Radiation from accelerating charges 1

- We now concentrate in computing the radiated power by an accelerated charge.

- From the Poynting vector definition, the radiated power per unit of surface area is:

$$W = \int_{-\infty}^{+\infty} dt' \frac{dt}{dt'} \vec{S}(\vec{x},t).\hat{n} = \int_{-\infty}^{+\infty} dt' [\kappa \vec{S}.\hat{n}]_{\mathrm{ret}}$$

- So $\quad \dfrac{dW}{dt} = [\kappa \vec{S}.\hat{n}]_{\mathrm{ret}}$

- From now on we will only consider radiation detected at very large R, this is the far field approximation. Only the second term (that involves the acceleration) is kept in the E, B fields associated to the Lienard-Wiechert potential.

Radiation from accelerating charges 2

- From $dP(t')/dA$, the instantaneous power radiated at time t' per unit solid angle $d\Omega$ is

$$\frac{dP(t')}{d\Omega} = [\kappa \vec{S} . \hat{n} R^2]_{ret}$$

- In the far field approximation we have

$$
\begin{aligned}
\vec{S} . \hat{n} &= \frac{c}{4\pi} [\vec{E} \times (\hat{n} \times \vec{E})] . \hat{n} \\
&= \frac{c}{4\pi} [E^2 - (\hat{n} . \vec{E})^2].
\end{aligned}
$$

- Let consider **n,E**

$$
\begin{aligned}
\hat{n} . \vec{E} &\propto \hat{n} . \{ \hat{n} \times [(\hat{n} - \vec{\beta}) \times \vec{\beta}] \} \\
&\propto \hat{n} . \{ (\hat{n} . \vec{\beta})(\hat{n} - \vec{\beta}) - [\hat{n} . (\hat{n} - \vec{\beta})] \vec{\beta} \} \\
&\propto \hat{n} . \{ (\hat{n} . \vec{\beta})(\hat{n} - \vec{\beta}) - (1 - \vec{\beta} . \hat{n}) \vec{\beta} \} \\
&= 0.
\end{aligned}
$$

Radiated power per unit of time

- So finally

$$\vec{S}.\hat{n} = \frac{c}{4\pi}E^2 = \frac{q^2}{4\pi c}\left[\frac{|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]|^2}{\kappa^6 R^2}\right]_{ret}$$

- and

$$\frac{dP(t')}{d\Omega} = [\kappa R^2 \vec{S}.\hat{n}]_{ret} = \frac{q^2}{4\pi c}\left[\frac{|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]|^2}{\kappa^5}\right]_{ret}.$$

- This is the power radiated per unit solid angle in terms of the charge proper time t'.

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Radiated energy per unit of time

- If we want to know *dP(t)/dt*, the power radiated per unit solid angle at time t it arrived at the enveloping surface, then we must trace back to the proper retarded time

- Note also that $dt = dt' \kappa_{ret}$, so if a particle is suddenly accelerated for a time $\Delta t' = \tau$, an observer will see a light pulse at time *t=R/c* with a duration $\Delta t = \kappa_{ret}\tau$.

- Energy is conserved: total energy radiated = total energy lost by the particle.

- The fact that $\tau \frac{dP(t')}{d\Omega} = \tau \kappa_{ret} \frac{dP(t)}{d\Omega}$ implied that energy radiated per unit of time is κ_{ret} times the energy lost by the particle in the far field per unit of time.

Instantaneous rate of radiation 1

- Hgf

$$P(t') = \frac{q^2}{4\pi c} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta) \frac{|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \vec{\dot{\beta}}]|^2}{\kappa^5}$$

- Work out the integrant

$$
\begin{aligned}
|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \vec{\dot{\beta}}]|^2 &= |(\hat{n}.\vec{\dot{\beta}})(\hat{n} - \vec{\beta}) - (1 - \vec{\beta}.\hat{n})\vec{\dot{\beta}}|^2 \\
&= |(\hat{n}.\vec{\dot{\beta}})(\hat{n} - \vec{\beta}) - \kappa\vec{\dot{\beta}}|^2 \\
&= (\hat{n}.\vec{\dot{\beta}})^2[1 - 2\hat{n}.\vec{\beta} + \beta^2] - 2\kappa\dot{\beta}(\hat{n}.\vec{\dot{\beta}})(\hat{n} - \vec{\beta}) + \kappa^2\dot{\beta}^2 \\
&= (\hat{n}.\vec{\dot{\beta}})^2[1 - 2\hat{n}.\vec{\beta} + \beta^2] - 2\kappa(\hat{n}.\vec{\dot{\beta}})(\hat{n}.\vec{\dot{\beta}} - \vec{\beta}.\vec{\dot{\beta}}) + \kappa^2\dot{\beta}^2
\end{aligned}
$$

- From $\vec{\beta}.\hat{n} = 1 - \kappa$ we get

$$|\hat{n} \times [(\hat{n} - \vec{\beta}) \times \vec{\dot{\beta}}]|^2 = -\gamma^{-2}(\hat{n}.\vec{\dot{\beta}})^2 + 2\kappa(\vec{\beta}.\vec{\dot{\beta}})(\hat{n}.\vec{\dot{\beta}}) + \kappa^2\dot{\beta}^2.$$

Instantaneous rate of radiation 2

- So

$$P(t') = \frac{q^2}{4\pi c} 2\pi \int_0^\pi d\theta \sin(\theta) \frac{1}{\kappa^5} [\kappa^2 \dot{\beta}^2 + 2\kappa(\vec{\dot{\beta}}.\hat{n})(\vec{\dot{\beta}}.\vec{\dot{\beta}}) - \frac{1}{\gamma^2}(\vec{\dot{\beta}}.\hat{n})^2]$$

$$= \frac{q^2}{2c} \int_0^\pi d\theta \sin\theta \left[\frac{\dot{\beta}^2}{\kappa^3} + \frac{2(\vec{\dot{\beta}}.\vec{\dot{\beta}})\dot{\beta}^i n_i}{\kappa^4} - \frac{1}{\gamma^2} \frac{\dot{\beta}^i \dot{\beta}^j n_i n_j}{\kappa^5} \right].$$

- Recall $\kappa \equiv 1 - \vec{\beta}.\hat{n} = 1 - \cos\theta$ and define the integrals

$$I \equiv \int_0^\pi \frac{\sin\theta d\theta}{(1 - \vec{\beta}.\hat{n})^3} = \int_{-1}^1 -\frac{du}{(1-\beta u)^3} = \frac{2}{(1-\beta^2)^2} = 2\gamma^4$$

$$J_i \equiv \int_0^\pi \frac{n_i \sin\theta d\theta}{(1 - \vec{\beta}.\hat{n})^4} = \frac{1}{3}\frac{\partial I}{\partial \beta^i} = \frac{8}{3}\frac{\beta_i}{(1-\beta^2)^3} = \frac{8}{3}\beta_i \gamma^6,$$

$$K_{ij} \equiv \int_0^\pi \frac{n_i n_j \sin\theta d\theta}{(1 - \vec{\beta}.\hat{n})^5} = \frac{1}{4}\frac{\partial J_i}{\partial \beta^j} = \frac{2}{3}\frac{\delta_{ij} + \frac{6\beta_i \beta_j}{1-\beta^2}}{(1-\beta^2)^3} = \frac{2}{3}\gamma^6[\delta_{ij} + 6\gamma^2 \beta_i \beta_j].$$

- So the power is $P(t') = \frac{q^2}{2c}[\dot{\beta}^2 I + 2(\vec{\dot{\beta}}.\vec{\dot{\beta}})\dot{\beta}^i J_i - \frac{1}{\gamma^2}\dot{\beta}^i \dot{\beta}^j K_{ij}].$

Instantaneous rate of radiation 3

- We finally have

$$
\begin{aligned}
P(t') &= \frac{q^2}{2c}\left[2\gamma^2\dot{\beta}^2 + \frac{16}{3}\gamma^6\beta^i\dot{\beta}_i(\vec{\beta}.\vec{\dot{\beta}}) - \frac{2}{3}\gamma^4(\delta_{ij} + 6\gamma^2\beta_i\beta_j)\dot{\beta}^i\dot{\beta}^j\right] \\
&= \frac{q^2}{2c}\left[2\gamma^4\dot{\beta}^2 + \frac{16}{3}\gamma^6(\vec{\beta}.\vec{\dot{\beta}})^2 - \frac{2}{3}\gamma^4[\dot{\beta}^2 + 6\gamma^2(\vec{\beta}.\vec{\dot{\beta}})^2]\right] \\
&= \frac{2q^2}{3c}\left[\gamma^4\dot{\beta}^2 + \gamma^6(\vec{\beta}.\vec{\dot{\beta}})^2\right] \\
&= \frac{2q^2}{3c}\gamma^6\left[(1-\beta^2)\dot{\beta}^2 + (\vec{\beta}.\vec{\dot{\beta}})^2\right] = \frac{2q}{3c}\gamma^6[\dot{\beta}^2 - \dot{\beta}^2\beta^2(1 - \cos^2\Phi) \\
&= \frac{2q^2}{3c}\gamma^6[1 - \dot{\beta}^2\beta^2\sin^2\Phi] = \frac{2q^2}{3c}\gamma^6[\dot{\beta}^2 - (\vec{\beta}\times\vec{\dot{\beta}})^2]
\end{aligned}
$$

- The last line is the relativistic generalization of Larmor's formula; to recover the Larmor formula just take the limit $\beta \to 0$.

Power radiated in linear accelerators 1

- In linear accelerators $\vec{\dot{\beta}} \parallel \vec{\beta}$.

- We need to evaluate the acceleration. Start from the momentum

$$\dot{p} = mc(\dot{\gamma}\beta + \gamma\dot{\beta}) = mc[(\gamma^3\beta\dot{\beta})\beta + \gamma\dot{\beta}]$$
$$= \gamma mc\left(\frac{\beta^2}{1-\beta^2} + 1\right)\dot{\beta} = \gamma^3 mc\dot{\beta}.$$

- Thus the radiated power is

$$P(t') = \frac{2}{3}\frac{q^2}{m^2c^3}\dot{p}^2 \ [\text{JDJ Eq. (14.27)}]$$

Lighter particles are
subject to higher loss

Power radiated in linear accelerators 2

- One important question is how does the emission of radiation influence the charge particle dynamics.

- The accelerator induce a momentum change of the form

$$\dot{p} = dE/dz$$

(where we assumed the acceleration is along the *z*-axis)

- Let $P_{ext} \equiv [dE/dt]_{ret}$ the power associated to the external force. The particle dynamics is affected when P_{ext} is comparable to the radiated power:

$$\frac{P_{rad}}{P_{ext}} = \frac{P(t')}{v\,dE/dz} = \frac{2}{3}\frac{q^2}{m^2c^3}\left[\frac{1}{v}\frac{dE}{dz}\right]_{ret} \sim 1.$$

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Power radiated in linear accelerators 3

- Consider a relativistic electron then

$$\frac{P_{rad}}{P_{ext}} = \frac{2}{3}\frac{e^2/(mc^2)}{mc^2}\left[\frac{dE}{dz}\right]_{ret}$$

- So $P_{rad} \simeq P_{ext}$ if $dE/dz \simeq mc^2/r_e = 0.511/(2.8 \times 10^{-15}) = 2 \times 10^{14}$ MeV/m compare to 100 MeV/m state-of-art conventional accelerator or to 30 Gev/m plasma-based

- So the effect seems to be negligible.

- This is actually part of the story some coherent effect can kick in an induce some distortion when considering highly charged electron bunches for instance…

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Power radiated in circular accelerators 1

- Now $\vec{\dot{\beta}} \perp \vec{\beta}$ and

$$\dot{\beta}^2 - (\vec{\beta} \times \vec{\dot{\beta}})^2 = \dot{\beta}^2(1 - \beta^2) = \frac{\dot{\beta}^2}{\gamma^2}$$

- The radiated power is

$$P(t') = \frac{2}{3}\frac{q^2 c}{R^2}(\beta\gamma)^4 = \frac{2}{3}\frac{q^2 c}{R^2}\beta^4\left[\frac{E}{mc^2}\right]^4$$

where E is the energy. Let's introduce $T = 2\pi R/(\beta c)$, and $P = \frac{\Delta E}{T}$.

- So radiative energy loss per turn is

$$\Delta E = PT = \frac{2}{3}\frac{q^2 c}{R^2}\beta^4\left[\frac{E}{mc^2}\right]^4\frac{2\pi R}{\beta c}$$

Power radiated in circular accelerators 2

- That is

$$\Delta E = \frac{4\pi}{3} \frac{q^2}{R} \beta^3 \left[\frac{E}{mc^2} \right]^4 \quad \text{[JDJ Eq. (14.32)]}$$

- For an e- synchrotron this becomes

$$\Delta E \simeq \frac{4\pi}{3} \frac{e^2}{R} \left(\frac{E}{mc^2} \right)^4.$$

- Take E=1 TeV, R=2 km we have

$$\Delta E \, [\text{eV}] = \frac{1}{3\epsilon_0} \frac{e}{R} \left(\frac{E}{mc^2} \right)^4 = 44.2 \text{ TeV} \, !!$$

- Conclusion bad idea to build circular accelerator for HEP but good as copious radiation sources (e.g. APS in Argonne).