

# e.m. Field tensor & covariant equation of motion

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- Define the tensor of dimension 2

$$F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = g^{\alpha\delta} \partial_\delta A^\beta - g^{\beta\delta} \partial_\delta A^\alpha \quad \swarrow \text{4 potential}$$

- $F$ , is the e.m. field tensor. It is easily found to be

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad F_{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

- In SI units,  $F$  is obtained by  $E \rightarrow E/c$

- The equation of motion is

$$\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^\alpha_\beta u^\beta.$$



# Invariant of the e.m. field tensor

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- Consider the following invariant quantities

$$F^{\mu\nu} F_{\mu\nu} = 2(E^2 - B^2), \text{ and } F^{\mu\nu} \mathcal{F}_{\mu\nu} = 4\vec{E} \cdot \vec{B}$$

- Usually one redefine these invariants as

$$\mathcal{I}_1 \equiv -\frac{1}{4}F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(B^2 - E^2), \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4}F^{\mu\nu} \mathcal{F}_{\mu\nu} = -\vec{E} \cdot \vec{B}.$$

- Which can be rewritten as  $\mathcal{I}_1 \equiv -\frac{1}{4}\text{tr}(F^2)$  and  $\mathcal{I}_2 \equiv -\frac{1}{4}\text{tr}(F\mathcal{F})$

where  $F \equiv F_{\mu}^{\nu} = F^{\mu\alpha}g_{\alpha\nu}$  and  $\mathcal{F} \equiv \mathcal{F}_{\mu}^{\nu} = \mathcal{F}^{\mu\alpha}g_{\alpha\nu}$ .

- Finally note the identities

$$F\mathcal{F} = \mathcal{F}F = -\mathcal{I}_2 I, \text{ and } F^2 - \mathcal{F}^2 = -2\mathcal{I}_1 I$$



# Eigenvalues of the e.m. field tensor

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- The eigenvalues are given by

$$F\Psi = \lambda\Psi \Rightarrow \mathcal{F}F\Psi = \lambda\mathcal{F}\Psi \Rightarrow \mathcal{F}\Psi = -\frac{\mathcal{I}_2}{\lambda}\Psi.$$

$$(F^2 - \mathcal{F}^2)\Psi = -2I\mathcal{I}_1\Psi = [\lambda^2 - (\mathcal{I}_2/\lambda)^2]\Psi,$$

- Characteristic polynomial  $\lambda^4 + 2\mathcal{I}_1\lambda^2 - \mathcal{I}_2^2 = 0$ .
- With solutions

$$\lambda_{\pm} = \sqrt{\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2} \pm \mathcal{I}_1}$$

$$\lambda_1 = -\lambda_2 = \lambda_-, \quad \lambda_3 = -\lambda_4 = i\lambda_+.$$



# Motion in an arbitrary e.m. field I

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- We now attempt to solve directly the equation

$$\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^\alpha{}_\beta u^\beta;$$

We consider a time independent e.m. field

following the treatment by Munos. Let  $\theta \equiv \frac{q\tau}{mc}$ .

- The equation of motion reduces to

$$\frac{dU}{d\theta} = Fu \quad \text{with solution } u = e^{\theta F} u(0).$$

- Where the matrix exponential is defined as

$$e^{\theta F} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} F^n.$$



# Motion in an arbitrary e.m. field II

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- The main work is now to compute the matrix exponential.
- To compute the power series of  $F$  one needs to recall the identities

$$F^2 = \mathcal{F}^2 - 2\mathcal{I}_1 I$$

- Because of this one can show that any power of  $F$  can be written as linear combination of  $F$ ,  $\mathcal{F}$ ,  $F^2$  and  $I$ :

$$F^3 = FF^2 = F\mathcal{F}^2 - 2\mathcal{I}_1 F = -\mathcal{I}_2 \mathcal{F} - 2\mathcal{I}_1 F;$$

$$F^4 = -\mathcal{I}_2 F\mathcal{F} - 2\mathcal{I}_1 F^2 = \mathcal{I}_2^2 I - 2\mathcal{I}_1 F^2;$$

$$F^5 = \mathcal{I}_2^2 F - 2\mathcal{I}_1 F^3 = (4\mathcal{I}_1^2 + \mathcal{I}_2^2)F + 2\mathcal{I}_1 \mathcal{I}_2 \mathcal{F};$$

etc...

- This means  $e^{\theta F} = \alpha I + \beta F + \gamma \mathcal{F} + \delta F^2$ .



# Motion in an arbitrary e.m. field III

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- Consider

$$e^{\theta F} = \alpha I + \beta F + \gamma \mathcal{F} + \delta F^2.$$

- We need to compute the coefficient of the expansion

$$t_0 \equiv \frac{1}{4} \text{Tr}[e^{\theta F}] = \alpha - \mathcal{I}_1 \delta,$$

$$t_1 \equiv \frac{1}{4} \text{Tr}[F e^{\theta F}] = -\mathcal{I}_1 \beta - \mathcal{I}_2 \gamma,$$

$$t_2 \equiv \frac{1}{4} \text{Tr}[F^2 e^{\theta F}] = -\mathcal{I}_1 \alpha + (2\mathcal{I}_1^2 + \mathcal{I}_2^2) \delta,$$

$$t_3 \equiv \frac{1}{4} \text{Tr}[F^3 e^{\theta F}] = 2(\mathcal{I}_1^2 + \mathcal{I}_2^2) \beta + \mathcal{I}_1 \mathcal{I}_2 \gamma.$$



# Motion in an arbitrary e.m. field IV

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- The solution for the coefficient is

$$\begin{aligned}\alpha &= \frac{(2\mathcal{I}_1^2 + \mathcal{I}_2^2)t_0 + \mathcal{I}_1 t_2}{\mathcal{I}_1^2 + \mathcal{I}_2^2}; & \beta &= \frac{t_3 + \mathcal{I}_1 t_1}{\mathcal{I}_1^2 + \mathcal{I}_2^2}; \\ \gamma &= -\frac{(2\mathcal{I}_1^2 + \mathcal{I}_2^2)t_1 + \mathcal{I}_1 t_3}{\mathcal{I}_2(\mathcal{I}_1^2 + \mathcal{I}_2^2)}; & \delta &= \frac{t_2 + \mathcal{I}_1 t_0}{\mathcal{I}_1^2 + \mathcal{I}_2^2}.\end{aligned}$$

- Now let evaluate the  $t_i$ 's
- The Trace is invariant upon change of basis. So consider a basis where  $F$  is diagonal, let  $F'$  be the diagonal form then

$$\text{Tr}[e^{\theta F}] = \text{Tr}[e^{\theta F'}] = \sum_{i=1}^4 e^{\theta \lambda_i}$$

- Recall than

$$\lambda_1 = -\lambda_2 = \lambda_-, \text{ and } \lambda_3 = -\lambda_4 = i\lambda_+ \quad \lambda_{\pm} = \sqrt{\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2} \pm \mathcal{I}_1}$$



# Motion in an arbitrary e.m. field $V$

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- The traces are then

$$t_0 = \frac{1}{4} \text{Tr}[e^{\theta F}] = \frac{1}{2} [\cosh(\theta \lambda_-) + \cos(\theta \lambda_+)]$$
$$t_k = \frac{1}{4} \text{Tr}[F^k e^{\theta F}] = \frac{\partial^k t_0}{\partial \theta^k}$$

- Now let evaluate the  $t_i$ 's

$$t_1 = \frac{1}{2} [\lambda_- \sinh(\theta \lambda_-) - \lambda_+ \sin(\theta \lambda_+)]$$
$$t_2 = \frac{1}{2} [\lambda_-^2 \cosh(\theta \lambda_-) - \lambda_+^2 \cos(\theta \lambda_+)]$$
$$t_3 = \frac{1}{2} [\lambda_-^3 \sinh(\theta \lambda_-) + \lambda_+^3 \sin(\theta \lambda_+)]$$



# Motion in an arbitrary e.m. field VI

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- The traces are then

$$\alpha = \frac{\lambda_+^2 \cosh(\theta\lambda_-) + \lambda_-^2 \cos(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}}; \quad \beta = \frac{\lambda_- \sinh(\theta\lambda_-) + \lambda_+ \sin(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}};$$
$$\gamma = \frac{|\mathcal{I}_2|}{\mathcal{I}_2} \frac{\lambda_- \sin(\theta\lambda_+) - \lambda_+ \sinh(\theta\lambda_-)}{2\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}}; \quad \delta = \frac{\cosh(\theta\lambda_-) - \cos(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}}.$$

- Substitute in the power expansion to yield

$$u(\theta) = \frac{1}{2\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}} \left[ (\lambda_+^2 I + F^2) \cosh(\theta\lambda_-) + (\lambda_-^2 I - F^2) \cos(\theta\lambda_+) \right. \\ \left. + \left( \lambda_- F - \frac{|\mathcal{I}_2|}{\mathcal{I}_2} \lambda_+ \mathcal{F} \right) \sinh(\theta\lambda_-) + \left( \lambda_+ F + \frac{|\mathcal{I}_2|}{\mathcal{I}_2} \lambda_- \mathcal{F} \right) \sin(\theta\lambda_+) \right] u(0).$$



# Motion in an arbitrary e.m. field VII

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- Remember that  $u(\theta) = \frac{2}{mc} \frac{dx}{d\theta}$ .
- Integrate for  $\theta \in [0, \theta)$

$$\begin{aligned} x(\tau) = & x(0) + \frac{mc}{q\mathcal{I}_2} \mathcal{F} u(0) + \frac{mc}{2q\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2}} \left[ \left( F - \frac{\lambda_+^2}{\mathcal{I}_2} \mathcal{F} \right) \cosh(\theta\lambda_-) \right. \\ & \left. - \left( F + \frac{\lambda_-^2}{\mathcal{I}_2} \mathcal{F} \right) \cos(\theta\lambda_+) + \frac{\lambda_+^2 I + F^2}{\lambda_-} \sinh(\theta\lambda_-) + \frac{\lambda_-^2 I - F^2}{\lambda_+} \sin(\theta\lambda_+) \right] u(0). \end{aligned}$$



# Motion in an arbitrary e.m. field VIII

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- Let's consider the special case  $\vec{E} = E\hat{x}$ ,  $\vec{B} = B\hat{y}$
- Then  $\vec{E} \perp \vec{B} \Rightarrow \mathcal{I}_2 = 0$ .
- So we just take the limit  $\mathcal{I}_2 \rightarrow 0$  in the equation motion derived in the previous slide which means:  
 $\lambda_- \rightarrow 0$ ;  $\lambda_+ \rightarrow \sqrt{2\mathcal{I}_1}$ ,  $\cosh(\theta\lambda_-) \rightarrow 1$  and  $\sinh(\theta\lambda_-)/\lambda_- \rightarrow \theta$ .
- So we obtain

$$x(\tau) = \frac{mc}{q\mathcal{I}_2}\mathcal{F}u(0) + \frac{mc}{2q\mathcal{I}_1} \left[ \left( F - \frac{2\mathcal{I}_1}{\mathcal{I}_2}\mathcal{F} \right) - F \cos(\theta\lambda_+) \right. \\ \left. + (2\mathcal{I}_1 I + F^2)\theta - \frac{1}{\sqrt{2\mathcal{I}_1}}F^2 \sin(\theta\sqrt{2\mathcal{I}_1}) \right] u(0).$$



# ExB drift

- With  $\Omega \equiv \frac{q}{mc} \sqrt{2\mathcal{I}_1}$  then

$$x(\tau) = \left( I + \frac{F^2}{2\mathcal{I}_1} u(0) \tau + \frac{mc}{2q\mathcal{I}_1} (1 - \cos \Omega\tau) - \frac{F}{\sqrt{2\mathcal{I}_1}} \sin \Omega\tau \right) F u(0)$$

- compute

$$F u(0) = \gamma_0 c \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \beta_{0x} \\ \beta_{0y} \\ \beta_{0z} \end{pmatrix} = \gamma_0 c \begin{pmatrix} E \\ E - \beta_{0z} B \\ 0 \\ \beta_{0x} B \end{pmatrix}$$

$$F^2 u(0) = \gamma_0 c \begin{pmatrix} E(E - \beta_{0z} B) \\ -2\mathcal{I}_1 \beta_{0x} \\ 0 \\ B(E - \beta_{0z} B) \end{pmatrix}$$



# ExB drift I

- With  $\Omega \equiv \frac{q}{mc} \sqrt{2\mathcal{I}_1}$  then

$$x(\tau) = \left( I + \frac{F^2}{2\mathcal{I}_1} u(0) \tau + \frac{mc}{2q\mathcal{I}_1} (1 - \cos \Omega\tau) - \frac{F}{\sqrt{2\mathcal{I}_1}} \sin \Omega\tau \right) F u(0)$$

- compute

$$F u(0) = \gamma_0 c \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \beta_{0x} \\ \beta_{0y} \\ \beta_{0z} \end{pmatrix} = \gamma_0 c \begin{pmatrix} E \\ E - \beta_{0z} B \\ 0 \\ \beta_{0x} B \end{pmatrix}$$

$$F^2 u(0) = \gamma_0 c \begin{pmatrix} E(E - \beta_{0z} B) \\ -2\mathcal{I}_1 \beta_{0x} \\ 0 \\ B(E - \beta_{0z} B) \end{pmatrix}$$



# ExB drift II

- The “projected” equation of motions

$$x = \frac{\gamma_0 mc^2}{2q\mathcal{I}_1} \left[ (E - B\beta_{0z})(1 - \cos \Omega\tau) + \sqrt{2\mathcal{I}_1} \beta_{0x} \sin \Omega\tau \right]$$

$$y = \gamma_0 v_{0y} \tau$$

$$z = \frac{\gamma_0 c E}{2\mathcal{I}_1} (B - E\beta_{0z}) \tau + \frac{\gamma_0 mc^2 B}{2q\mathcal{I}_1} \left[ \beta_{0x}(1 - \cos \Omega\tau) - \frac{E - B\beta_{0z}}{\sqrt{2\mathcal{I}_1}} \sin \Omega\tau \right]$$

$$t = \frac{\gamma_0 B}{2\mathcal{I}_1} (B - E\beta_{0z}) \tau + \frac{\gamma_0 mc E}{2q\mathcal{I}_1} \left[ \beta_{0x}(1 - \cos \Omega\tau) - \frac{E - B\beta_{0z}}{\sqrt{2\mathcal{I}_1}} \sin \Omega\tau \right]$$

- the particle has a velocity perpendicular to  $\vec{E}$  and  $\vec{B}$  fields.
- This is the so-called **ExB** drift and the drift velocity of the particle is  $v_d = cE/B$ .



# ExB drift III

