# Particle in a static E-field

- Consider a charge particle q interacting with a E-field ***E***=*E**x***.
- Particle's initial conditions ***p***$(t=0)=p_0$***y***.
- The Lorentz force is:

$$\dot{p}_x = qE, \text{ and}, \dot{p}_y = 0$$

- integrate

$$p_x = qEt, \text{ and}, p_y = p_0$$

- So that $p^2 = (qEt)^2 + p_0^2.$

- total energy a time *t* is:

$$\mathcal{E}^2(t) = c^2\left[(qEt)^2 + p_0^2\right] + m^2c^4 = (cqEt)^2 + \mathcal{E}_0$$

$$\mathcal{E}_0 \equiv \mathcal{E}(t=0)$$

# Particle in a static E-field

- The velocity is then

$$v_x = \frac{dx}{dt} = c\frac{cqEt}{\sqrt{(cqEt)^2 + \mathcal{E}_0^2}}$$

- Integrate

$$x(t) = \frac{1}{qE}\sqrt{(cqEt)^2 + \mathcal{E}_0^2}.$$

- Similarly for *y* axis we have

$$\frac{dy}{dt} = \frac{c^2 p_0}{\sqrt{(cqEt)^2 + \mathcal{E}_0^2}},$$

- Integrate $\int_0^\xi \frac{d\tilde{\xi}}{\tilde{\xi}^2+1} = \sinh^{-1}(\xi)$

$$y = \frac{p_0 c}{qE}\sinh^{-1}\left(\frac{cqEt}{\mathcal{E}_0}\right).$$

# Particle in a static E-field

- Explicit *t* as a function of *y*

$$cqEt = \sinh\left(\frac{qEy}{p_0 c}\right)$$

- And insert in *x(t)*

$$\begin{aligned} x &= \frac{\mathcal{E}_0}{qE}\sqrt{\sinh^2\left(\frac{qEy}{p_0 c}\right)+1} \\ &= \frac{\mathcal{E}_0}{qE}\cosh\left(\frac{qEy}{p_0 c}\right). \end{aligned}$$

- Note the NR limit

$$x = \frac{mc^2}{qE}\cosh\left(\frac{qEy}{mv_0 c}\right) \simeq \frac{qE}{2mv_0^2}y^2 + \text{const.}$$

# Particle in a static E-field

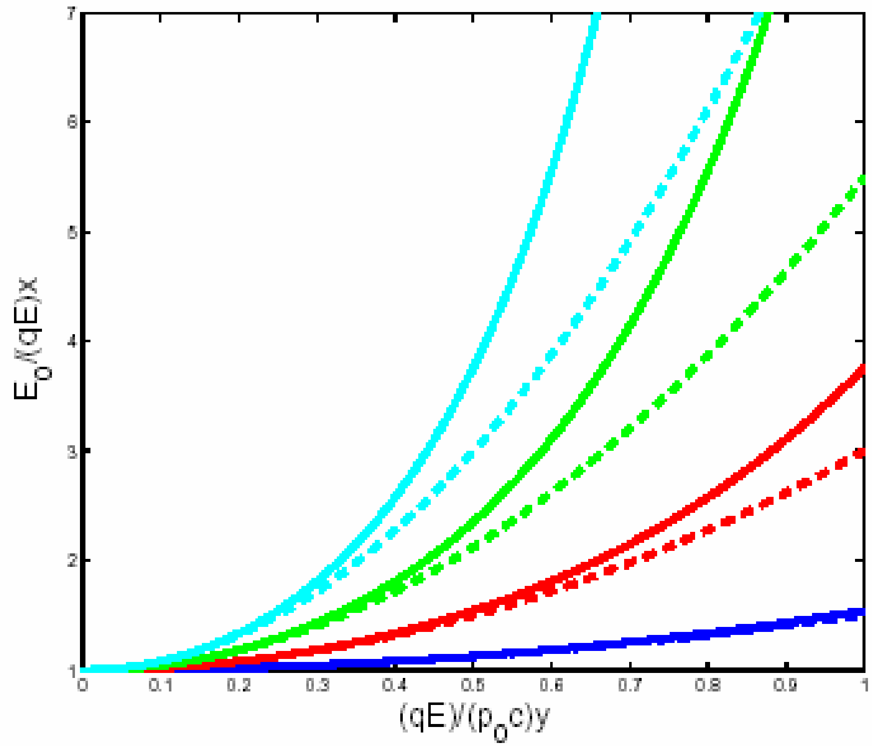



Figure 3.1: Trajectories (in normalized coordinate) in uniform constant E-field: $\hat{x} = \cosh(\kappa\hat{y})$, with $\kappa = 1, 2, 3, 4$. dashed are corresponding parabolic approximation $\hat{x} = 1 + \frac{1}{2}(\kappa\hat{y})^2$.

# Particle in a static B-field

- Consider a charge particle q interacting with a B-field ***B***=*E**z***.
- Partic $\overrightarrow{x}(x^1, x^2, x^3)$ nditions ***p***$(t=0)=p_0$***y***.
- The Lorentz force is:

$$\dot{\overrightarrow{p}} = \frac{q}{c}\overrightarrow{v} \times \overrightarrow{B}; \quad \overrightarrow{p} = \frac{\mathcal{E}}{c^2}\overrightarrow{v} \Rightarrow \dot{\overrightarrow{v}} = \frac{cq}{\mathcal{E}}\overrightarrow{v} \times \overrightarrow{B}.$$

- From

$$\overrightarrow{v} \times \overrightarrow{B} = v_y B\hat{x} - v_x B\hat{y},$$

- We get the 3 equations

$$\begin{aligned}
\dot{v}_x &= \frac{cqB}{\mathcal{E}} v_y, \\
\dot{v}_y &= -\frac{cqB}{\mathcal{E}} v_x, \\
\dot{v}_z &= 0.
\end{aligned}$$

# Particle in a static B-field

- These are a system of 3 coupled ODEs of the form

$$\dot{v}_x = \omega v_y, \ \dot{v}_y = -\omega v_x, \ \dot{v}_z = 0.$$

with $\omega \equiv \frac{cqB}{\mathcal{E}}$

- We can cast the transverse equations into one equation

$$\frac{d}{dt}(v_x + i v_y) = -i\omega(v_x + i v_y),$$

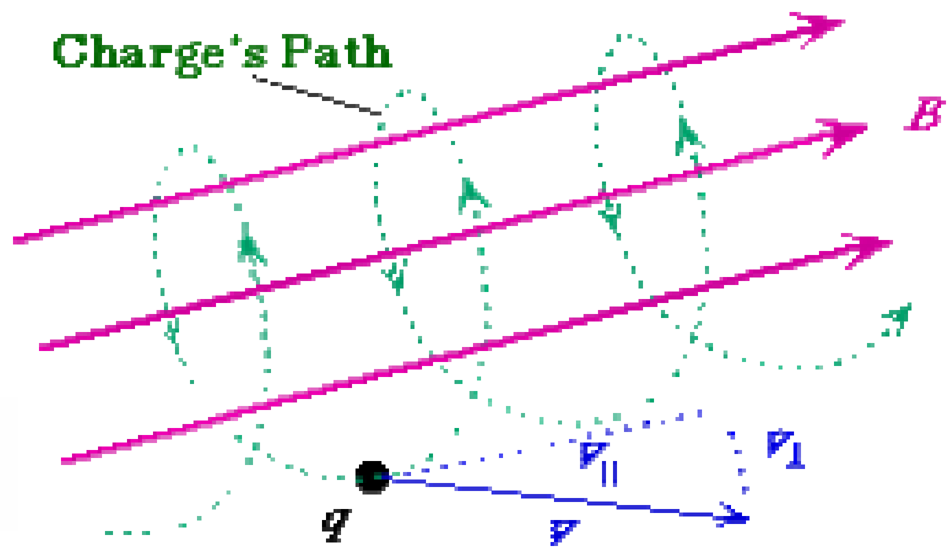
- Whose solution is $v_x + i v_y = v_\perp e^{-i(\omega t + \alpha)}$ so finally ($v_\parallel = v_z$ )

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_\perp \cos(\omega t + \alpha) \\ -v_\perp \sin(\omega t + \alpha) \\ v_\parallel \end{pmatrix}; \text{ with } v_\perp = \sqrt{v_x^2 + v_y^2}, \text{ and,}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + R\sin(\omega t + \alpha) \\ y_0 + R\cos(\omega t + \alpha) \\ z_0 + v_\parallel t \end{pmatrix}; \text{ with } R \equiv \frac{v_\perp}{\omega} = \frac{v_\parallel \mathcal{E}}{cqB}.$$

# Particle in a static B-field

- This is the equation of an helix with axis along ***z*** and radius *R*




- *R* is the gyro-radius

$$R = \frac{v_{\|}\mathcal{E}}{cqB} = \frac{p_{\perp}c}{qB}.$$

- $\omega$ is the gyro-frequency

$$\omega = \frac{qcB}{\mathcal{E}} = \frac{qcB}{\gamma mc^2} \Rightarrow \frac{qB}{\gamma mc}$$

- In SI units:

$$\omega = \frac{qB}{\gamma m}, \text{ and } R = \frac{\gamma m v_{\perp}}{qB}.$$