}x^{3/2}\right)$$

$$x = 3\xi/2 \quad a = \xi/2,$$

$$\boxed{\int_{-\infty}^{+\infty} x e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]} dx = \frac{2\pi}{(3\xi/2)^{1/3}} A_i'\left[\left(\frac{3\xi}{2}\right)^{2/3}\right] = -\frac{1}{i}\frac{2}{\sqrt{3}}K_{2/3}(\xi).}$$

Case of Circular motion V

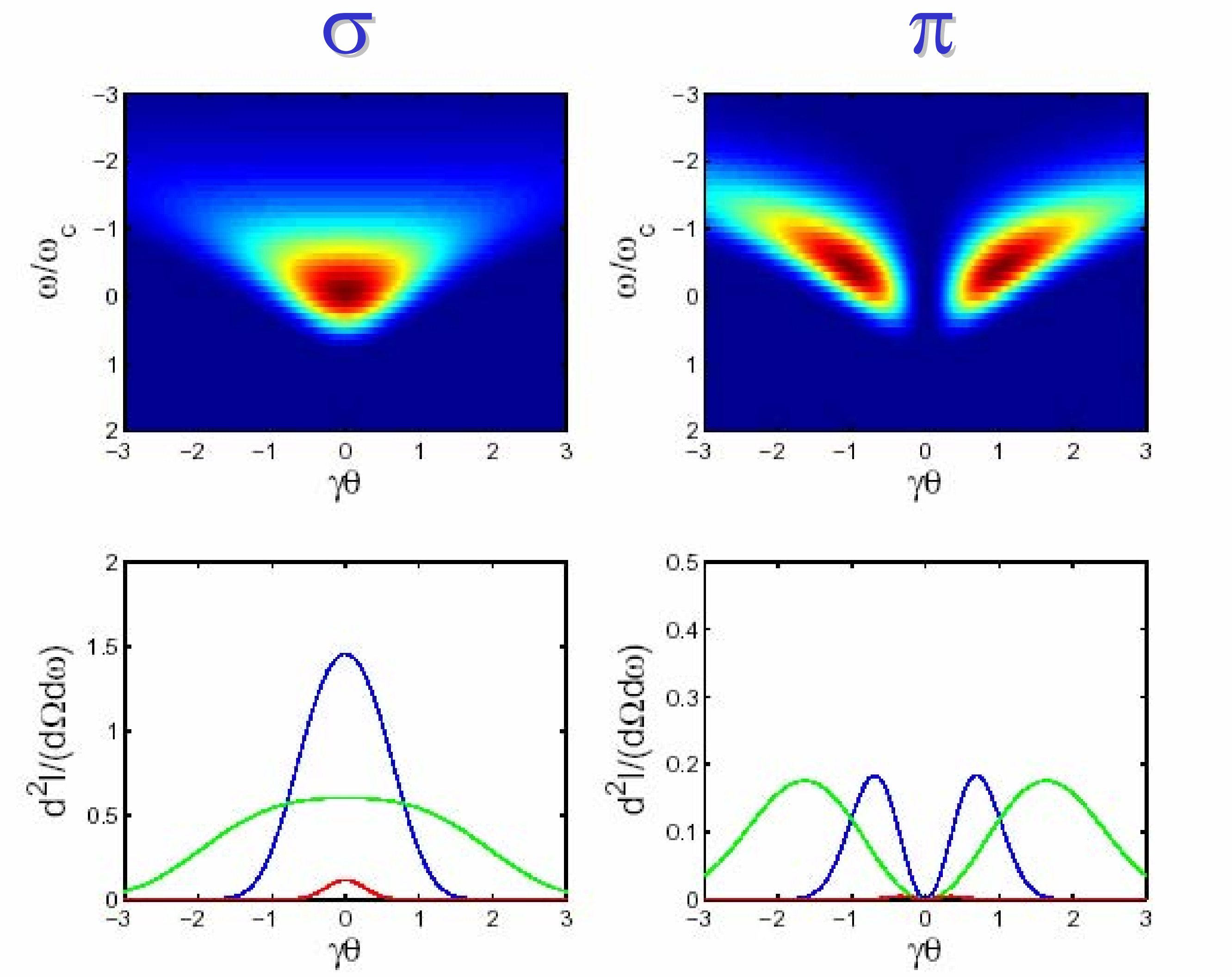
- Finally the *angular spectral fluence* takes the form

$$
\begin{aligned}
\frac{d^2 I}{d\Omega d\omega} &= |A_{\|}(\omega)|^2 + |A_{\perp}(\omega)|^2 \\
&= \frac{q^2}{3\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2 (\gamma^{-2}+\theta^2)^2 \left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2}+\theta^2} K_{1/3}^2(\xi)\right]
\end{aligned}
$$

- or, introducing $\xi = \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2} \equiv \frac{1}{2}\frac{\omega}{\omega_c}[1+\gamma^2\theta^2]^{3/2}$:

$$
\boxed{\frac{d^2 I}{d\Omega d\omega} = \frac{3q^2}{\pi^2 c}\xi^2 \frac{1}{\gamma^{-2}+\theta^2}\left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2}+\theta^2} K_{1/3}^2(\xi)\right]}
$$

Case of Circular motion VI



Case of Circular motion VII

$\sigma+\pi$