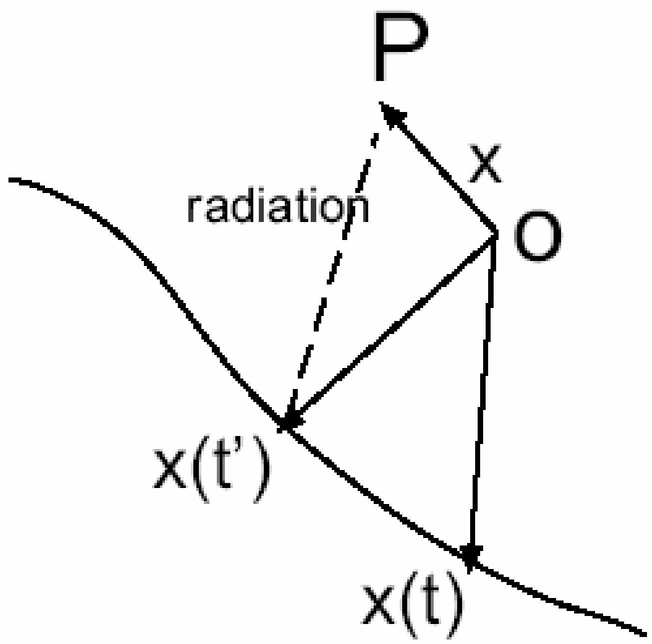
Radiation from accelerating charges

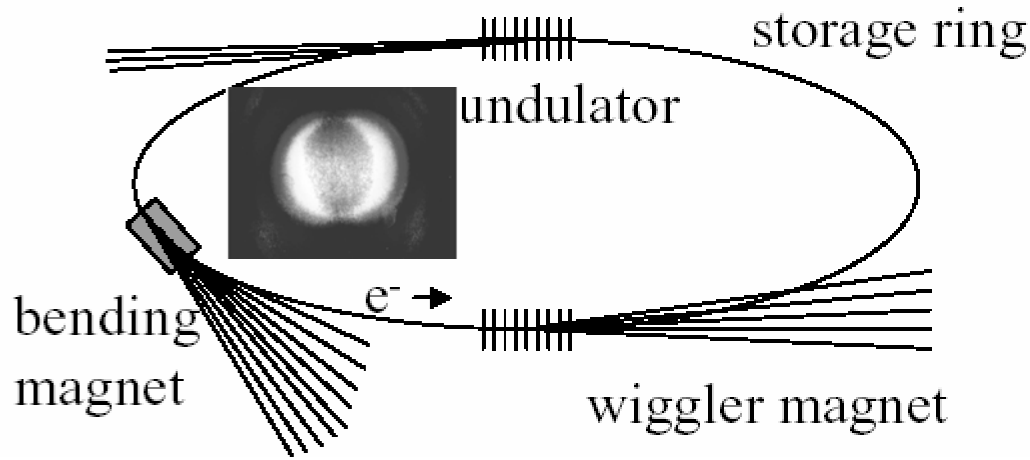
- Radiation emitted by particle at time t' reach observer at time t.
- Retardation occurs due to the finite value of speed of light
- Position of the particle at time t not relevant, need to know the particle history

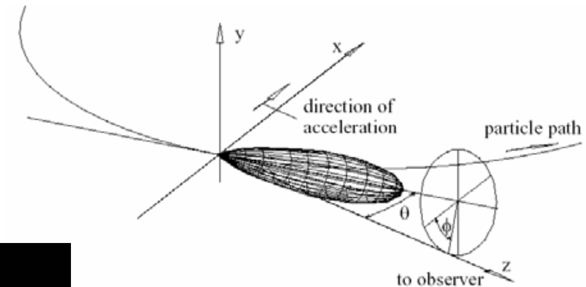


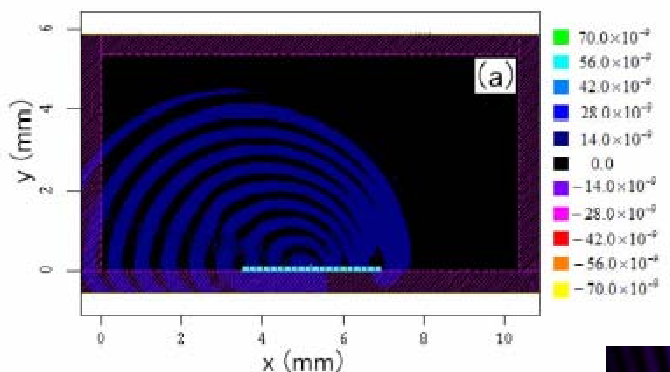
- **Mark your calendar 11/30:** Dr. Rui Li of Jefferson Laboratory will discuss retardation effect in synchrotron radiation and the inherent associated problem in charged particle beams.

Radiation from accelerating charges: application

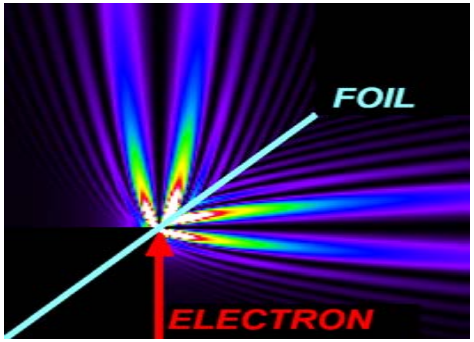
- Synchrotron radiation
- Undulator radiation
- Smith Purcell





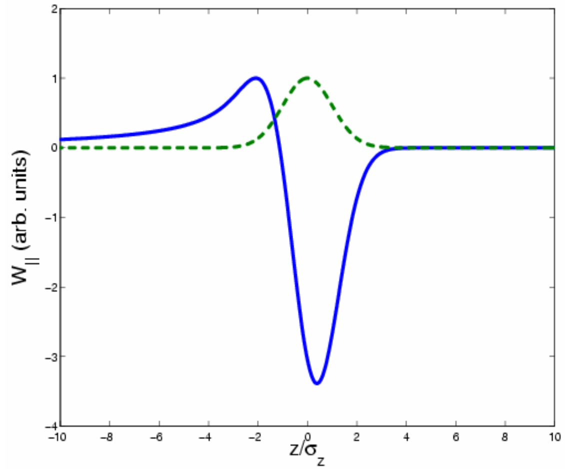


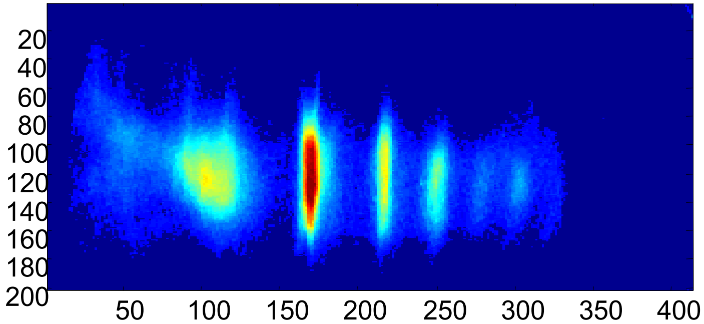
- Transition radiation

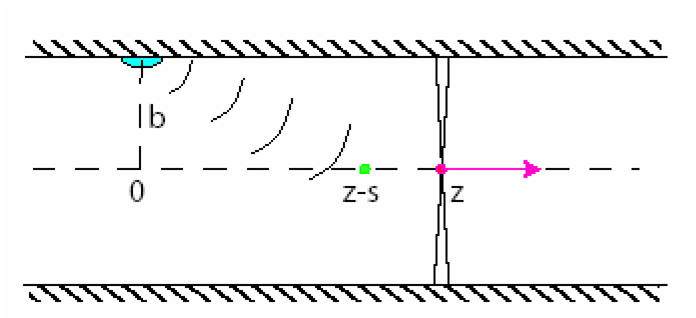


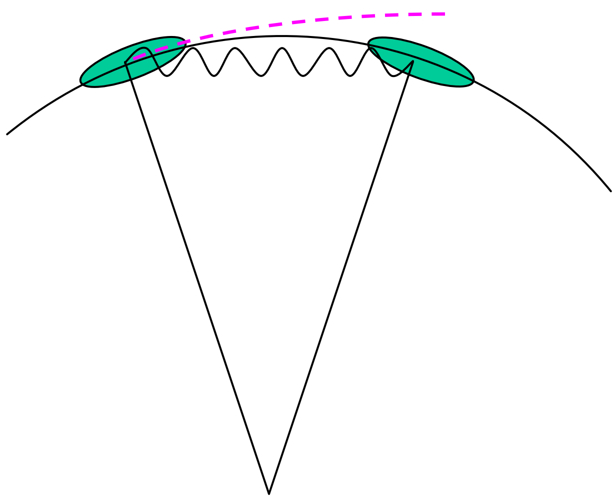
Radiation from accelerating charges: problems…

- Wakefield
- Coherent and
  Incoherent synchrotron radiation









Overtaking length

$$L_0 = (24\sigma_z \rho^2)^{1/3}$$

4-potential associated to a moving charge I

- Start with Maxwell's equation (inhomogeneous)

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta \qquad j^\beta \equiv (c\rho, \vec{J})$$

- Consider the Lorenz' gauge $\partial_\alpha A^\alpha = 0$

$$\partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = \partial_\alpha \partial^\alpha A^\beta = \frac{4\pi j^\beta}{c}$$

- Which can be rewritten

$$\Box A^\beta = \frac{4\pi}{c} j^\beta(x)$$

- Solution of the equation is in term of Green's function

$$\Box_x D(x, x') = \delta^{(4)}(x - x')$$

$$\delta^{(4)}(x - x') \equiv \delta(x_0 - x'_0)\delta^{(3)}(\vec{x} - \vec{x}').$$

If free space: $D(x, x') = D(x - x')$.

4-potential associated to a moving charge II

- With $z^\alpha = x^\alpha - x'^\alpha,\ D(x-x')$ previous eq. rewrites

$$\Box_z D(z) = \delta^{(4)}(z).$$

- Solve using Fourier transform: define

$$\begin{aligned} D(z) &= \frac{1}{(2\pi)^4}\int d^4k\tilde{D}(k)e^{-ikz}, \\ \delta^4(z) &= \frac{1}{(2\pi)^4}\int d^4k e^{-ikz} \end{aligned}$$

- In Fourier space the d'Alembert equation reduces to

$$\tilde{D}(k) = -\frac{1}{k_\beta k^\beta}$$

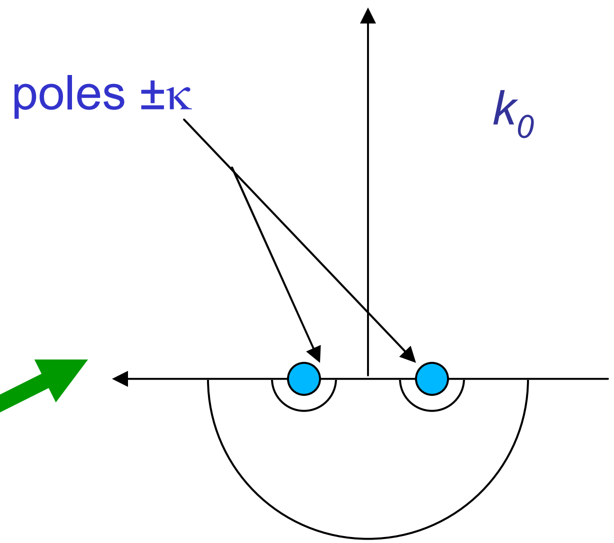
where $k^\beta \equiv (k_0, \vec{\kappa})$

$$z = (z_0, \vec{R}).$$

4-potential associated to a moving charge III

- The Green function is obtained by inverse Fourier transform:

$$
\begin{aligned}
D(z) &= \frac{1}{(2\pi)^4}\int d^4k(-)\frac{e^{-ikz}}{k_0^2-\kappa^2} \\
&= -\frac{1}{(2\pi)^4}\int d^3\kappa e^{i\vec{\kappa}\vec{R}}\int dk_0\frac{e^{-ik_0z_0}}{k_0^2-\kappa^2}
\end{aligned}
$$



- Consider the integral of k_0

$$
\begin{aligned}
\int_{\infty}^{+\infty} dk_0\frac{e^{-ik_0z_0}}{k_0^2-\kappa^2} &= \oint_C dk_0\frac{e^{-ik_0z_0}}{k_0^2-\kappa^2} = -2i\pi\sum \mathrm{Res}\left(\frac{e^{-ik_0z_0}}{k_0^2-\kappa^2}\right) \\
&= -\frac{2\pi}{\kappa}\sin(kz_0)
\end{aligned}
$$

4-potential associated to a moving charge IV

- So D becomes
$$D(z) = \frac{1}{(2\pi)^3}\int d^3k \frac{\sin(\kappa z_0)}{\kappa} e^{i\vec{\kappa}.\vec{R}} \quad (z_0 > 0)$$
$$= \frac{\Theta(z_0)}{(2\pi)^3}\int d^3k \frac{\sin(\kappa z_0)}{\kappa} e^{i\vec{\kappa}.\vec{R}}$$
- Introducing $d^3\kappa = k^2 d\kappa \sin(\theta) d\theta d\phi$ the angular part can be integrated:
$$\int \sin\theta d\theta d\phi e^{i\vec{\kappa}.\vec{R}} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta e^{i\kappa z\cos\theta} = 2\pi\left[\frac{e^{i\kappa z\cos\theta}}{-i\kappa z}\right]_0^{\pi}$$
$$= 4\pi\frac{\sin(\kappa R)}{\kappa R}.$$
-
$$D(z) = \frac{\Theta(z_0)}{(2\pi)^3}\int d\kappa \frac{4\pi}{R}\sin(\kappa R)\sin(\kappa z_0)$$
$$= \frac{\Theta(z_0)}{2\pi^2 R}\int_0^{\infty} d\kappa \sin(\kappa R)\sin(\kappa z_0)$$
$$= -\frac{1}{4\pi R}\frac{1}{2\pi}\int_0^{+\infty}\left[e^{ik(R+z_0)} - e^{ik(R-z_0)} - e^{-ik(R-z_0)} + e^{ik(R+z_0)}\right]$$
$$= \frac{\Theta(z_0)}{4\pi R}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[-e^{ik(R+z_0)} - e^{ik(R-z_0)}\right]$$
$$= \frac{\Theta(z_0)}{4\pi R}\left[\delta(z_0 - R) + \delta(z_0 + R)\right] = \frac{\Theta(z_0)}{4\pi R}\delta(z_0 - R)$$

4-potential associated to a moving charge V

- since the condition $z_0 > 0$ implies $\delta(z_0 + R) = 0$.

$$D(x - x') = \frac{\Theta(z_0)}{4\pi R}\delta(x - x' - R)$$

- Using the identity $\delta[(x - x_1)(x - x_2)] = \frac{\delta(x-x_1)+\delta(x-x_2)}{|x_1-x_2|}$

$$\begin{aligned}\delta[(x - x')^2] &= \delta[(x - x_0)^2 - |x - x'|^2] \\ &= \delta[(x_0 - x_0' - R)(x_0 - x_0' + R)] \\ &= \frac{1}{2R}[\delta(x_0 - x_0' - R) + \delta(x_0 - x_0' + R)]\end{aligned}$$

- *D* becomes

$$D(x - x') = \frac{1}{2\pi}\Theta(x_0 - x_0')\delta[(x - x')^2].$$

4-potential associated to a moving charge VI

- And the retarded potential is

$$A^\alpha(x) = \text{const.} + \frac{4\pi}{c}\int d^4x' D(x-x')J^\alpha(x')$$

- Retarded and **advanced**…

Liénard-Wiechert I

- And the retarded potential is

$$A^\alpha(x) = \frac{4\pi}{c}\int d^4x' D(x-x')j^\alpha(x'),$$

- The 4-current

$$j^\alpha(x') = ec\int d\tau v^\alpha(\tau)\delta^{(4)}[x' - r(\tau)]$$

- Expliciting

$$\begin{aligned} A^\alpha(x) &= 2e\int d\tau d^4x'\Theta(x_0 - x_0')\delta[(x-x')^2]v^\alpha(\tau)\delta^{(4)}[x - r(\tau)] \\ &= 2e\int d\tau\Theta(x_0 - x_0')v^\alpha(\tau)\delta[(x - r(\tau))^2] \end{aligned}$$

- Or, using $\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{\left|\frac{\partial f}{\partial x}\right|_{x=x_i}}.$,

$$\boxed{A^\alpha(x) = 2e\int d\tau\delta(\tau - \tau_0)\Theta(x_0 - x_0')v^\alpha(\tau)\left|\frac{-1}{2v^\beta(\tau)[x - r(\tau)]_\beta}\right|}$$

Liénard-Wiechert II

- So the 4-potential takes the form

$$A^\alpha(x) = \left.\frac{ev^\alpha(\tau)}{v^\beta[x - r(\tau)]_\beta}\right|_{\tau=\tau_0}$$

- Which can be re-expressed

$$\Phi(\vec{x}, t) = \left[\frac{e}{(1 - \vec{\beta}.\hat{n})R}\right]_{ret}, \text{ and, } \vec{A}(\vec{x}, t) = \left[\frac{e\vec{\beta}}{(1 - \vec{\beta}.\hat{n})R}\right]_{ret}$$

ret means the quantity in bracket have to be evaluated at the retarded time t'