Introduction

- Starting from the radiation field, we have

$$\frac{dP(t')}{d\Omega} = [\kappa R^2 \overrightarrow{S}.\hat{n}]_{ret} = \frac{q^2}{4\pi c}\left[\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]|^2}{\kappa^5}\right]_{ret} .$$

- What we did Lesson 16 was to perform the integration

$$P(t') = \frac{q^2}{4\pi c}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta)\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]|^2}{\kappa^5}$$

- Now we turn to computing the angular distribution (so we actually do not want to perform the integration)…

Angular distribution of radiation emitted by an accelerated charge

- Starting from the radiation field, we have

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2}{\kappa^5}$$
$$= \frac{q^2}{4\pi c}\frac{\kappa^2\dot{\beta}^2+2\kappa(\dot{\vec{\beta}}.\hat{n})(\vec{\beta}.\dot{\vec{\beta}})-\gamma^{-2}(\dot{\vec{\beta}}.\hat{n})^2}{\kappa^5}$$

where we used $|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2 = -\gamma^{-2}(\hat{n}.\dot{\vec{\beta}})^2+2\kappa(\vec{\beta}.\dot{\vec{\beta}})(\hat{n}.\dot{\vec{\beta}})+\kappa^2\dot{\beta}^2.$

Angular distribution for linear motion 1

- Introducing θ, we have:

$$\vec{\beta}.\hat{n} = \beta\cos\theta,\ \dot{\vec{\beta}}.\hat{n} = \dot{\beta}\cos\theta,\ \kappa = 1 - \vec{\beta}.\hat{n} = 1 - \beta\cos\theta,$$

- And the numerator becomes

$$\begin{aligned} & \dot{\beta}^2[\kappa^2 + 2\kappa\beta\cos\theta - (1-\beta^2)\cos^2\theta] \\ = \ & \dot{\beta}^2[(\kappa^2 + 2\kappa\beta\cos\theta + \beta^2\cos^2\theta) - \cos^2\theta] \\ = \ & \dot{\beta}^2[(\kappa + \beta\cos\theta)^2 - \cos^2\theta] = \dot{\beta}^2\sin^2\theta. \end{aligned}$$

- So the radiated power writes

$$\frac{dP(t')}{d\Omega} = \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

Angular distribution for linear motion 2

- The power distribution has maxima given by

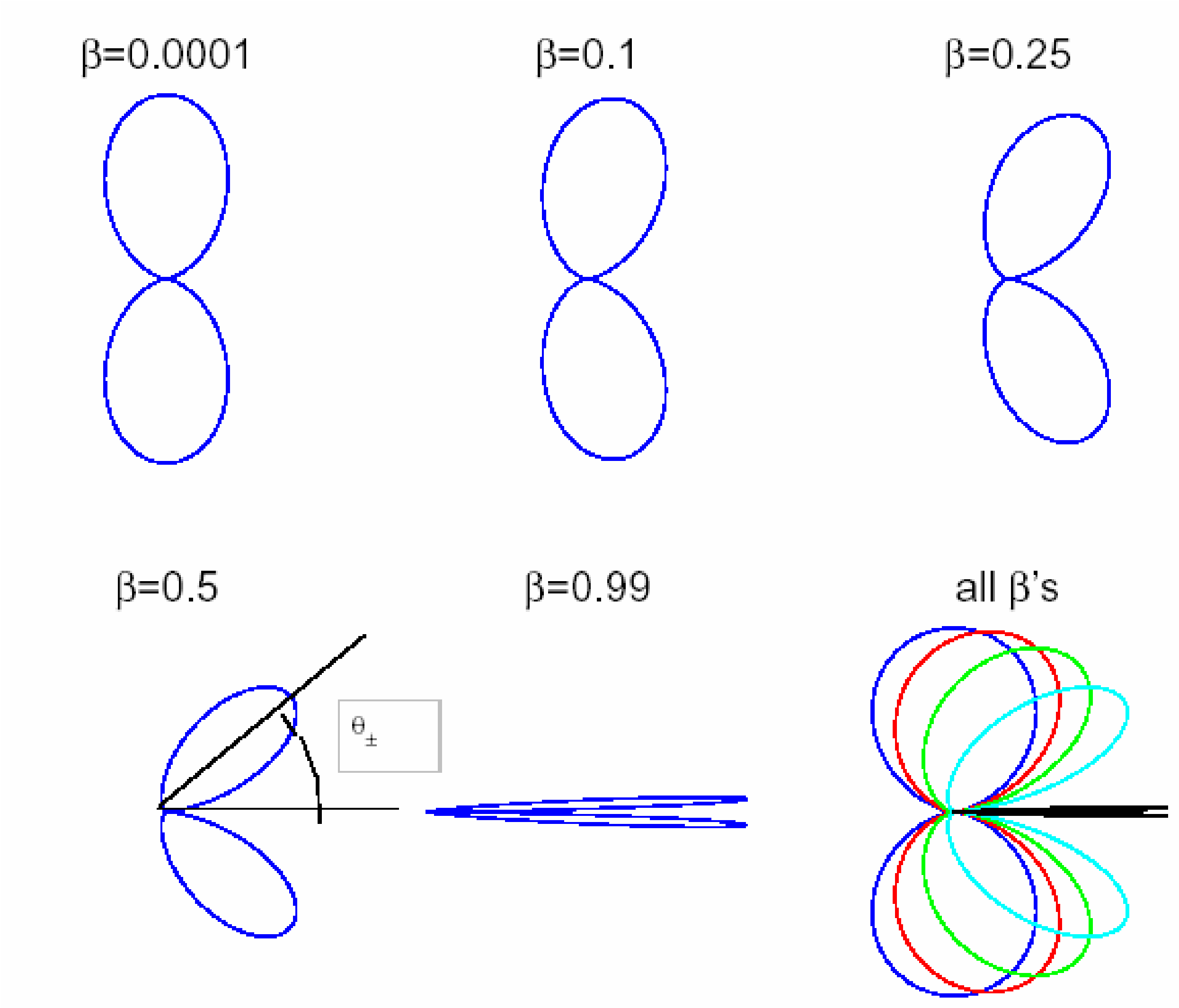
$$
\begin{aligned}
0 &= \frac{d}{d\theta}\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right) \\
&= \frac{\sin(\theta)\left(2\cos(\theta)+3\beta\,(\cos(\theta))^2-5\beta\right)}{(1-\beta\cos\theta)^4}
\end{aligned}
$$

- With solutions $[\cos\theta]_\pm = \frac{1}{3\beta}[-1\pm(1+15\beta^2)^{1/2}]$

- Only cos θ_+ is possible so:

$$\theta_\pm = \pm\arccos\left[\frac{1}{3\beta}[-1+(1+15\beta^2)^{1/2}]\right] \xrightarrow{\beta\to 1} \pm\frac{1}{2\gamma}$$

Angular distribution for linear motion 3



Angular distribution for linear motion 4

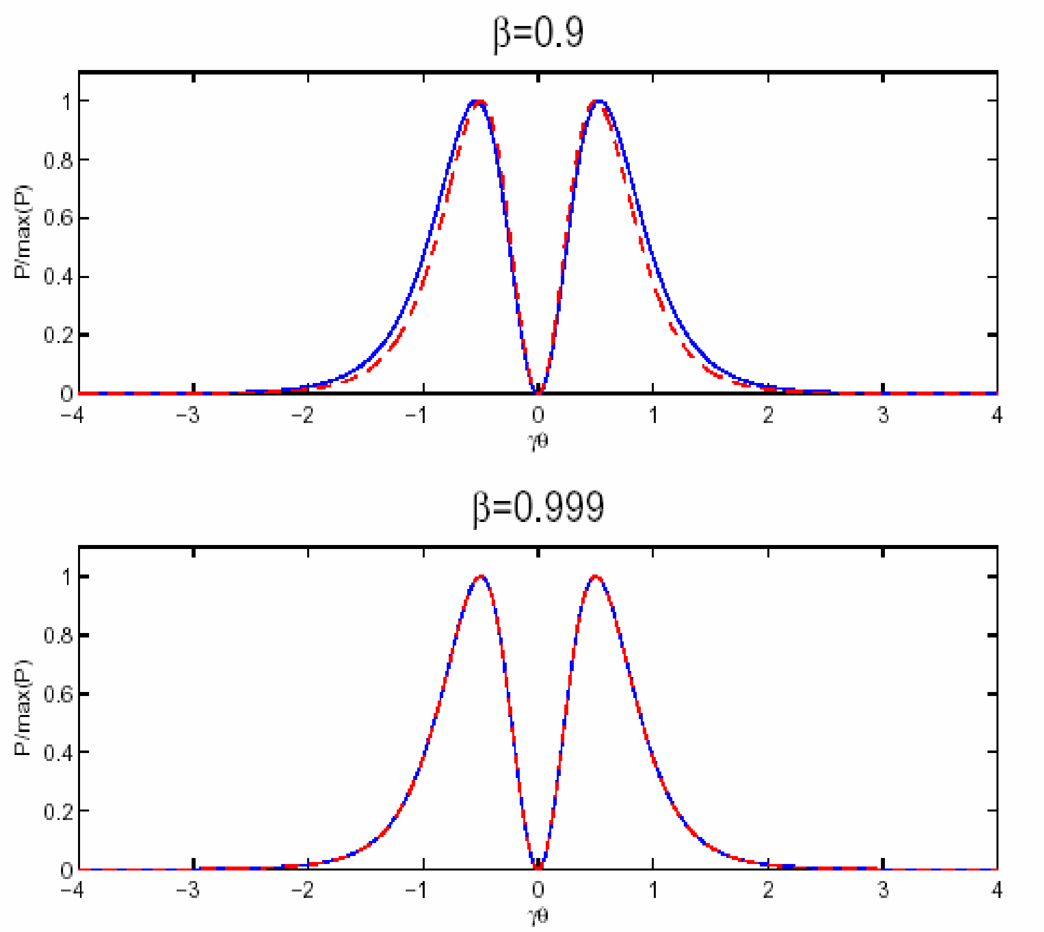
- Small angle approximation for ultra-relativistic case:

$$\frac{dP(t')}{d\Omega} \simeq \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{\theta^2}{(1-\beta(1-\frac{\theta^2}{2}))^5}$$

$$= \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{32\theta^2}{2(1-\beta)+\beta\theta^2))^5}$$

$$\simeq \frac{8}{\pi}\frac{\dot{\beta}^2}{c^2}\frac{\gamma^{10}\theta^2}{(1+\gamma^2\theta^2)^5}$$

JDJ equation 14.41



Angular distribution for circular motion 1

- We have:

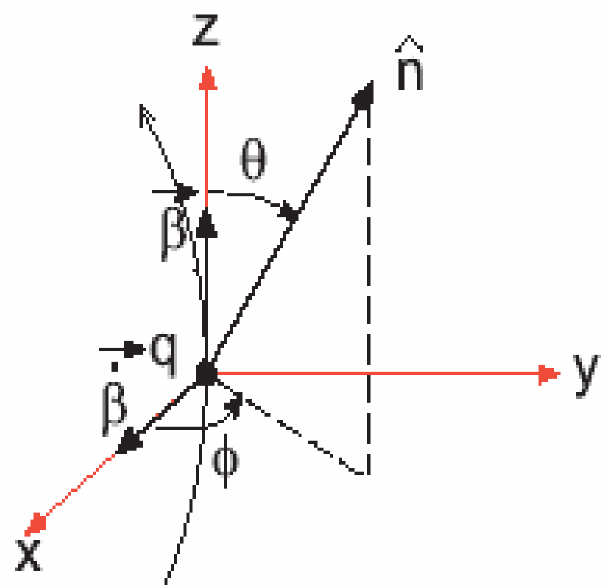
$$\hat{z} = \cos\theta\hat{n} - \sin\theta\hat{\theta}$$
$$\hat{x} = \sin\theta\cos\phi\hat{n} + \cos\theta\sin\phi\hat{\theta} - \sin\phi\hat{\phi}$$

- That is

$$\vec{\beta}.\hat{n} = \beta\cos\theta,\ \vec{\beta}.\vec{\dot{\beta}} = 0, \text{ and } \vec{\dot{\beta}}.\hat{n} = \dot{\beta}\sin\theta\cos\phi$$

- Which gives:

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^2}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right]$$



- In the ultra-relativistic limit (small angle approximation):

$$\frac{dP(t')}{d\Omega} = \frac{8}{\pi}\frac{q^2}{c^2}\frac{\dot{\beta}^2}{(1+\gamma^2\theta^2)^3}\gamma^6\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right]$$

Angular distribution for circular motion 2

- Note that $$P_{Linear} = \frac{2}{3} q^2 m^2 c^3 \dot{p}^2$$ $$P_{Circular} = \frac{2}{3} q^2 c \gamma^4 \dot{\beta}^2 = \frac{2}{3} q^2 m^2 c^3 \gamma^2 \dot{p}^2$$
- So we have $$\frac{P_{Circular}}{P_{Linear}} = \gamma^2$$

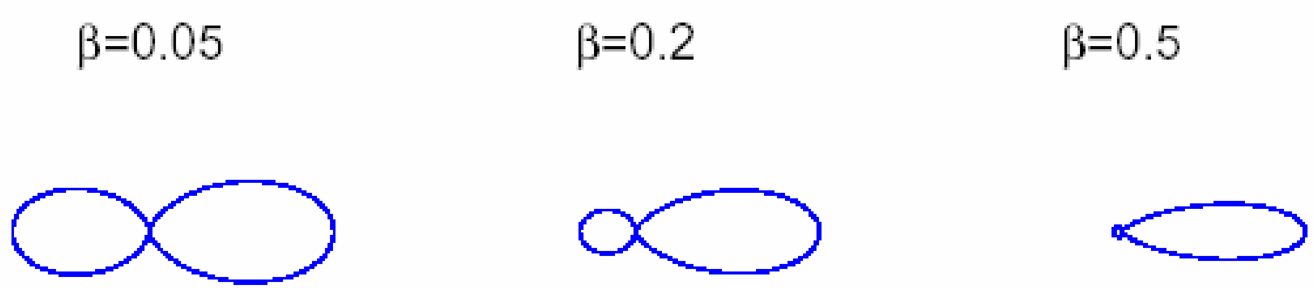


Figure 4.9: Distribution evaluated in the plan $\phi = 0$.