Non-uniform B-field: adiabatic invariance

- We now consider the case of a non uniform (still time-independent) magnetic field
- We suppose the magnetic field non-uniformity is slow, i.e. small compare to the gyro-radius
- Then motion is said to be adiabatic and there exist an invariant called the adiabatic invariant:

$$\boxed{J = \oint \overrightarrow{P}_{\perp} . \overrightarrow{dl}}$$

$\overrightarrow{dl}$ is the line element along the particle trajectory.

Non-uniform B-field: adiabatic invariance

- Let's explicit $P_\perp$:

$$\begin{aligned} J &= \oint (\gamma m \overrightarrow{v}_\perp + \frac{q}{c}\overrightarrow{A}).\overrightarrow{dl} \\ &= (\gamma m \omega_B a)(2\pi a) + \frac{q}{c}\int_S \overrightarrow{B}.\hat{n} dS \end{aligned}$$

$$\Rightarrow J = 2\pi\gamma m \omega_B a^2 - \frac{q}{c}\pi B a^2$$

- but $\gamma m \omega_B = \frac{q}{c}B$ so

$$\boxed{J = \frac{q}{c}\pi B a^2.}$$

Non-uniform B-field: adiabatic invariance

- The previous equation implies that

$$\Phi_B = \int_S \overrightarrow{B}.\overrightarrow{dS} = \pi B a^2$$

is an adiabatic invariant

- xxxx

Magnetic mirror

- The previous equation implies that

- Using the adiabatic invariant

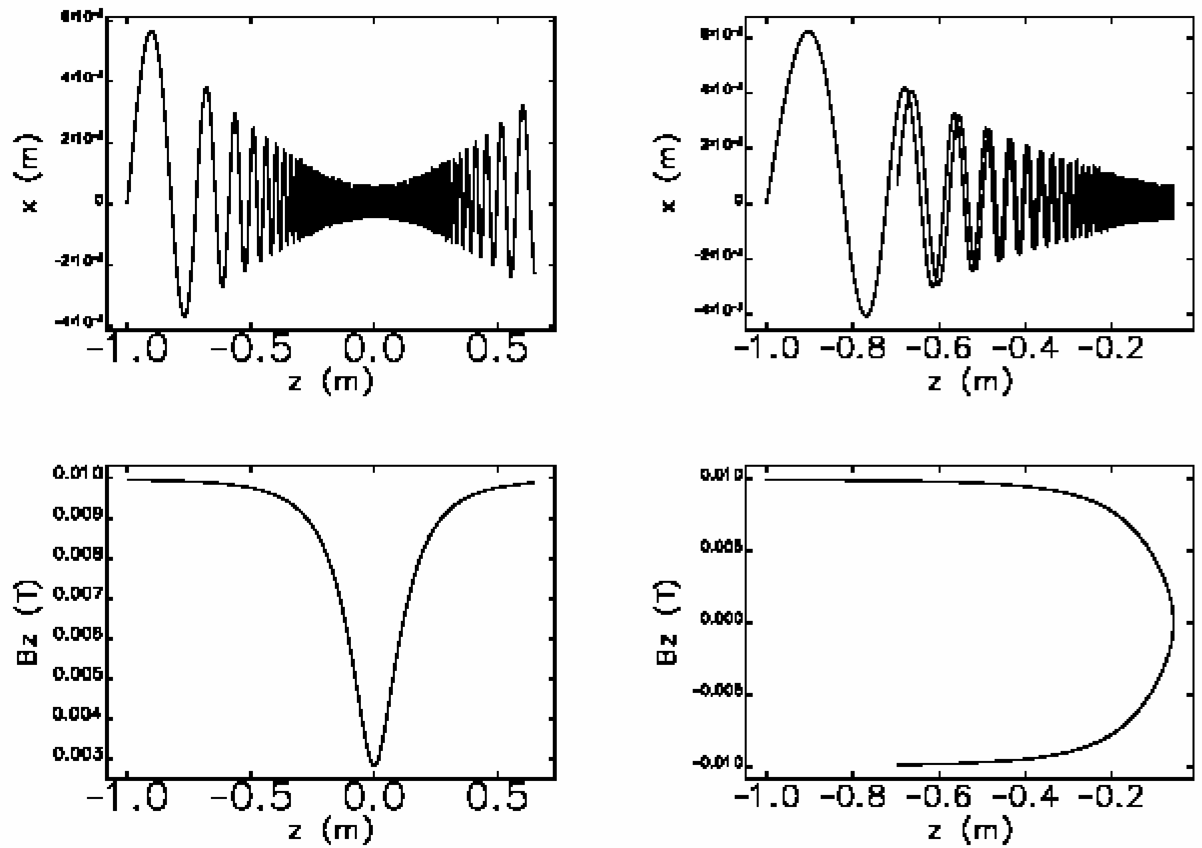
$$\frac{V_0^2 \sin^2(\theta)}{B_0} = \frac{v_\perp^2}{B_m}$$

$$v_\perp^2 = v_0^2 - v_\parallel$$

$$\sin^2(\theta) = \frac{B_0}{B_m}$$

Magnetic mirror

θ=4.75 (left) and θ=5.25 (right).



Trajector (top) in a non-uniform B-field for two cases of injection angle

Non-adiabatic invariance: the solenoid

- Consider a short magnetic solenoidal lens

Let $\overrightarrow{B}(z=0) \equiv B_c \hat{z}.$

- In cylindrical coordinate, compute the θ-component of the Lorentz force (this gives the angular momentum p_θ)

$$\begin{aligned} F_\theta &= \frac{q}{c} v_z B_r = \frac{dp_\theta}{dt};\; p_\theta(t=0) = p_\theta(z=0) = 0 \\ &\Rightarrow p_\theta = \frac{q}{c}\int_0^\infty B_r v_z dt = \frac{q}{c}\int_0^\infty B_r dz \end{aligned}$$

Non-adiabatic invariance: the solenoid

- Integrating over a Gauss-surface

$$\int_S \overrightarrow{B}\overrightarrow{dS} = 0 \;=\; -\pi r^2 B_c + 2\pi r \int_0^\infty B_r dz$$
$$\Rightarrow \int_0^\infty B_r dz = \frac{1}{2} B_c r.$$

- Consequently the charge pick-up the angular:

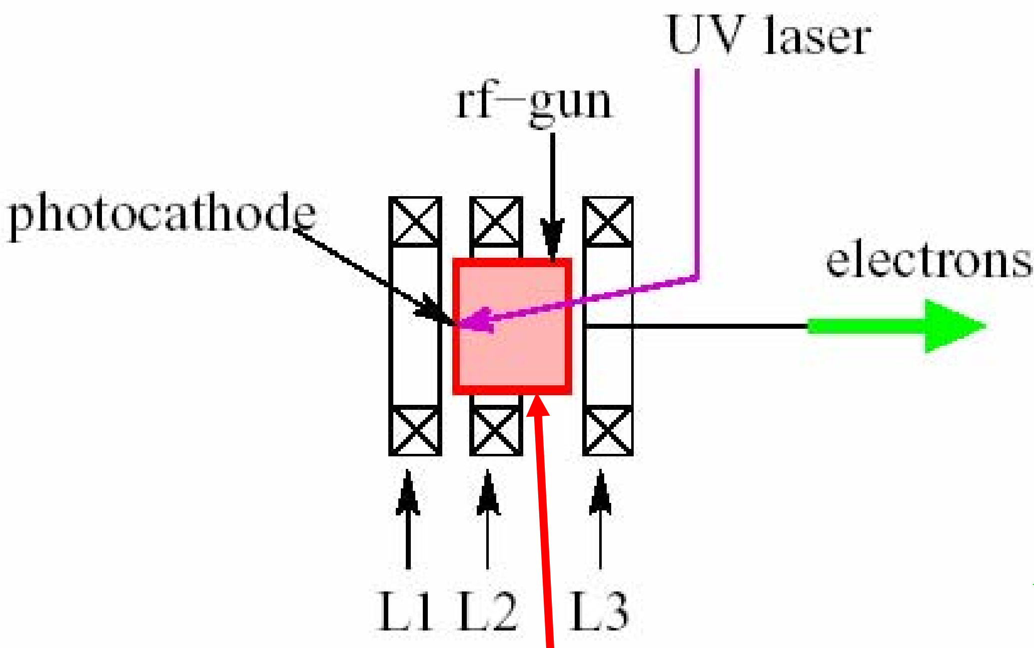
$$\frac{p_\theta}{p_c} = \frac{1}{2}\frac{qB_c}{p_c c} r = \frac{r}{2\rho}$$

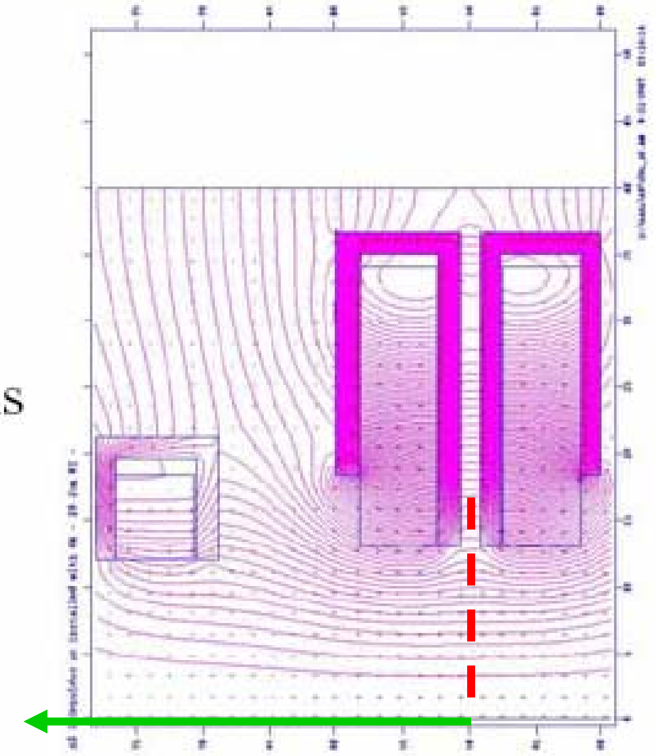
- With $\rho^{-1} \equiv \frac{qB_c}{p_c c}.$

Generation of angular-momentum dominated beams

magnetic field maps

L1, L2, L3: magnetic solenoidal lenses





Radio-frequency gun

e.m. Field tensor & covariant equation of motion

- As we showed we expect
  - Quadratic with e- radial position $L \equiv \vec{r} \times \vec{p} \propto r^2$
  - Full conversion of CAM to MAM as the electrons exit the magnetic field (A becomes zero)