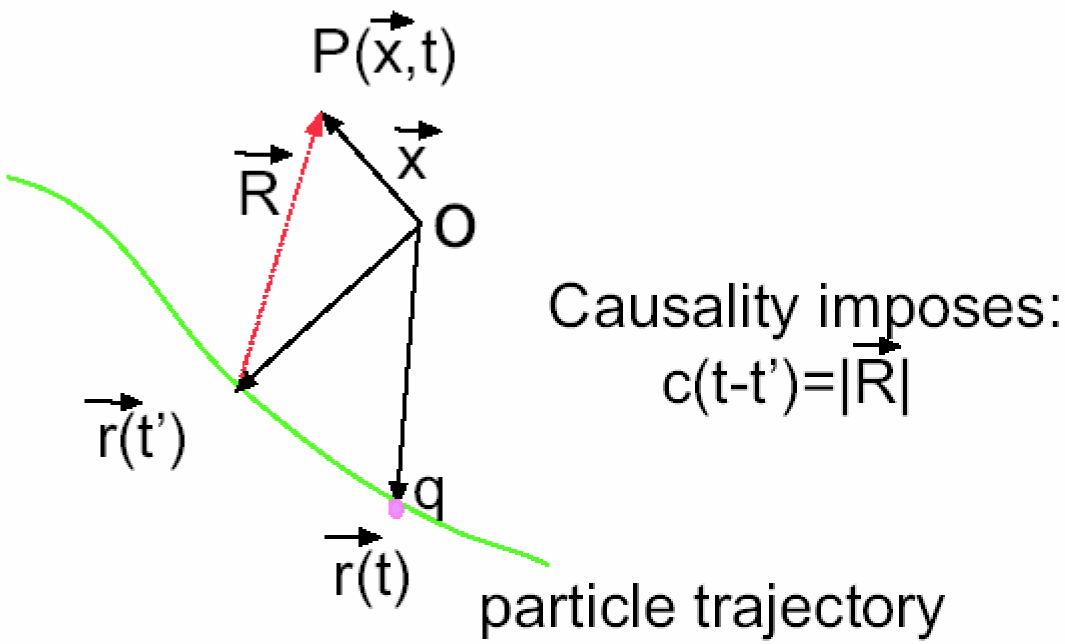
Radiation from accelerating charges

- We showed the Liénard Wiechert potential are

$$\begin{pmatrix} \Phi(\vec{x},t) \\ \vec{A}(\vec{x},t) \end{pmatrix} = \left[\frac{q}{(1-\vec{\beta}.\hat{n})R}\begin{pmatrix} 1 \\ \vec{\beta} \end{pmatrix}\right]_{ret}.$$



Radiation from accelerating charges: application

- In principle the field are easily obtained from the potentials using

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

- Problem: here all the quantities have to be evaluated at a retarded time…

- So we need to express $\vec{\nabla}$ and $\partial/\partial t$ in term of retarded quantities.

$\partial/\partial t$ as function of retarded quantities

- Consider $$\boxed{R(t') = c(t - t') \Rightarrow \frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right)}$$
- On another hand, the chain rule gives
$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t}$$
- Which can be expressed from
$$\frac{1}{2}\frac{\partial R^2}{\partial t'} = R\frac{\partial R}{\partial t'} = \vec{R}.\frac{\partial \vec{R}}{\partial t'} \ , \ \text{so}, \ R\frac{\partial R}{\partial t'} = -\vec{v}.\vec{R}.$$
- So $$\boxed{\frac{\partial R}{\partial t} = -c\vec{\beta}.\hat{n}\frac{\partial t'}{\partial t}}$$
- The two highlighted equation result in $$\boxed{\frac{\partial t'}{\partial t} = \frac{1}{1 - \vec{\beta}.\hat{n}} \equiv \frac{1}{\kappa} \quad \Rightarrow \frac{\partial}{\partial t} = \frac{1}{\kappa}\frac{\partial}{\partial t'}}$$

∇ as function of retarded quantities

- Consider $$\vec{\nabla} R = \vec{\nabla}[c(t-t')] = -c\vec{\nabla} t'$$
- Let $\vec{\nabla}_{t'}$ be the gradient evaluated at t'. Then (chains rule)

$$\begin{aligned} \vec{\nabla} R &= \vec{\nabla}_{t'} R + \frac{\partial R}{\partial t'}\vec{\nabla} t' \\ &= \hat{n} - c\vec{\beta}.\hat{n}\vec{\nabla} t' \end{aligned}$$

- The two previous equations result in

$$\vec{\nabla} t' = \frac{-\hat{n}}{c(1-\vec{\beta}.\hat{n})}$$

- That is

$$\vec{\nabla} = \vec{\nabla}_{t'} - \frac{\hat{n}}{c\kappa}\frac{\partial}{\partial t'}$$

Electric field I

- In term of retarded quantities the E-field is

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}_{t'}\Phi + \frac{\hat{n}}{c\kappa}\frac{\partial \Phi}{\partial t'} - \frac{1}{\kappa c}\frac{\partial \vec{A}}{\partial t'},$$

- with $\Phi = \frac{e}{\kappa R}$.
- We have $\vec{\nabla}_{t'}\Phi = \frac{-e}{(\kappa R)^2}[R\vec{\nabla}_{t'}\kappa + \kappa\vec{\nabla}_{t'}R]$

$$\vec{\nabla}_{t'}R = \hat{n}, \text{ and } \vec{\nabla}_{t'}\kappa = \vec{\nabla}_{t'}(1-\vec{\beta}.\hat{n}) = -\vec{\nabla}_{t'}(\vec{\beta}.\hat{n}) = -(\vec{\beta}.\vec{\nabla}_{t'})\hat{n}.$$

- So finally

$$\begin{aligned} \Rightarrow \vec{\nabla}_{t'}\kappa &= -(\vec{\beta}.\vec{\nabla}_{t'})\frac{\vec{R}}{R} \\ &= -\frac{R(\vec{\beta}.\vec{\nabla}_{t'})\vec{R} - \vec{R}(\vec{\beta}.\vec{\nabla}_{t'})R}{R^2} \\ &= -\frac{\vec{\beta} - \hat{n}(\vec{\beta}.\hat{n})}{R}. \end{aligned}$$

Electric field II

- And finally
$$\begin{aligned}\overrightarrow{\nabla}_{v}\Phi &= \frac{-e}{(\kappa R)^2}[-\overrightarrow{\beta}+\hat{n}(\overrightarrow{\beta}.\hat{n})+(1-\overrightarrow{\beta}.\hat{n})\hat{n}] \\ &= -\frac{e}{(\kappa R)^2}[\hat{n}-\overrightarrow{\beta}]\end{aligned}$$

- Now let's consider $\frac{\hat{n}}{c\kappa}\frac{\partial}{\partial t'}\Phi$

- This gives $\frac{\partial\Phi}{\partial t'}=-e\frac{\partial}{\partial t'}\left(\frac{1}{\kappa R}\right)=\frac{-e}{(\kappa R)^2}[\kappa\dot{R}+R\dot{\kappa}]$

- With $\dot{R}=-c\overrightarrow{\beta}.\hat{n}$, and $\dot{\kappa}=-\overrightarrow{\dot{\beta}}.\hat{n}-\dot{\hat{n}}.\overrightarrow{\beta}$.

- and
$$\begin{aligned}\dot{\hat{n}} &= \frac{\partial}{\partial t'}\frac{\overrightarrow{R}}{R}=\frac{R\overrightarrow{\dot{R}}-\dot{R}\overrightarrow{R}}{R^2} \\ &= \frac{-\overrightarrow{v}+(\overrightarrow{v}.\hat{n})\hat{n}}{R}=-c\frac{\overrightarrow{\beta}-(\overrightarrow{\beta}.\hat{n})\hat{n}}{R}.\end{aligned}$$

Electric field III

- Thus
$$\dot{\Phi} = -\frac{e}{(\kappa R)^2}[-\vec{\dot{\beta}}.\vec{R} + c\beta^2 - c\vec{\beta}.\hat{n}].$$

- From the two latest highlighted equation we get
$$\vec{\nabla}_{r'}\Phi + \frac{\hat{n}}{c\kappa}\frac{\partial \Phi}{\partial t'} = -\frac{e}{(\kappa R)^2}\{\hat{n} - \vec{\beta} + \frac{\hat{n}}{c\kappa}[+\vec{\dot{\beta}}.\vec{R} - c\beta^2 + c\vec{\beta}.\hat{n}]\}$$

- Now we consider $\frac{\partial \vec{A}}{\partial t}$
$$\frac{\partial \vec{A}}{\partial t} = \vec{\dot{A}}\frac{\partial t'}{\partial t} = \frac{1}{\kappa}\vec{\dot{A}}$$

- The t-derivative of A is :
$$\vec{\dot{A}} = \vec{\dot{\beta}}\Phi + \vec{\beta}\dot{\Phi} = +\frac{e}{R\kappa}\vec{\dot{\beta}}\frac{e\vec{\beta}}{(R\kappa)^2}[\vec{\dot{\beta}}\vec{R} + c\vec{\beta}\hat{n} - \beta^2 c]$$

Electric field IV

- So the E- field is finally given by

$$\begin{aligned}\vec{E}(t') &= -\vec{\nabla}\Phi(t') - \frac{1}{c}\frac{\partial \vec{A}}{\partial t'} \\ &= \frac{e}{(\kappa R)^2\kappa}\left[(\hat{n}-\vec{\beta})\kappa + \frac{\hat{n}}{c}(\dot{\vec{\beta}}.\vec{R} + c\vec{\beta}.\hat{n} - c\beta^2)\right. \\ &\quad \left. - \frac{\vec{\beta}}{c}(\dot{\vec{\beta}}.\vec{R} + c\vec{\beta}.\hat{n} - c\beta^2) - \frac{e}{cR\kappa^2}\dot{\vec{\beta}}\right]\end{aligned}$$

- Which simplifies to

$$\begin{aligned}\vec{E}(t') &= \frac{e}{\kappa^3 R^2}\left[\frac{\hat{n}}{c}\dot{\vec{\beta}}.\vec{R} + (1-\beta^2)\hat{n} - \frac{\vec{\beta}}{c}\dot{\vec{\beta}}.\vec{R} + \vec{\beta} - \vec{\beta}\beta^2\right] \\ &= \frac{e}{\kappa^3 R^2}\left[(1-\beta^2)(\hat{n}-\vec{\beta})\right] + \frac{e}{cR\kappa^3}\left[\dot{\vec{\beta}}.\hat{n}(\hat{n}-\vec{\beta}) - \dot{\vec{\beta}}\kappa\right]\end{aligned}$$

Electric field V

- So finally the $\vec{E}$ and $\vec{B}$ fields are given by:

$$\vec{E}(t') = \left[\frac{e}{\kappa^3 R^2 \gamma^2}(\hat{n} - \vec{\beta})\right]_{ret} + \left[\frac{e}{\kappa^3 R}\{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]\}\right]_{ret}$$

$$\vec{B}(t') = [n \times \vec{E}]_{ret}$$

where the identity $\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}] = \dot{\vec{\beta}}.\hat{n}(\hat{n} - \vec{\beta}) - \dot{\vec{\beta}}(1 - \vec{\beta}.\hat{n})$ was used.

$$\vec{E}(\vec{x}, t) = q\left[\frac{\hat{n} - \vec{\beta}}{\gamma^2 \kappa^3 R^2}\right]_{ret} + \frac{q}{c}\left[\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{\kappa^3 R}\right]_{ret}$$

Near field
Velocity fields

Far field
Radiation fields

Electric field of a uniformly moving charge I

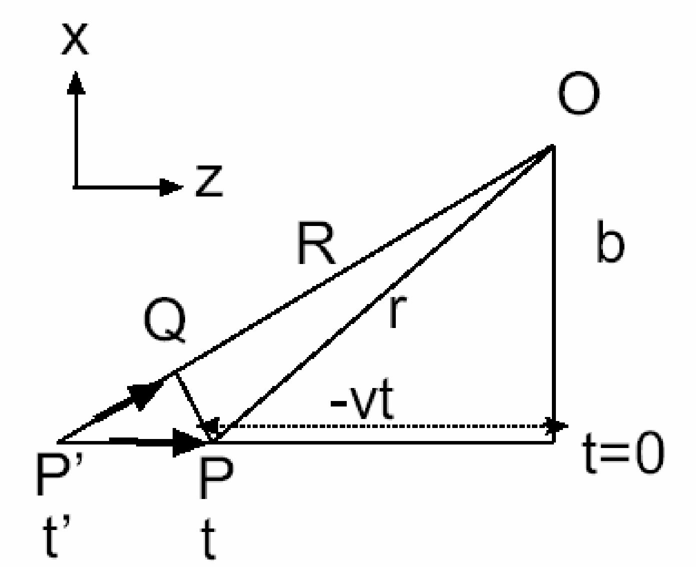
- $d\beta/dt = 0$ so

$$\vec{E}(\vec{x},t) = q\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2\kappa^3R^2}\right]_{ret}.$$

- Does this agree with what we learnt?

$$\vec{E}_\perp(\vec{x},t) = \frac{\gamma q b}{(b^2+\gamma^2v^2t^2)^{3/2}}.$$

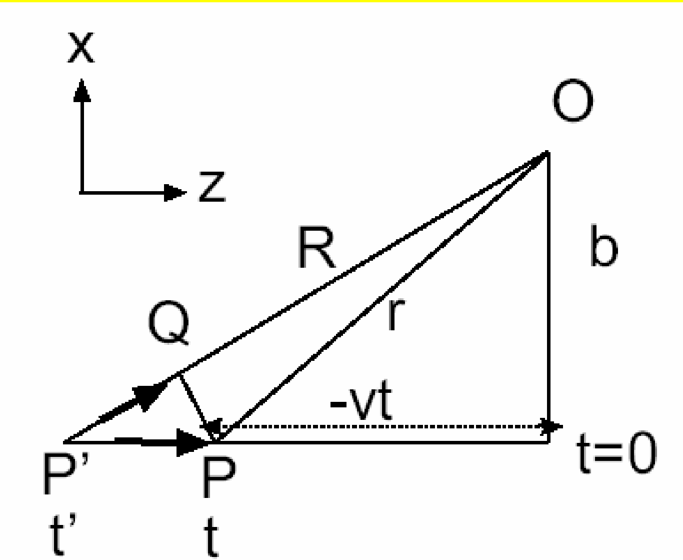
- Yes! Consider the drawing:



Electric field of a uniformly moving charge II

- Geometric considerations give

$$PP' = v(t-t') = \beta R,\ P'Q = PP'\cos(\theta) = \beta R\cos(\theta),$$
$$PQ = \beta R\sin(\theta) = \beta R\frac{b}{R} = \beta b;\ QO = R - PP' = (1-\vec{\beta}.\hat{n})R$$
$$r^2 = QO^2 + PQ^2 \rightarrow (1-\vec{\beta}\hat{n})^2R^2 = r^2 - \beta^2b^2 = (vt)^2 + b^2 - \beta^2b^2 \text{ So,}$$
$$r^2 = \gamma^{-2}[b^2+\gamma^2v^2t^2] = [\kappa^2R^2]_{ret}$$
$$\text{and } \hat{x}.(\hat{n}-\vec{\beta})_{ret} = \sin(\theta) = \frac{b}{R} \text{ so that}$$



$$E_x = q\left[\frac{\hat{x}.(\hat{n}-\vec{\beta})}{\gamma^2\kappa^3R^2}\right]_{ret} = q\frac{b\gamma}{[b^2+\gamma^2v^2t^2]^{3/2}}.$$