Northern Illinois University, PHY 571, Fall 2006

# Part V: Scattering

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Two types of scattering:

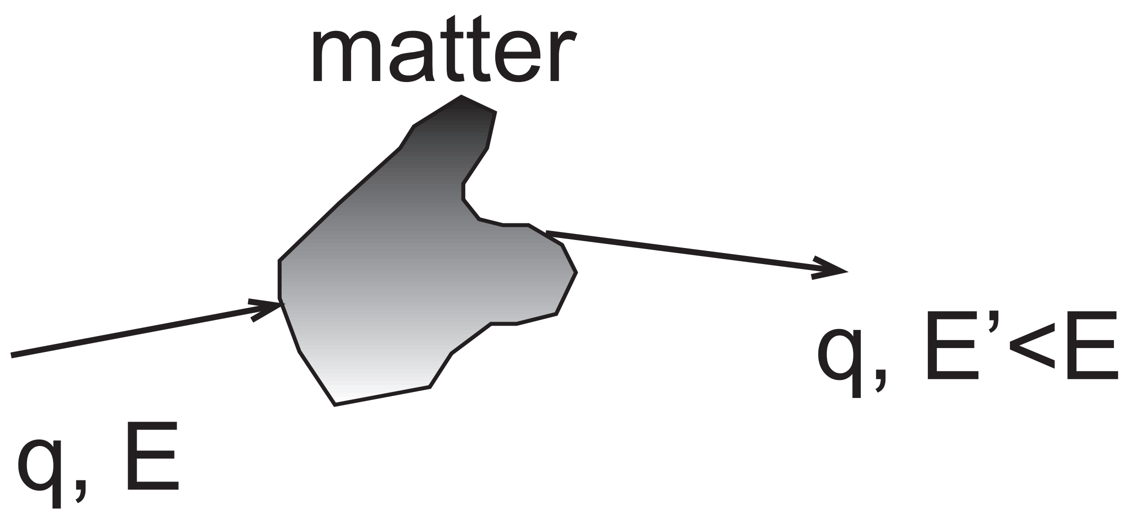
- e- ($q = -e, m_e = 9.1 \times 10^{-31}$ kg)

$\longrightarrow$ high energy loss, small deflection,

- nuclei ($q = Ze, m_n \gg m_e$)

$\longrightarrow$ low energy loss, large deflection

There are more e- than nuclei (factor $Z$) so Z more time e- scattering...



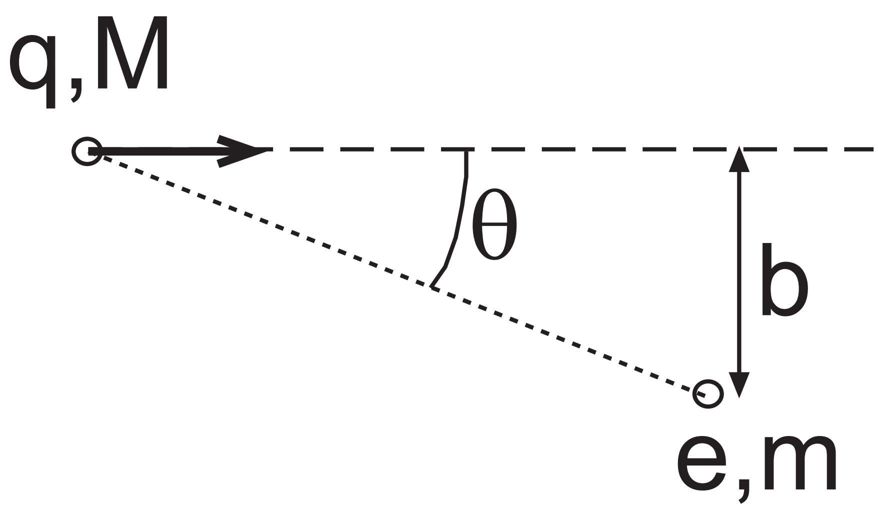
Energy transfer:

Impulse approximation (IA):

- incident particle is not deflected by collision,
- target particle is stationary during collision.

E-field at target is: (we ignore magnetic field since $e$ is stationary (in IA).

$$\overrightarrow{E}(x=-b, y=z=0, t) = -\gamma q \frac{b\hat{x} + vt\hat{z}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{1}$$



The momentum transfer from $q$ to $e$ is:

$$\Delta \overrightarrow{p} = \int_{-\infty}^{+\infty} dte\overrightarrow{E} = -qe\gamma \int_{-\infty}^{+\infty} dt \frac{b\widehat{x} + vt\widehat{z}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \tag{2}$$

$$= -qe\gamma \int_{-\infty}^{+\infty} dt \frac{b\widehat{x}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \frac{-qe}{vb}\widehat{x} \int_{-\infty}^{+\infty} \frac{du}{(1+u^2)^{3/2}},$$

$$= \frac{-qe}{vb}\widehat{x} \left[ \frac{u}{\sqrt{1+u^2}} \right]_{-\infty}^{+\infty} = -\frac{2qe}{vb}\widehat{x}. \tag{3}$$

The associated energy change is:

$$\Delta T = \sqrt{(\Delta p_e c)^2 + (mc^2)^2} - mc^2 \simeq \frac{2}{m}\left(\frac{qe}{vb}\right)^2 \tag{4}$$

where the RHS approximation is written in the non-relat. (NR) limit ($\Delta p \ll mc$). For e-: $\Delta T_e \equiv \Delta T \propto \frac{e^2}{m}$. For nuclei $\Delta T_n \propto \frac{q_n^2}{m_n} \propto \frac{Z^2 e^2}{m_n}$. Hence

$$\frac{\Delta T_n}{\Delta T_e} = \left(\frac{q_n}{e}\right)^2 \frac{m_e}{m_n} = Z^2 \frac{m_e}{m_n} = \frac{Z}{1836} \ll 1. \tag{5}$$

So we see that e- are much more efficient that nuclei at extracting energy from incident particles. But when is the IA valid? Let's check the assumptions:

1- incident particle travels on straight path:

$$\theta = \frac{\Delta p_e}{\gamma M v} = \frac{2qe}{\gamma M v^2 b} = \frac{2}{\gamma}\frac{qe/b}{Mv^2} = 2\frac{\text{E electrostat.}}{\text{incident Energy}} = 2\frac{\mathcal{V}}{E}. \tag{6}$$

So $\theta \ll 1 \Rightarrow \mathcal{V} \ll E$.

2- target remains stationary $\Rightarrow$ recoil distance during collision is $d \ll b$. The interation time is $\tau \sim \frac{b}{\gamma v}$, and corresponding recoil distance is $d \sim \frac{\Delta p_e}{m}\tau$ so

$$d \ll b \Rightarrow \frac{\Delta p_e}{m}\frac{b}{\gamma v} \ll 1 \Rightarrow \frac{2qe/b}{\gamma m v^2} \ll 1. \tag{7}$$

this is a stronger condition than the one coming from $\theta \ll 1$ by a factor $M/m$.

So let's keep the stronger condition (and rewrite it by making the classical radius of e- appearing):

$$\frac{2}{\gamma}\frac{q}{e}\frac{e^2/(mc^2)}{b}\frac{c^2}{v^2} \ll 1 \tag{8}$$

So IA is valid when:

$$\frac{2}{\beta\gamma}\frac{q}{e}\frac{r_e}{b} \ll 1 \tag{9}$$

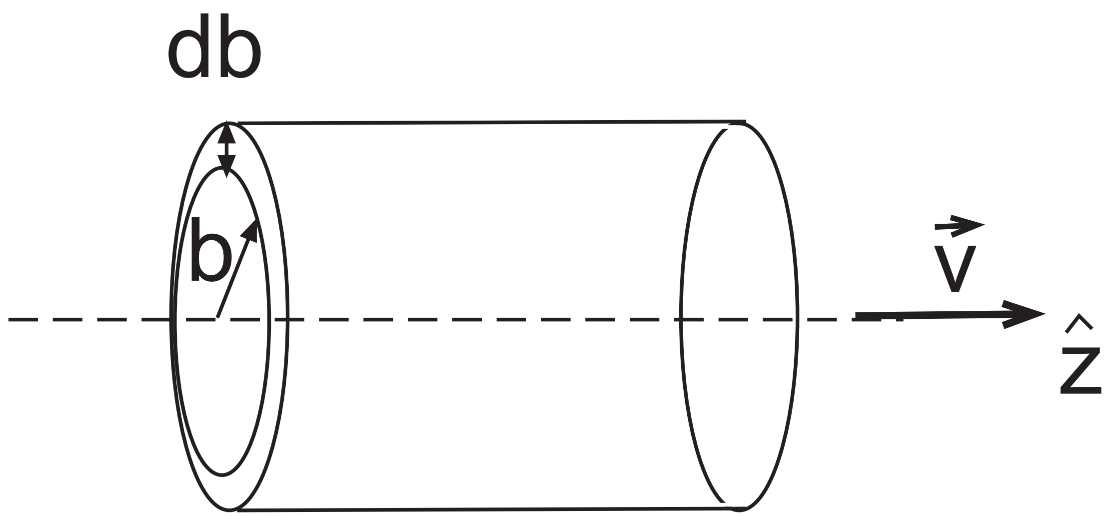
The NR limit implies:

$$\begin{aligned}\frac{\Delta p_e}{mc} &\ll 1 \Rightarrow \frac{2qe}{mvcb} \ll 1 \Rightarrow 2\frac{q}{e}\frac{e^2/(mc^2)}{\beta b} \ll 1 \\ &\Rightarrow \frac{2}{\beta}\frac{q}{e}\frac{r_e}{b} \ll 1.\end{aligned} \tag{10}$$

Same as condition for IA to be valid but with $\gamma \to 1$.

Now consider the case when the charge $q$ passes through a bulk material (many e-), let $n_e$ be the electron density. We have to add all their respective energy gain to deduce their influence on $q$'s slowing down. We do not need to consider the nuclei to a good approximation. The total number of e- in a cylindrical shell of radius $b$ and thickness $db$ is:

$$N_e = n_e(vdt)(2\pi bdb) \tag{11}$$



The differential energy loss by the charge $q$ is:

$$\frac{d^2T_q}{dtdb} = -2\pi n_e v b \left[\frac{2}{m}\left(\frac{qe}{bv}\right)^2\right] \tag{12}$$

(minus comes from energy is lost by $q$). Integrating over $b$ gives:

$$\begin{aligned} \frac{dT_q}{dt} &= -4\pi n_e \frac{(qe)^2}{mv}\int_{b_{min}}^{b_{max}} \frac{db}{b} && (13) \\ &= -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{b_{max}}{b_{max}}. && (14) \end{aligned}$$

We cannot integrate from 0 to $\infty$. When $b \to 0$, IA is no more valid. IA is valid when

$$\frac{2}{\beta^2\gamma}\frac{q}{e}\frac{r_e}{b} \ll 1 \tag{15}$$

$\Rightarrow$ Take $b_{min} \equiv \frac{2}{\beta^2\gamma}\frac{q}{e}r_e = \frac{qe}{\gamma m v^2}$.

Choice of $b_{max}$: e- are bounded with energy $E_e$. Their orbits have angular frequency $\omega_e = E_e/h$. We must have the collision time $\tau \ll \omega_0^{-1}$ otherwise target not stationary and IA not applicable. This gives:

$$\tau \sim \frac{b}{\gamma v}; \;\; \frac{b_{max}}{\gamma v} = \frac{1}{\omega_0} \tag{16}$$
$$\Rightarrow b_{max} = \frac{\gamma v}{\omega_0}.$$

Then

$$\frac{dT_q}{dt} = -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\frac{\gamma v}{\omega_0}}{\frac{qe}{\gamma m v^2}} = -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\gamma v}{\omega_0 b_{min}}, \tag{17}$$
$$= -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\gamma^2 m v^3}{q e \omega_0} \tag{18}$$

Note that $d\Delta T_q/dt = dE/dt$ ($E$ is total energy of $q$), and $(1/v)d/dt = d/dz$ so we can write:

$$\frac{dE}{dz} \;=\; -4\pi n_e \frac{(qe)^2}{mv^2} \ln \frac{\gamma^2 mv^3}{qe\omega_0} \text{ [JDJ Eq (13.9)].} \tag{19}$$

This equation has been derived under the IA. Compare to Bohr's results (1915) more carefully derived:

$$\frac{dE}{dz} \;=\; -4\pi n_e \frac{(qe)^2}{mv^2} \left[ \ln \frac{1.123\gamma^2 mv^2}{qe\langle\omega\rangle} - \frac{1}{2}\frac{v^2}{c^2} \right], \tag{20}$$

where $\langle\omega\rangle$ represents the average angular frequency of the bound electron in taget. The agreement between Bohr's and the equation we derived is no bad: the IA seems to contain the essential physics.

## Influence of Dielectric Screening

For particles not too relativistic the observed energy loss is accurately given by the Bohr's formula for all kinds of particles in all types of media. For ultra-relativistic particles observed energy loss less than what predicted with Bohr formula $\Rightarrow$ reduction of energy loss is due to "density" effects.

In dense media, dielectric polarization alter the particle's field compared to free-space

Problem of finding field in the medium can be solved using the Fourier transform.

Consider a dielectric medium, $\epsilon = \epsilon(\omega)$, $\mu = 1$. In Gaussian units we have:

$$\Box A^{\alpha} = \frac{4\pi}{c} J^{\alpha} \tag{21}$$

where $\Box \equiv \partial_{\alpha}\partial^{\alpha} = \frac{\epsilon}{c^2} - \partial_t^2 \nabla^2$. $A^{\alpha} = (\Phi, \overrightarrow{A})$ and $J^{\alpha} = (\rho c/\epsilon, \overrightarrow{J})$. Define

$$\begin{aligned}
F(\overrightarrow{x}, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega F(\overrightarrow{x}, \omega) e^{-i\omega t} \\
F(\overrightarrow{x}, \omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega F(\overrightarrow{x}, t) e^{+i\omega t} \\
F(\overrightarrow{x}, \omega) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} d\overrightarrow{k} F(\overrightarrow{k}, \omega) e^{\overrightarrow{k}.\overrightarrow{x}} \\
F(\overrightarrow{k}, \omega) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} d\overrightarrow{k} F(\overrightarrow{x}, \omega) e^{-\overrightarrow{k}.\overrightarrow{x}}
\end{aligned}$$

two first eqns: Fourier transform in time, two last eqns: Fourier transform in space.

The source of the field is the incident point charge ($q$) so we have:

$$\begin{aligned} \rho(\overrightarrow{x}, t) &= q\delta(\overrightarrow{x} - \overrightarrow{v}t) \\ J(\overrightarrow{x}, t) &= \overrightarrow{v}\rho(\overrightarrow{x}, t) \end{aligned}$$

The time and space Fourier transform is:

$$\begin{aligned} \rho(\overrightarrow{k}, \omega) &= \frac{q}{(2\pi)^2}\int d\overrightarrow{x}\int dt[q\delta(\overrightarrow{x} - \overrightarrow{v}t)]e^{-i(\overrightarrow{k}.\overrightarrow{x}-\omega t)} \\ &= \frac{q}{(2\pi)^2}\int dte^{-i(\overrightarrow{k}.\overrightarrow{v}-\omega)t} = \frac{q}{2\pi}\delta(\omega - \overrightarrow{k}.\overrightarrow{v}) \end{aligned} \tag{22}$$

So finally:

$$\rho(\overrightarrow{k}, \omega) = \frac{q}{2\pi}\delta(\omega - \overrightarrow{k}.\overrightarrow{v}); \text{ and } \overrightarrow{J}(\overrightarrow{k}, \omega) = \frac{q\overrightarrow{v}}{2\pi}\delta(\omega - \overrightarrow{k}.\overrightarrow{v}) \tag{23}$$

Now transform the wave equation in the Fourier space:
$\Box A^\alpha \rightarrow (k^2 - \epsilon(\omega)\frac{\omega^2}{c^2})A^\alpha = \frac{4\pi}{c}J^\alpha$
So that:

$$A^\alpha = \frac{4\pi J^\alpha}{c[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} \tag{24}$$

or

$$\begin{aligned} \overrightarrow{A} &= \frac{4\pi\overrightarrow{J}}{c[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} = \frac{\overrightarrow{v}}{c}\epsilon(\omega)\Phi(\overrightarrow{k},\omega) \\ \Phi &= \frac{4\pi\rho}{\epsilon(\omega)[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} = \frac{2q\delta(\omega - \overrightarrow{k}.\overrightarrow{v})}{\epsilon(\omega)[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]}. \end{aligned}$$

The electric field is then given by
$\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \frac{1}{c}\frac{\partial A}{\partial t}$ or $\overrightarrow{E}(\overrightarrow{k},\omega) = i(\frac{\omega}{c^2}\epsilon(\omega)\overrightarrow{v} - \overrightarrow{k})\Phi$ and
$\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A} \rightarrow i\overrightarrow{k} \times \overrightarrow{A} = i\frac{\epsilon(\omega)}{c}\overrightarrow{k} \times \overrightarrow{v}\Phi$. Hence

$$\begin{pmatrix} \overrightarrow{E}(\overrightarrow{k},\omega) \\ \overrightarrow{B}(\overrightarrow{k},\omega) \end{pmatrix} = i \begin{pmatrix} \frac{\omega\epsilon(\omega)}{c^2}\overrightarrow{v} - \overrightarrow{k} \\ \frac{\epsilon(\omega)}{c}\overrightarrow{k} \times \overrightarrow{v} \end{pmatrix} \Phi(\overrightarrow{k},\omega) \tag{25}$$

We want to find the flow of energy away from the incident particle's trajectory $\Rightarrow$ find the Poynting flux $\Rightarrow$ find $\overrightarrow{E}(\overrightarrow{x},\omega)$ and $\overrightarrow{B}(\overrightarrow{x},\omega)$:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{i}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} d\overrightarrow{k}\,[\frac{\omega}{c}\epsilon(\omega)\frac{\overrightarrow{v}}{c} - \overrightarrow{k}]\Phi(\overrightarrow{k},\omega)e^{+i\overrightarrow{k}.\overrightarrow{x}} \tag{26}$$

Let's specify the problem: consider $\overrightarrow{x} = b\widehat{x}$ and take $\overrightarrow{v} = v\widehat{z}$.

$$
\begin{aligned}
\overrightarrow{E}(\overrightarrow{x},\omega) &= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int\int dk_x dk_y dk_z \left[\frac{\omega}{c}\epsilon(\omega)\frac{\overrightarrow{v}}{c} - \overrightarrow{k}\right]\frac{\delta(\omega - k_z v)}{k^2 - \epsilon\frac{\omega^2}{c^2}}e^{ik_x b} \\
&= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int\int dk_x dk_y dk_z \left[-k_x\widehat{x} - k_y\widehat{y} + \left(\frac{\omega}{c}\frac{v}{c}\epsilon(\omega) - k_z\right)\widehat{z}\right] \\
&\quad \times\frac{\delta(\omega - k_z v)}{k^2 - \epsilon\frac{\omega^2}{c^2}}e^{ik_x b}.
\end{aligned}
$$

the term in $k_y\widehat{y}$ has no contribution to the integral of $k_y$. Let's integrate over $dk_z$:

$$
\begin{aligned}
\overrightarrow{E}(\overrightarrow{x},\omega) &= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int dk_x dk_y \left[-k_x\widehat{x} + \frac{\omega}{v}\left(\frac{v^2}{c^2}\epsilon - 1\right)\widehat{z}\right] \\
&\quad \times\frac{e^{ik_x b}}{k_x^2 + k_y^2 + \left(\frac{\omega}{v}\right)^2(1 - \epsilon\frac{v^2}{c^2})}
\end{aligned}
\tag{27}
$$

Let $\lambda \equiv \frac{\omega}{v}\sqrt{1-\epsilon\frac{v^2}{c^2}}$ and $\mathcal{I} \equiv \int\int dk^2 \frac{e^{ik_x b}}{k_x^2+k_y^2+\lambda^2}$. Then the E-field takes the form:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{1}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\left[-\frac{d\mathcal{I}}{db}\widehat{x} + \frac{i\omega}{v}(\epsilon\beta^2-1)\mathcal{I}\widehat{z}\right] \tag{28}$$

The integration over $dk_y$ gives:

$$\int_{-\infty}^{+\infty} dk_y \frac{1}{k_x^2+k_y^2+\lambda^2} = \frac{\arctan\left(\frac{k_y}{\sqrt{k_x^2+\lambda^2}}\right)}{\sqrt{k_x^2+\lambda^2}}\Bigg|_{-\infty}^{+\infty} = \frac{\pi}{\sqrt{k_x^2+\lambda^2}} \tag{29}$$

So

$$\begin{aligned}\mathcal{I} &= \pi\int_{-\infty}^{\infty} dk_x \frac{e^{ik_x b}}{\sqrt{k_x^2+\lambda^2}} = \pi\int_0^{\infty} dk_x \frac{e^{ik_x b}+e^{-ik_x b}}{\sqrt{k_x^2+\lambda^2}} \\ &= 2\pi\int_0^{+\infty} dk_x \frac{\cos(k_x b)}{\sqrt{k_x^2+\lambda^2}} = 2\pi K_0(b\lambda).\end{aligned} \tag{30}$$

and $\frac{d\mathcal{I}}{db} = -2\pi\lambda K_1(b\lambda)$.

So finally the E-field re-writes:

$$\overrightarrow{E}(\overrightarrow{x},\omega) \;=\; \sqrt{\frac{2}{\pi}}\frac{q}{v}\left[\frac{\lambda}{\epsilon}\lambda K_1(b\lambda)\widehat{x} - i\frac{\omega}{v}(\epsilon\beta - 1)K_0(b\lambda)\widehat{z}\right]. \qquad (31)$$

Now let's find the B-field

$$\begin{aligned}
\overrightarrow{B}(\overrightarrow{k},\omega) &= i\frac{\epsilon(\omega)}{c}\overrightarrow{k}\times\overrightarrow{v}\Phi(\overrightarrow{k},\omega) = i\epsilon\frac{v}{c}(-k_x\widehat{y} + k_y\widehat{x})\Phi(\overrightarrow{k},\omega) \\
&= \frac{i}{(2\pi)^{3/2}}\int\int\int dk_x dk_y dk_z \left[-k_x\widehat{y} + k_y\widehat{x}\right]\frac{\epsilon v}{c}\frac{2q\delta(\omega - \overrightarrow{k}.\overrightarrow{v})}{\epsilon[k^2 - \epsilon\frac{\omega^2}{c^2}]}e^{i\overrightarrow{k}.\overrightarrow{x}} \\
&= -\frac{i}{(2\pi)^{3/2}}2q\frac{v}{c}\widehat{y}\int\int\int dk_x dk_y dk_z k_x \frac{\delta(\omega - k_z v)}{k_x^2 + k_y^2 + k_z^2 - \epsilon\frac{\omega^2}{c^2}}e^{i\overrightarrow{k}.\overrightarrow{x}}.
\end{aligned}$$

the term in $k_y\widehat{x}$ has no contribution to the integral in $dk_y$. Integrate over $dk_z$:

$$\overrightarrow{B} \;=\; -\frac{i}{(2\pi)^{3/2}}2q\frac{v}{c}\widehat{y}\int\int dk_x dk_y k_x\frac{e^{ik_x b}}{k_x^2 + k_y^2 + \lambda^2} = -\frac{i}{(2\pi)^{3/2}}2q\frac{1}{c}\widehat{y}\frac{d\mathcal{I}}{db}.$$

By inspection this is the same as $\widehat{x}$-component of $\overrightarrow{E}$ thus

$$\overrightarrow{B} = \sqrt{\frac{2}{\pi}}\frac{q}{c}\lambda K_1(b\lambda)\widehat{y}. \tag{32}$$

Now that we have $\overrightarrow{E}$ and $\overrightarrow{B}$ we are in position of computing the e.m. field energy flowing out of a cylindrical surface of radius $b$ extending from $-\infty$ to $+\infty$ in $z$:

$$\frac{d\mathcal{E}_f}{dz} = 2\pi b \int_{-\infty}^{\infty} \overrightarrow{S}.\widehat{n}dt = 2\pi b \frac{1}{4\pi} \int_{-\infty}^{+\infty} (\overrightarrow{E} \times \overrightarrow{B}).\widehat{n}dt \tag{33}$$

we have:

$$\begin{aligned}(\overrightarrow{E} \times \overrightarrow{B}).\widehat{n} &= [(E_x\widehat{x} + E_z.\widehat{z}) \times B_y\widehat{y}].\widehat{n} \\ &= (E_xB_y\widehat{z} - E_zB_y\widehat{x}).\widehat{n} = -E_zB_y \end{aligned} \tag{34}$$

So

$$
\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= -\frac{b}{2}\int_{-\infty}^{\infty} E_z B_y dt \\
&= -\frac{b}{4\pi}\int_{-\infty}^{\infty} dt \left[\int_{-\infty}^{\infty} d\omega E_z(\omega) e^{-i\omega t}\right]\left[\int_{-\infty}^{\infty} d\omega' B_y(\omega') e^{-i\omega' t}\right] \\
&= -\frac{b}{2}\int_{-\infty}^{+\infty} E_z(\omega) B_y(-\omega) d\omega = -\frac{b}{2}\int_{-\infty}^{+\infty} E_z(\omega) B_y^*(\omega) d\omega \\
&= \mathcal{R}e\left(-b\int_{0}^{+\infty} E_z(\omega) B_y^*(\omega) d\omega\right) \qquad (35)
\end{aligned}
$$

Expliciting $E_z$ and $B_y$ we have:

$$
\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= -b\mathcal{R}e\left\{\int_{0}^{+\infty} d\omega \left[-i\sqrt{\frac{2}{\pi}}\frac{q}{v}\frac{\omega}{v}(1/\epsilon-\beta^2)K_0(\lambda b)\right]\right. \\
&\qquad \left.\times\left[\sqrt{\frac{2}{\pi}}\frac{q}{c}\lambda^* K_1(\lambda^* b)\right]\right\} \qquad (36)
\end{aligned}
$$

$$\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= \mathcal{R}e\left\{\int_0^{+\infty} d\omega \frac{2}{\pi}\frac{q^2}{v^2}[i\omega(1/\epsilon-\beta^2)\lambda^* b]K_0(\lambda b)K_1(\lambda^* b)\right\} \\
&= \frac{2}{\pi}\frac{q^2}{v^2}\mathcal{R}e\left\{\int_0^{+\infty} d\omega(i\omega\lambda^* b)(1/\epsilon-\beta^2)K_0(\lambda b)K_1(\lambda^* b)\right\} \quad (37)
\end{aligned}$$

The equation was first derived by Fermi. Note that $\lambda$ or $\epsilon$ need to be complex to have $\frac{d\mathcal{E}_f}{dz} \neq 0$.

To proceed with our calculation we now need to introduce a model for $\epsilon(\omega)$. Use the same model as the one used to study Thomson Scattering: we model the bound target electron as a damped harmonic oscillator:

$$\overrightarrow{x}(\omega) = \frac{-\frac{e}{m}\overrightarrow{E}(\omega)}{\omega_0^2-\omega^2-i\omega\Gamma} \quad (38)$$

The dipole moment is just $-e\overrightarrow{x}$ and the polarization is defined as the dipole moment density that is $-n_e e\overrightarrow{x}$:

$$\begin{aligned}\overrightarrow{P}(\omega) &= \frac{n_e e^2}{m}\frac{\overrightarrow{E}(\omega)}{\omega_0^2-\omega^2-i\omega\Gamma} \\ &= \frac{\epsilon(\omega)-1}{4\pi}\overrightarrow{E}(\omega).\end{aligned} \tag{39}$$

So we can write

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2-\omega^2-i\omega\Gamma} \tag{40}$$

wherein $\omega_p \equiv \sqrt{4\pi n_e e^2/m}$ is the plasma frequency. Now we just plug this into $\frac{d\mathcal{E}_f}{dz}$ and perform the integral.

Integral not so simple to perform. We follow JDJ's suggestion and use the "narrow resonance approximation"

$$\omega \simeq \omega_0 \Rightarrow b\lambda = b\frac{\omega}{v}\sqrt{1-\epsilon\beta^2} \sim b\frac{\omega_0}{v}\sqrt{1-\epsilon\beta^2} \tag{41}$$

So

$$b\frac{\omega_0}{v} \simeq 2\pi\frac{b}{\lambda_e} \tag{42}$$

$b\lambda \ll 1$ if $b <$ an atomic radius.Then, using the small argument approximation for the modified bessel function we have: (see JDJ Eq. 3.103)

$$\begin{aligned} b\lambda^* K_1(b\lambda^*) &\sim b\lambda^*\frac{1}{b\lambda^*} \sim 1 \\ K_0(b\lambda) &\sim \ln 2 - \ln(b\lambda) - \gamma = \ln\left(\frac{2e^{-\gamma}}{2\gamma}\right) = \ln\left(\frac{1.123}{b\lambda}\right) \end{aligned} \tag{43}$$

$\gamma = 0.577$ Euler constant.

$$\begin{aligned}\frac{d\mathcal{E}_f}{dz} &= \frac{2}{\pi}q^2v^2\mathcal{R}e\left\{\int_0^{+\infty} d\omega i\omega(1/\epsilon-\beta^2)\ln\left(\frac{1.123}{b\lambda}\right)\right\} \\ &\equiv \frac{2}{\pi}q^2v^2\mathcal{R}e(\mathcal{I}) \end{aligned} \tag{44}$$

where $\mathcal{I} \equiv \int_0^{+\infty} d\omega i\omega(\frac{\epsilon-1}{\epsilon})\ln\left(\frac{1.123}{b\lambda}\right)$ (we took $\beta = 1$). Explicit $\epsilon(\omega)$ [recall that $b\lambda = \frac{b\omega_0}{c}\sqrt{1-\epsilon}$ ]:

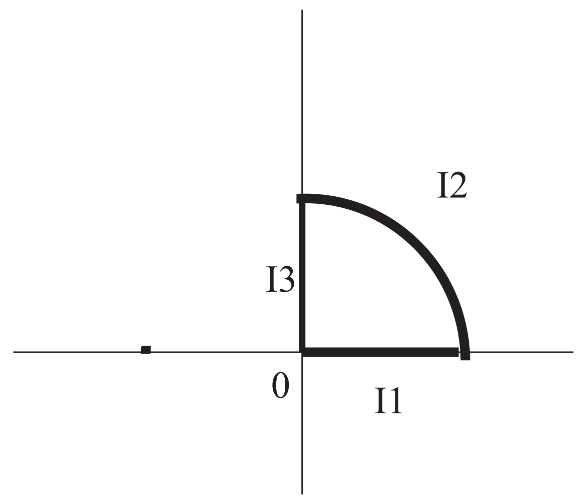
$$\begin{aligned}\mathcal{I} &= i\int_0^{+\infty} d\omega\omega\left(\frac{-\omega_p^2}{\omega_p^2+\omega_0^2-\omega^2-i\omega\Gamma}\right)\left[\ln\left(\frac{1.123c}{b\omega_p}\right)-\ln\omega+\right. \\ &\left.+\frac{1}{2}\ln(\omega^2-\omega_0^2+i\omega\Gamma)\right] \end{aligned} \tag{45}$$

Now let's perform the integration in the complex plane...

Two sources of poles: $-\omega_0^2 + \omega^2 - i\omega\Gamma = 0$ from the ln(...), and $\omega_p^2 + \omega_0^2 - \omega^2 - i\omega\Gamma = 0$ from denominator of $\frac{1-\epsilon}{\epsilon}$. All the poles are in the lower part of the complex plane. Consider the integral along $\mathcal{C}$. This gives:

$$I_1 + I_2 + I3 = 0$$

, so $\mathcal{I} = iI_1 = i(-I_2 - I_3)$. The $i$ comes from the fact we drop the $i$ when evaluating the integrals $I_n$.



Let's evaluate the integrals:

$$I_3 = \int_{+i\infty}^{0} d\omega\omega(\ldots)\ln(\ldots) \tag{46}$$

Let $\omega \equiv i\Omega$ with $\Omega \in \mathbb{R}$ then:

$$\begin{aligned} I_3 &= -\int_{\infty}^{0} d\Omega\Omega(\ldots)\ln(\ldots) = \int_{0}^{\infty} d\Omega\Omega \frac{\omega_0^2 + \Omega^2 + \Omega\Gamma}{\omega_p^2 + \omega_0^2 + \Omega^2 + \Omega\Gamma} \\ &\quad \times \left(\ln\frac{1.123c}{b\omega_p} - \ln i\Omega + \frac{1}{2}\ln(-\Omega^2 - \omega_0^2 - \Omega\Gamma)\right) \end{aligned} \tag{47}$$

the bracket simplifies:

$$(\ldots) = \ln\frac{1.123c}{b\omega_p} - \ln i - \ln\Omega + \frac{1}{2}\ln -1 + \frac{1}{2}\ln(\Omega^2 + \omega_0^2 + \Omega\Gamma) \tag{48}$$

the $\ln i$ and $1/2\ln(-1)$ cancel each other. So $I_3$ becomes:

$$I_3 = \int_0^\infty d\Omega\Omega\frac{\omega_0^2+\Omega^2+\Omega\Gamma}{\omega_p^2+\omega_0^2+\Omega^2+\Omega\Gamma} \times\left(\ln\frac{1.123c}{b\omega_p}-\ln\Omega+\frac{1}{2}\ln(\Omega^2+\omega_0^2+\Omega\Gamma)\right) \tag{49}$$

So $I_3$ is real, so $iI_3$ is pure imaginary and therefore its contribution to $\mathcal{R}e\mathcal{I}$ is zero.

Now consider $I_2$, let $\omega\equiv Re^{i\theta}$, then

$$I_2 = \lim_{R\to\infty}\int_0^{\pi/2} id\theta Re^{i\theta}Re^{i\theta}\frac{\omega_p^2}{\omega_p^2+\omega_0^2-R^2e^{2i\theta}-iRe^{i\theta}\Gamma} \times\left(\ln\frac{1.123c}{b\omega_p}-\ln Re^{i\theta}+\frac{1}{2}\ln(-\omega_0^2+R^2e^{2i\theta}+iRe^{i\theta}\Gamma)\right) \tag{50}$$

Taking the limit $R = \longrightarrow \infty$, we get:

$$I_2 \;=\; \int_0^{\pi/2} i d\theta \omega_p^2 \ln \frac{1.123c}{b\omega_p} = i\frac{\pi\omega_p^2}{2} \ln \frac{1.123c}{b\omega_p} \tag{51}$$

So finally energy loss is:

$$\frac{d\mathcal{E}_f}{dz} = \frac{2}{\pi} q^2 v^2 \mathcal{R}e(\mathcal{I}) = -\frac{q^2\omega_p^2}{c^2} \ln \frac{1.123c}{b\omega_p}. \tag{52}$$

where we have taken $v = c$. On another hand we have derived at the beginning of Part V the energy loss under the impulse approximation to be:

$$\frac{d\mathcal{E}_f}{dz} = -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\gamma v^2}{\omega_0 b} = -\frac{q^2\omega_p^2}{c^2} \ln \frac{\gamma c}{b\omega_0}. \tag{53}$$

note that we have actually derived $\frac{d\mathcal{E}_f}{dt}$, we also took $v = c$ in the latter equation.

The influence of dielectric screening is two-folds:

1- It removes the dependence of energy loss on atomic structure $\omega_0$ is replaced by $\omega_p$ which only depend on the density number of e- (and not on their binding energy).

2- It reduces the energy loss from highly relativistics incident charge, the $\gamma$ in the argument of ln is gone.

# Cerenkov radiation

We now consider density effect in the extreme limit $b\lambda \gg 1$ and look at the energy deposited in the target. The large argument approximation for the modified bessel function gives: $K_0(b\lambda) = K_1(b\lambda) = \sqrt{\frac{\pi}{2}}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}$. So the fields are:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{q}{v}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}\left(\frac{\lambda}{\epsilon}\widehat{x} - i\frac{\omega}{v}(\frac{1}{\epsilon} - \beta^2)\widehat{z}\right), \tag{54}$$

$$\overrightarrow{B}(\overrightarrow{x},\omega) = \frac{q}{v}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}\widehat{y}. \tag{55}$$

to get radiation $\lambda$ or $\epsilon \in \mathbb{C}$. Let's take $\epsilon \in \mathbb{R}$ (no dielectric screening). Then

$$\lambda = \frac{\omega}{v}\sqrt{1 - \epsilon(\omega)\beta^2} \tag{56}$$

To have $\lambda \in \mathbb{I}$, $1 - \epsilon\beta^2 < 0 \Rightarrow \epsilon\beta^2 > 1$, this is the Cerenkov condition.

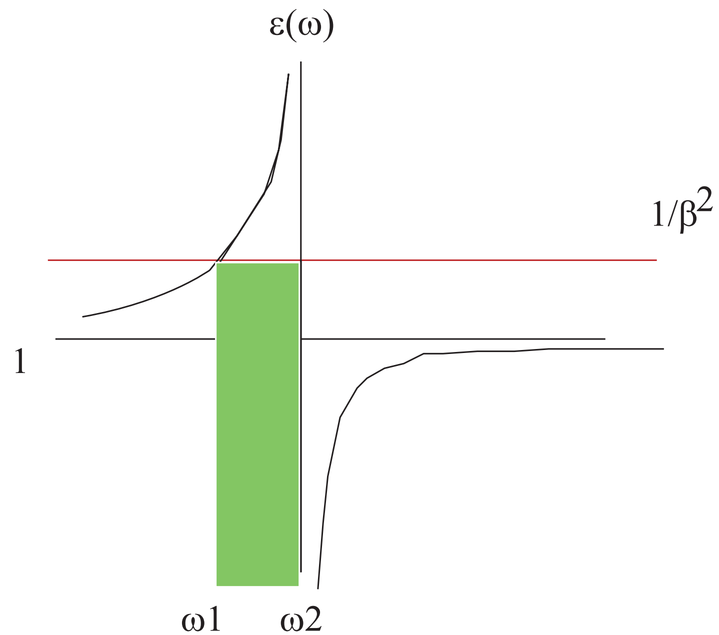
Now replace the field in the expression for $\frac{d\mathcal{E}_f}{dz}$:

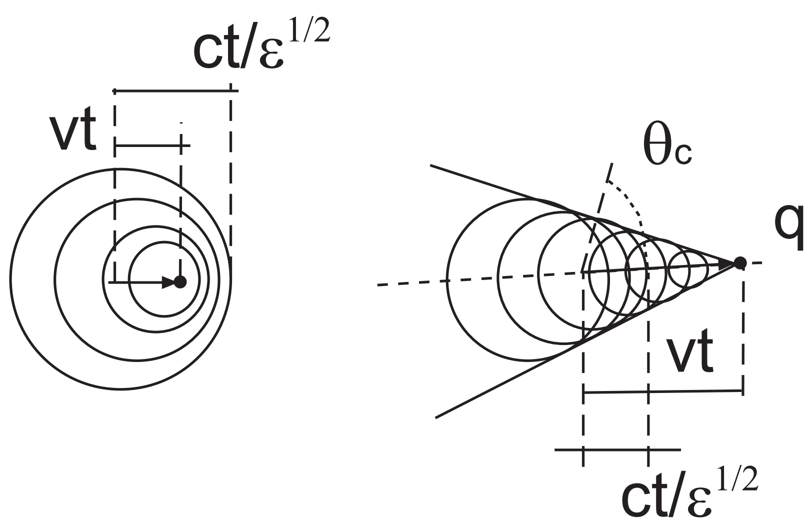
$$\begin{aligned}\frac{d\mathcal{E}_f}{dz} &= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^\infty d\omega(i\omega\lambda^* b)(\frac{1}{\epsilon}-\beta^2)K_0(b\lambda)K_1(b\lambda^*)\right) && (57)\\ &= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^\infty d\omega(i\omega\lambda^* b)(\frac{1}{\epsilon}-\beta^2)\frac{e^{-b(\lambda+\lambda^*)}}{b\sqrt{\lambda\lambda^*}}\right) && (58)\\ &= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^\infty i\omega\sqrt{\frac{\lambda^*}{\lambda}}(\frac{1}{\epsilon}-\beta^2)\right) && (59)\end{aligned}$$

but radiation only when $\lambda \in \mathbb{I}$ that is for a frequency band $\omega \in [\omega_l, \omega_0]$.

$$\frac{d\mathcal{E}_f}{dz} = \frac{q^2}{v^2}\int_{\omega_L}^{\omega_0} d\omega\omega\left(1 - \frac{1}{\epsilon\beta^2}\right) \qquad (60)$$

This is Frank-Tamm (1937) equation.



The propagation direction of the wave is given by $\overrightarrow{k}$ and $k$ is perpendicular to $\overrightarrow{E}$ and $\overrightarrow{B}$. So if $\theta_c \angle(\overrightarrow{v}, \overrightarrow{k})$ then

$$\begin{aligned} \cos\theta_c &= \frac{|E_x|}{|E|} = \frac{E_x}{\sqrt{E_x^2 + E_z^2}} \\ &= \frac{\frac{\lambda}{\epsilon}}{\left[\left(\frac{\lambda}{\epsilon}\right)^2 - \frac{\omega^2}{v^2}\left(\frac{1}{\epsilon} - \beta^2\right)\right]^{1/2}}. \end{aligned} \tag{61}$$

introducing $\lambda^2 = (\omega/v)^2(1 - \epsilon\beta^2)$, we finally obtain:

$$\cos\theta_c = \frac{1}{\sqrt{1 - 1 + \beta^2\epsilon}} = \frac{1}{\beta\sqrt{\epsilon}} = \frac{c_m}{v} \tag{62}$$

wherein $c_m \equiv c/\sqrt{\epsilon}$ is the velocity of light in the medium; $c_m < c$ so $\cos\theta_c < 1$ and $\theta \in \mathbb{R}$.

The shock wave feature should be derivable from the e.m. potential,

$$\left(k^2 - \frac{\omega^2}{c_m^2}\right)\sqrt{\epsilon}\Phi(\overrightarrow{k},\omega) = \frac{4\pi}{\sqrt{\epsilon}}\rho(\overrightarrow{k},\omega)$$
$$\left(k^2 - \frac{\omega^2}{c_m^2}\right)\sqrt{\epsilon}\overrightarrow{A}(\overrightarrow{k},\omega) = \frac{4\pi}{c_m}\overrightarrow{J}(\overrightarrow{k},\omega)$$

So in the medium, $A^\alpha$ takes the same form as in vacuum, under the renormalization $q \to q/\sqrt{\epsilon}$, $c \to c_m$. Using these potential we can directly get the Lienard-Wiechert potentials:

$$\begin{pmatrix} \sqrt{\epsilon}\Phi(\overrightarrow{x},t) \\ \overrightarrow{A}(\overrightarrow{x},t) \end{pmatrix} = \frac{q}{\sqrt{\epsilon}}\frac{1}{[\kappa R]_{ret}}\begin{pmatrix} 1 \\ \frac{\overrightarrow{v}}{c_m} \end{pmatrix} \tag{63}$$

Let $\overrightarrow{\zeta} = \overrightarrow{x} - \overrightarrow{v}t$, $\overrightarrow{R} = \overrightarrow{x} - \overrightarrow{x}(t') = \overrightarrow{x} - \overrightarrow{v}t'$. So $\overrightarrow{R} = \overrightarrow{x} - \overrightarrow{v}t + \overrightarrow{v}(t-t') = \overrightarrow{\zeta} + v(t-t')$.

So $t-t' = \frac{R(t')}{c_m} = \frac{|\overrightarrow{\zeta}+\overrightarrow{v}(t-t')|}{c_m}$.
$\Rightarrow (t-t')^2 = \frac{1}{c_m^2}[\zeta^2+2\overrightarrow{\zeta}.\overrightarrow{v}(t-t')+v^2(t-t')^2]$. $\Rightarrow (v^2-c_m^2)(t-t')^2+2\overrightarrow{\zeta}.\overrightarrow{v}(t-t')+\zeta^2=0$; solve to get

$$(t-t')_{\pm} = \frac{-\overrightarrow{\zeta}.\overrightarrow{v} \pm \sqrt{(\overrightarrow{\zeta}.\overrightarrow{v})^2-(v^2-c_m^2)\zeta^2}}{v^2-c_m^2}. \tag{64}$$

For cherenkov radiation $v > c_m$ to obtain $t-t' > 0 \in \mathbb{R}$ we need: $\overrightarrow{\zeta}.\overrightarrow{v} < 0$ and $(\overrightarrow{\zeta}.\overrightarrow{v})^2 > (v^2-c_m^2)\zeta^2$, which means $\zeta v \cos theta < 0$ or $theta > \pi/2$, and $\cos^2\theta > 1-c_m^2/v^2$. So

$$\theta > \arccos(-\sqrt{1-(c_m/v)^2}), \tag{65}$$

which lies in $[\pi/2, \pi]$.

So potential and fields exist at time t only within a cone which the apex lies at $\zeta = \overrightarrow{x} - \overrightarrow{t}$ (i.e. the present position of incident charge) and for which the apex angle is $\pi - \arccos(-\sqrt{1-(c_m/v)^2})$. The 4-potential is $A^\alpha = A^\alpha_- + A^\alpha_+$ where the $\pm$ corresponds to $(t-t')_\pm$. Now,

$$\begin{aligned}
[\kappa R]_{ret} &= |(1 - 1/c_m \overrightarrow{v}.\hat{n})\overrightarrow{R}| = |\overrightarrow{R} - \frac{\hat{n}}{c_m}\overrightarrow{v}.[\overrightarrow{\zeta} + \overrightarrow{v}(t-t')]| \\
&= |\overrightarrow{R} - \frac{\hat{n}}{c_m}\overrightarrow{\zeta}.\overrightarrow{v} - \frac{\hat{n}}{c_m}v^2(t-t')| \\
&= |\hat{n}[c_m(t-t') - \frac{\overrightarrow{\zeta}.\overrightarrow{v}}{c_m} - \frac{v^2}{c_m}(t-t')]| \\
&= \frac{1}{c_m}|(c_m^2 - v^2)(t-t') - \overrightarrow{\zeta}.\overrightarrow{v}|. \qquad (66)
\end{aligned}$$

Expliciting $(t-t')$ in the latter equation (using 64), we get:

$$[\kappa R]_{ret} \;=\; \frac{\zeta}{c_m}\sqrt{c_m^2 - v^2 \sin^2\theta} = \zeta\sqrt{1-\frac{v^2}{c_m^2}\sin^2\theta}. \tag{67}$$

both for $(t-t')_{\pm}$. So the potentials are given by:

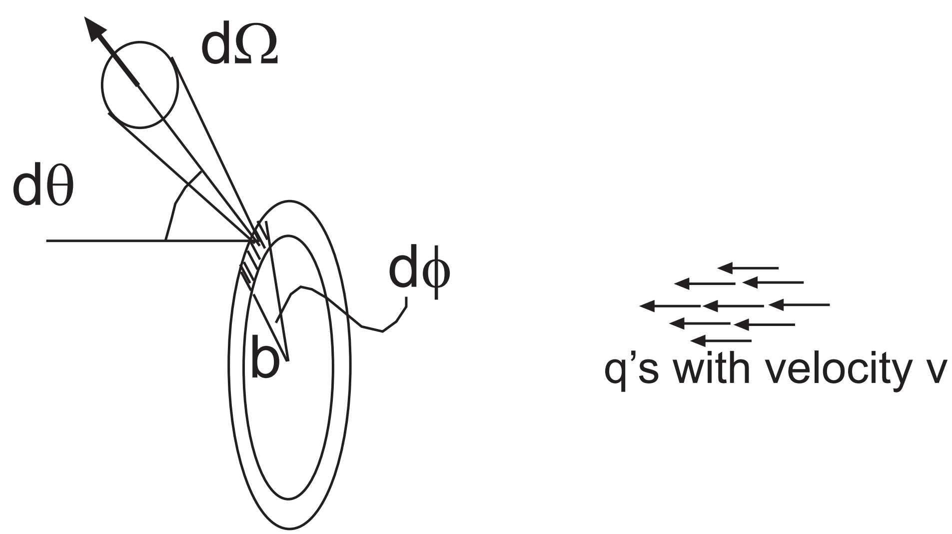
$$\begin{pmatrix} \sqrt{\epsilon}\Phi(\overrightarrow{x},t) \\ \overrightarrow{A}(\overrightarrow{x},t) \end{pmatrix} = \frac{2q}{\sqrt{\epsilon}} \frac{1}{\zeta\sqrt{1-\frac{v^2}{c_m^2}\sin^2\theta}} \begin{pmatrix} 1 \\ \frac{\overrightarrow{v}}{c_m} \end{pmatrix} \tag{68}$$

The potentials have a singularity (a hock front) at $\sin^2\theta = (c_m/v)^2$, which corresponds to the earlier results $\cos^2\theta = 1-(c_m/v)^2$. Note that when the frequency-dependence of $\epsilon$ is introduced the shock wave-front is smeared.

## Scattering

Thus far we have only looked at energy loss from charges incident to a target. Now let's look at momentum transfer that is scattering. Let $N'$ be the number of incident particle scattered from $bdbd\phi$ into $d\Omega$ per unit time; we have:

$$d^2N = nvbdbd\phi = N'd\Omega \Rightarrow bdbd\phi = \frac{N'}{nv}d\Omega = \frac{d\sigma}{d\Omega}d\Omega; \Rightarrow \frac{d\sigma}{d\Omega}\frac{N'}{nv}. \quad (69)$$



$$bdb = \frac{d\sigma}{d\Omega}\sin\theta d\theta \Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta} \tag{70}$$

Under the impulse approximation we have $\sin\theta \sim \theta \Rightarrow \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$. But

$$|\theta| = \frac{\Delta p}{p} = \frac{2qe}{\gamma b M v^2} \tag{71}$$

for target e$^-$ (cf beginning of part V). so $b = \frac{2qe}{\gamma\theta M v^2} \Rightarrow \left|\frac{db}{d\theta}\right| = \frac{2qe}{\gamma\theta^2 M v^2}$.
So finally,

$$\frac{d\sigma}{d\Omega} = \frac{b\theta}{\theta^2}\left|\frac{db}{d\theta}\right| = \left(\frac{2qe}{\gamma M v^2}\right)^2 \frac{1}{\theta^4} \tag{72}$$

For target Nuclei:

$$\frac{d\sigma}{d\Omega} = \frac{1}{\theta^4}\left(\frac{2qZe}{\gamma M v^2}\right)^2 \tag{73}$$

this is the small-angle Rutherford formula. scattering by nuclei is $Z^2$ times stronger than by e-.

There are $Z$ times more e- than nuclei, so the net effect is that nuclei scattering is $Z$ times stronger than e- scattering.

Average deflection angle in a material: To get the mean-square deflection angle, evaluate:

$$\langle \theta^2 \rangle = \frac{\int d\Omega \theta^2 d\sigma / d\Omega}{\int d\Omega d\sigma / d\Omega} \simeq \frac{\int d\theta \theta^3 1/\theta^4}{\int d\theta \theta 1/\theta^4} \tag{74}$$

$$= \frac{\int_{\theta_{min}}^{\theta_{max}} d\theta 1/\theta}{\int d\theta 1/\theta^3} = \frac{\ln \frac{\theta_{max}}{\theta_{min}}}{\frac{1}{2}(1/\theta_{min}^2 - 1/\theta_{max}^2)} \tag{75}$$

So $\langle \theta^2 \rangle \simeq 2\theta_{min}^2 \ln \frac{\theta_{max}}{\theta_{min}}$ for a single scattering event. This is just few times $\theta_{min}^2$ which is a small number.

Let's estimate $\theta_{min}$ from physical arguments: $b_{max} \simeq a$, the atomic radius because atomic electrons almost completely screen the nucleus if $b > a$. So

$$\theta_{min} = \frac{2qZe}{\gamma b_m a x M v^2} \simeq \frac{2qZe}{\gamma a M v^2} \tag{76}$$

$$\sim \frac{e^2}{a m_p c^2} \sim \frac{e^2/(m_e c^2)}{a m_p c^2} \sim \frac{r_e}{1836a} \ll 1 \tag{77}$$

So to achieve a sizeable deflection angle, the incident charge needs either to undergo manu small-angle scattering of a few large-angle scattering.

• Case of many small-angle scattering:
Net effect: charge $q$ random-walk through the target $\langle\Theta^2\rangle = N\langle\theta^2\rangle$ and:

$$\Rightarrow \frac{d\langle\Theta^2\rangle}{dz} = n\sigma\langle\theta^2\rangle \simeq 2n\sigma\theta_{min}^2 \ln(\theta_{max}/\theta_{min}) \tag{78}$$

The distribution of angle after many small-scattering event (random-walk) is given by:

$$P_{RW}(\theta_p) \propto e^{\frac{-\theta_p^2}{2\langle\Theta_p^2\rangle}}.$$

• Case of few large-angle scattering:
Consider the distribution of scattering angle for a single scattering event:

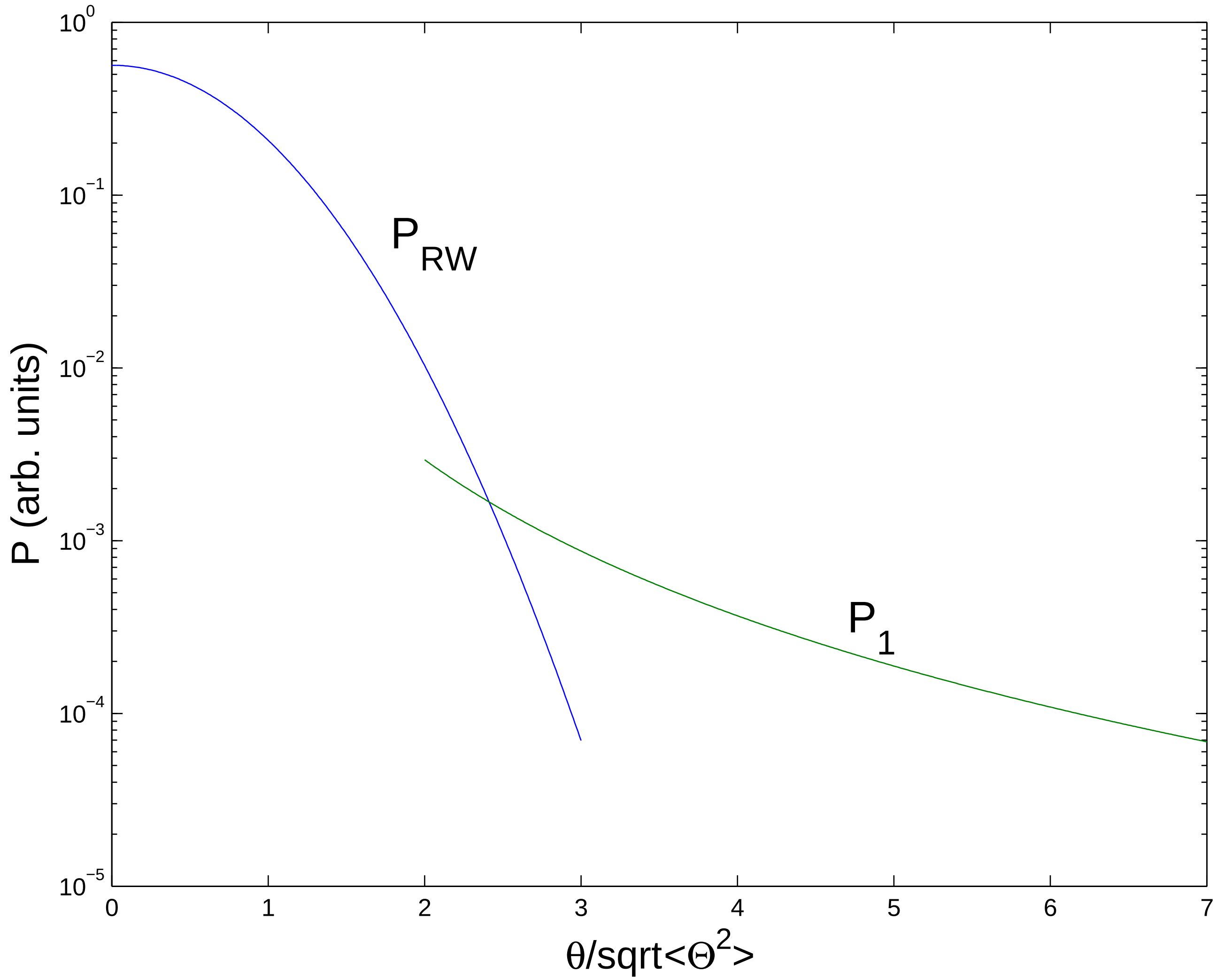
$$\frac{d\sigma}{d\Omega}d\Omega = \left(\frac{2qZe}{\gamma M v^2}\right)\frac{1}{\theta^4}d\phi\theta d\theta \tag{79}$$

In terms of projected angle $\theta_p = \theta \sin\phi$, this takes the form:

$$\frac{d\sigma}{d\Omega}d\Omega = \left(\frac{2qZe}{\gamma M v^2}\right)\frac{1}{\theta_p^3}d\theta_p \sin^2\phi d\phi \tag{80}$$

Upon integration over $\phi$ we find that the distribution scales as:

$$P_1(\theta_p)d\theta_p \propto \frac{d\theta_p}{\theta_p^3} \leftrightarrow P_1(\theta_p) \propto \frac{1}{\theta_p^3}$$



10^0
10^−1
10^−2
10^−3
10^−4
10^−5
P (arb. units)
P_RW
P_1
0
1
2
3
4
5
6
7
θ/sqrt<Θ^2>