Power radiated in linear accelerators

- In linear accelerators, $\beta || \vec{\beta}$.

- We need to evaluate the acceleration. Start from the momentum:

\[
\dot{p} = mc(\gamma \beta + \gamma \dot{\beta}) = mc[(\gamma^3 \beta \dot{\beta}) \beta + \gamma \dot{\beta}]
\]
\[
= \gamma mc \left( \frac{\beta^2}{1 - \beta^2} + 1 \right) \dot{\beta} = \gamma^3 mc \dot{\beta}.
\]

- Thus the radiated power is

\[
P(t') = \frac{2}{3} \frac{q^2}{m^2 c^3} \dot{p}^2 \quad [\text{JDJ Eq. (14.27)}]
\]

Lighter particles are subject to higher loss.
One important question is how does the emission of radiation influence the charge particle dynamics.

The accelerator induces a momentum change of the form

\[ \dot{p} = \frac{dE}{dz} \]

(where we assumed the acceleration is along the z-axis)

Let \( P_{\text{ext}} \equiv \left[ \frac{dE}{dt} \right]_{\text{ret}} \) the power associated to the external force. The particle dynamics is affected when \( P_{\text{ext}} \) is comparable to the radiated power:

\[ \frac{P_{\text{rad}}}{P_{\text{ext}}} = \frac{P(t')}{vdE/dz} = \frac{2}{3} \frac{q^2}{m^2c^3} \left[ \frac{1}{v} \frac{dE}{dz} \right]_{\text{ret}} \sim 1. \]
Power radiated in linear accelerators 3

• Consider a relativistic electron then

\[
\frac{P_{rad}}{P_{ext}} = \frac{2e^2/(mc^2)}{3mc^2} \left[ \frac{dE}{dz} \right]_{rel}
\]

• So \( P_{rad} \approx P_{ext} \) if \( dE/dz \approx mc^2/r_e = 0.511/(2.8 \times 10^{-15}) = 2 \times 10^{14} \text{ MeV/m} \)
  compare to 100 MeV/m state-of-art conventional accelerator or to 30 Gev/m plasma-based

• So the effect seems to be negligible.

• This is actually part of the story some coherent effect can kick in an
  induce some distortion when considering highly charged electron
  bunches for instance…
Power radiated in circular accelerators 1

• Now \( \vec{\beta} \perp \vec{\beta} \) and

\[
\dot{\beta}^2 - (\vec{\beta} \times \vec{\beta})^2 = \beta^2 (1 - \beta^2) = \frac{\dot{\beta}^2}{\gamma^2}
\]

• The radiated power is

\[
P(t') = \frac{2}{3} \frac{q^2 c}{R^2} (\beta \gamma)^4 = \frac{2}{3} \frac{q^2 c}{R^2} \beta^4 \left[ \frac{E}{mc^2} \right]^4
\]

where \( E \) is the energy. Let’s introduce \( T = 2\pi R/(\beta c) \), and \( P = \frac{\Delta E}{T} \).

• So radiative energy loss per turn is

\[
\Delta E = PT = \frac{2}{3} \frac{q^2 c}{R^2} \beta^4 \left[ \frac{E}{mc^2} \right]^4 \frac{2\pi R}{\beta c}
\]
Power radiated in circular accelerators 2

• That is
\[ \Delta E = \frac{4\pi q^2}{3R} \beta^3 \left[ \frac{E}{mc^2} \right]^4 \]  
[JDJ Eq. (14.32)]

• For an e- synchrotron this becomes
\[ \Delta E \simeq \frac{4\pi e^2}{3R} \left( \frac{E}{mc^2} \right)^4. \]

• Take \( E=1 \) TeV, \( R=2 \) km we have
\[ \Delta E \ [\text{eV}] = \frac{1}{3\varepsilon_0} \frac{e}{R} \left( \frac{E}{mc^2} \right)^4 = 44.2 \text{ TeV} !! \]

• Conclusion:
  – bad idea to build electron circular accelerator for HEP
  – but good as copious radiation sources (e.g. APS in Argonne).
Angular distribution of radiation emitted by an accelerated charge

- Starting from the radiation field, we have

\[
\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c} \frac{\left| \hat{n} \times [(\hat{n} - \vec{\beta}) \times \vec{\beta}] \right|^2}{\kappa^5} = \frac{q^2 \kappa^2 \beta^2 + 2\kappa (\vec{\beta} \cdot \hat{n})(\vec{\beta} \cdot \vec{\beta}) - \gamma^{-2}(\vec{\beta} \cdot \hat{n})^2}{4\pi c \kappa^5}
\]

where we used

\[
\left| \hat{n} \times [(\hat{n} - \vec{\beta}) \times \vec{\beta}] \right|^2 = -\gamma^{-2}(\hat{n} \cdot \vec{\beta})^2 + 2\kappa (\vec{\beta} \cdot \hat{n})(\hat{n} \cdot \vec{\beta}) + \kappa^2 \beta^2.
\]
Angular distribution for linear motion 1

- Introducing $\theta$, we have:

$$\vec{\beta} \cdot \hat{n} = \beta \cos \theta, \quad \vec{\beta}' \cdot \hat{n} = \beta' \cos \theta, \quad \kappa = 1 - \vec{\beta} \cdot \hat{n} = 1 - \beta \cos \theta,$$

- And the numerator becomes

$$\dot{\beta}^2 [\kappa^2 + 2\kappa \beta \cos \theta - (1 - \beta^2) \cos^2 \theta]$$

$$= \dot{\beta}^2 [(\kappa^2 + 2\kappa \beta \cos \theta + \beta^2 \cos^2 \theta) - \cos^2 \theta]$$

$$= \dot{\beta}^2 [(\kappa + \beta \cos \theta)^2 - \cos^2 \theta] = \dot{\beta}^2 \sin^2 \theta.$$

- So the radiated power writes

$$\frac{dP(t')}{d\Omega} = \frac{q^2 \dot{\beta}^2}{4\pi c^2} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}$$
Angular distribution for linear motion 2

- The power distribution has maxima given by

\[
0 = \frac{d}{d\theta} \left( \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5} \right) = \frac{\sin(\theta) \left( 2 \cos(\theta) + 3 \beta \left( \cos(\theta) \right)^2 - 5 \beta \right)}{(1 - \beta \cos \theta)^4}
\]

- With solutions

\[
[\cos \theta]_\pm = \frac{1}{3\beta} \left[ -1 \pm (1 + 15\beta^2)^{1/2} \right]
\]

- Only \( \cos \theta_+ \) is possible so:

\[
\theta_\pm = \pm \arccos \left[ \frac{1}{3\beta} \left[ -1 + (1 + 15\beta^2)^{1/2} \right] \right] \xrightarrow{\beta \to 1} \pm \frac{1}{2\gamma}
\]
Angular distribution for linear motion 3

\[ \beta = 0.0001 \quad \beta = 0.1 \quad \beta = 0.25 \]

\[ \beta = 0.5 \quad \beta = 0.99 \quad \text{all } \beta \text{'s} \]
Angular distribution for linear motion 4

- Small angle approximation for ultra-relativistic case:

\[
\frac{dP(t')}{d\Omega} = \frac{q^2 \beta^2}{4\pi c^2} \frac{\theta^2}{(1 - \beta (1 - \frac{\theta^2}{2}))^5} = \frac{q^2 \beta^2}{4\pi c^2} \frac{32\theta^2}{2(1 - \beta + \beta \theta^2))^5}
\]

\[
\frac{8 \dot{\beta}^2}{\pi c^2} \frac{\gamma^1 \theta^2}{(1 + \gamma^2 \theta^2)^5}
\]

JDJ equation 14.41
Angular distribution for circular motion 1

• We have:
  \[\ddot{z} = \cos \theta \hat{n} - \sin \theta \hat{\theta}\]
  \[\ddot{x} = \sin \theta \cos \phi \hat{n} + \cos \theta \sin \phi \hat{\theta} - \sin \phi \hat{\phi}\]

• That is
  \[\vec{\beta} \cdot \hat{n} = \beta \cos \theta, \vec{\beta} \cdot \hat{\beta} = 0, \text{ and } \vec{\beta} \cdot \hat{\phi} = \beta \sin \theta \cos \phi\]

• Which gives:
  \[\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^2 (1 - \beta \cos \theta)^3} \left[ \frac{\beta^2}{\gamma^2 (1 - \beta \cos \theta)^2} - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 (1 - \beta \cos \theta)^2} \right]\]

• In the ultra-relativistic limit (small angle approximation):
  \[\frac{dP(t')}{d\Omega} = \frac{8 q^2}{\pi c^2 (1 + \gamma^2 \theta^2)^3} \gamma^6 \left[ 1 - \frac{4 \gamma^2 \theta^2 \cos^2 \phi}{(1 + \gamma^2 \theta^2)^2} \right]\]
Angular distribution for circular motion 2

- Note that

\[ P_{\text{Linear}} = \frac{2}{3} q^2 m^2 c^3 p^2 \]
\[ P_{\text{Circular}} = \frac{2}{3} q^2 c \gamma \beta^2 = \frac{2}{3} q^2 m^2 c^3 \gamma^2 \beta^2 \]

\[ \frac{P_{\text{Circular}}}{P_{\text{Linear}}} = \gamma^2 \]

- So we have

\[ \beta = 0.05 \quad \beta = 0.2 \quad \beta = 0.5 \]

Figure 4.9: Distribution evaluated in the plan \( \phi = 0 \).