PHYS 571: Electromagnetism Theory II

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Contents

List of Figures

List of Tables

Chapter 1

Electromagnetic resonance in cylindrical cavities and waveguides

1.1 Maxwell's Equations in MKSA units

In a medium with dielectric permittivity ϵ and mangetic permeability μ, Maxwell's equation takes the form

$$\begin{aligned}
\overrightarrow{\nabla}.\overrightarrow{D} &= \rho, \text{ Coulomb law} &(1.1)\\
\overrightarrow{\nabla}.\overrightarrow{B} &= 0, \text{ no magnetic charges} &(1.2)\\
\overrightarrow{\nabla}\times\overrightarrow{E} &= -\partial_t\overrightarrow{B}, \text{ Faraday law} &(1.3)\\
\overrightarrow{\nabla}\times\overrightarrow{H} &= \overrightarrow{J}+\partial_t\overrightarrow{D}, \text{ Ampère- Maxwell law} &(1.4)
\end{aligned}$$

where

- $\overrightarrow{D} \equiv \epsilon\overrightarrow{E} + \overrightarrow{P}$ is the electric displacement, $\overrightarrow{E}$ the electric field $\overrightarrow{P}$ is the polarization.
- ρ, is the charge density,
- B is the induction, $\overrightarrow{H} \equiv \overrightarrow{B}/\mu - \overrightarrow{M}$ is the magnetic field, μ the magnetic permeability of the medium and $\overrightarrow{M}$ the magnetization.
- $\overrightarrow{J}$ is the current.

Let's now consider a perfectly conducting resonant cavity which is cylindrical, and which is filled with a homogeneous, isotropic, non-conducting, non-dissipative medium. Then in the medium,

$$\overrightarrow{D} = \epsilon\overrightarrow{E}, \text{ and } \overrightarrow{B} = \mu\overrightarrow{H}. \tag{1.5}$$

Given the boundary conditions at the cavity walls:

$$\hat{n}.\overrightarrow{B} = \hat{n}\times\overrightarrow{E} = 0, \tag{1.6}$$

Maxwell's equations (MKSA units) can be written as:

$$\overrightarrow{\nabla}.\overrightarrow{D} = 0 \Rightarrow \overrightarrow{\nabla}.\overrightarrow{E} = 0; \overrightarrow{\nabla}.\overrightarrow{B} = 0; \tag{1.7}$$

$$\overrightarrow{\nabla} \times \overrightarrow{E} + \partial_t \overrightarrow{B} = 0; \tag{1.8}$$

$$\overrightarrow{\nabla} \times \overrightarrow{H} - \partial_t \overrightarrow{D} = 0 \Rightarrow \nabla \times \overrightarrow{B} - \mu\epsilon \partial_t \overrightarrow{E} = 0. \tag{1.9}$$

1.2 Relation between axial and transverse components of the em field in a resonant cavity

We now specialize the problem to cylindrical cavities with revolution axis $\hat{z}$ and we write the electric and magnetic field in the form

$$\begin{bmatrix} \overrightarrow{E}(r,\phi,z,t) \\ \overrightarrow{B}(r,\phi,z,t) \end{bmatrix} = \begin{bmatrix} \overrightarrow{E}(r,\phi) \\ \overrightarrow{B}(r,\phi) \end{bmatrix} e^{\pm ikz - i\omega t}. \tag{1.10}$$

Separating transverse and axial components:

$$\begin{bmatrix} \overrightarrow{E} \\ \overrightarrow{B} \end{bmatrix} = \begin{bmatrix} \overrightarrow{E}_t \\ \overrightarrow{B}_t \end{bmatrix} + \begin{bmatrix} E_z \\ B_z \end{bmatrix} \hat{z}; \tag{1.11}$$

$$\begin{bmatrix} E_z \\ B_z \end{bmatrix} = \hat{z}. \begin{bmatrix} \overrightarrow{E} \\ \overrightarrow{B} \end{bmatrix}; \text{ and } \begin{bmatrix} \overrightarrow{E}_t \\ \overrightarrow{B}_t \end{bmatrix} = \left(\hat{z} \times \begin{bmatrix} \overrightarrow{E} \\ \overrightarrow{B} \end{bmatrix} \right) \times \hat{z}. \tag{1.12}$$

From Maxwell's equations,

$$\left[\hat{z} \times (\overrightarrow{\nabla} \times \overrightarrow{E}) \right] \times \hat{z} - i\omega \overrightarrow{B}_t = 0, \tag{1.13}$$

$$\left[\hat{z} \times (\overrightarrow{\nabla} \times \overrightarrow{B}) \right] \times \hat{z} + i\mu\epsilon\omega \overrightarrow{E}_t = 0. \tag{1.14}$$

In general,

$$\hat{z} \times (\overrightarrow{\nabla} \times \overrightarrow{V}) = \overrightarrow{\nabla}(\hat{z}.\overrightarrow{V}) - (\hat{z}.\overrightarrow{\nabla})\overrightarrow{V} = \overrightarrow{\nabla} V_z - \partial_z \overrightarrow{V} \tag{1.15}$$

$$\begin{aligned} \left[\hat{z} \times (\overrightarrow{\nabla} \times \overrightarrow{V}) \right] \times \hat{z} &= \overrightarrow{\nabla} V_z \times \hat{z} - \partial_z (\overrightarrow{V} \times \hat{z}) && (1.16) \\ &= -\hat{z} \times \overrightarrow{\nabla}_t V_z + \hat{z} \times \partial_z \overrightarrow{V}_t. && (1.17) \end{aligned}$$

So,

$$\begin{aligned} i\omega \overrightarrow{B}_t &= \hat{z} \times \partial_z \overrightarrow{E}_t - \hat{z} \times \overrightarrow{\nabla}_t E_z, && (1.18) \\ -i\omega\mu\epsilon \overrightarrow{E}_t &= \hat{z} \times \partial_z \overrightarrow{B}_t - \hat{z} \times \overrightarrow{\nabla}_t B_z. && (1.19) \end{aligned}$$

Apply "$\hat{z} \times \partial_z$" to 1.18:

$$
\begin{aligned}
i\omega \hat{z} \times \partial_z \overrightarrow{B}_t &= \hat{z} \times (\hat{z} \times \partial_z^2 \overrightarrow{E}_t) - \quad \hat{z} \times (\hat{z} \times \overrightarrow{\nabla}_t \partial_z E_z) && (1.20)\\
&= -\partial_z^2 \overrightarrow{E}_t - \qquad (-)\overrightarrow{\nabla}_t \partial_z E_z && (1.21)\\
&= +k^2 \overrightarrow{E}_t \pm ik \overrightarrow{\nabla}_t E_z. && (1.22)
\end{aligned}
$$

$$
\Rightarrow \hat{z} \times \partial_z \overrightarrow{B}_t = \frac{-i}{\omega}(k^2 \overrightarrow{E}_t \pm ik \overrightarrow{\nabla}_t E_z). \tag{1.23}
$$

Similarly:

$$
\hat{z} \times \partial_z \overrightarrow{E}_t = \frac{+i}{\mu\epsilon\omega}(k^2 \overrightarrow{B}_t \pm ik \overrightarrow{\nabla}_t B_z). \tag{1.24}
$$

Insert 1.24 into 1.18:

$$
i\omega B_t = \frac{i}{\mu\epsilon\omega}(k^2 \overrightarrow{B}_t \pm ik \overrightarrow{\nabla}_t B_z) - \hat{z} \times \overrightarrow{\nabla}_t E_z \tag{1.25}
$$

$$
\begin{aligned}
i(\mu\epsilon\omega^2 - k^2)\overrightarrow{B}_t &= \mp k \overrightarrow{\nabla}_t B_z + \mu\epsilon\omega \hat{z} \times \overrightarrow{\nabla}_t E_z && (1.26)\\
\Rightarrow \overrightarrow{B}_t &= \frac{i}{\mu\epsilon\omega^2 - k^2}(\pm k \overrightarrow{\nabla}_t B_z + \mu\epsilon\omega \hat{z} \times \overrightarrow{\nabla}_t E_z). && (1.27)
\end{aligned}
$$

Insert 1.23 into 1.19:

$$
-i\mu\epsilon\omega E_t = \frac{-i}{\omega}(k^2 \overrightarrow{E}_t \pm ik \overrightarrow{\nabla}_t E_z) - \hat{z} \times \overrightarrow{\nabla}_t B_z \tag{1.28}
$$

$$
\begin{aligned}
-i(\mu\epsilon\omega^2 - k^2)\overrightarrow{E}_t &= \pm k \overrightarrow{\nabla}_t E_z - \omega \hat{z} \times \overrightarrow{\nabla}_t B_z && (1.29)\\
\Rightarrow \overrightarrow{E}_t &= \frac{i}{\mu\epsilon\omega^2 - k^2}(\pm k \overrightarrow{\nabla}_t E_z - \omega \hat{z} \times \overrightarrow{\nabla}_t B_z). && (1.30)
\end{aligned}
$$

Eqn 1.27 and 1.30 generally pertain as long as the transverse cross-section is z-independent as illustrated in JDJ Fig. 8.3.

1.3 Wave Equation

Multiplying Faraday's law by $\overrightarrow{\nabla}\times$ gives

$$
\begin{aligned}
\overrightarrow{\nabla} \times (\overrightarrow{\nabla} \times \overrightarrow{E}) + \partial_t \overrightarrow{\nabla} \times \overrightarrow{B} &= \overrightarrow{\nabla}(\overrightarrow{\nabla}.\overrightarrow{E}) - \nabla^2 \overrightarrow{E} + \mu\epsilon\partial_t^2 \overrightarrow{E} && (1.31)\\
&= -\nabla_t^2 \overrightarrow{E} + k^2 \overrightarrow{E} - \mu\epsilon\omega^2 \overrightarrow{E} = 0 && (1.32)
\end{aligned}
$$

$$
\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right] \overrightarrow{E} = 0. \tag{1.33}
$$

Similarly multiplying Ampère-Maxwell's law by $\overrightarrow{\nabla}\times$ gives

$$\begin{aligned}\overrightarrow{\nabla}\times(\overrightarrow{\nabla}\times\overrightarrow{B})-\mu\epsilon\partial_t\overrightarrow{\nabla}\times\overrightarrow{E} &= \overrightarrow{\nabla}(\overrightarrow{\nabla}.\overrightarrow{B})-\nabla^2\overrightarrow{B}+\mu\epsilon\partial_t^2\overrightarrow{B} \qquad (1.34)\\ &= -\nabla_t^2\overrightarrow{B}+k^2\overrightarrow{B}-\mu\epsilon\omega^2\overrightarrow{B}=0 \qquad (1.35)\end{aligned}$$

$$\left[\nabla_t^2+(\mu\epsilon\omega^2-k^2)\right]\overrightarrow{B}=0. \qquad (1.36)$$

Equations 1.33 and 1.36 can be casted as

$$\left[\nabla_t^2+(\mu\epsilon\omega^2-k^2)\right]\left[\begin{array}{c}\overrightarrow{E}\\ \overrightarrow{B}\end{array}\right]=0. \qquad (1.37)$$

which is sometime referred to as the wave equation.

1.4 Recipe for EM-field calculations

The previous derivations suggest a prescription for computing the em field in a resonant cavity.

1. Find $E_z(r,\phi)$ and $B_z(r,\phi)$ from $[\nabla_t^2+(\mu\epsilon\omega^2-k^2)]\left[\begin{array}{c}\overrightarrow{E}\\ \overrightarrow{B}\end{array}\right]=0.$

2. Find $\overrightarrow{E}_t(r,\phi)$ and $\overrightarrow{B}_t(r,\phi)$ from

$$\left[\begin{array}{c}\overrightarrow{E}_t(r,\phi)\\ \overrightarrow{B}_t(r,\phi)\end{array}\right]=\frac{i}{\mu\epsilon\omega^2-k^2}\left\{\pm k\overrightarrow{\nabla}_t\left[\begin{array}{c}E_z(r,\phi)\\ B_z(r,\phi)\end{array}\right]-\omega\hat{z}\times\overrightarrow{\nabla}_t\left[\begin{array}{c}B_z(r,\phi)\\ -\mu\epsilon E_z(r,\phi)\end{array}\right]\right\}.$$

3. The total field is:

$$\left[\begin{array}{c}\overrightarrow{E}\\ \overrightarrow{B}\end{array}\right]=\left[\begin{array}{c}\overrightarrow{E}_t\\ \overrightarrow{B}_t\end{array}\right]+\left[\begin{array}{c}E_z\\ B_z\end{array}\right]\hat{z}.$$

4. Incorporate the boundary conditions ($\mathcal{S}$: cavity side surface):

$$\begin{aligned}\hat{n}\times\overrightarrow{E}=0\Rightarrow E_z\big|_{\mathcal{S}}=0\text{ and, }\overrightarrow{E}_t=0\text{ at end plates}\\ \hat{n}.\overrightarrow{B}=0\Rightarrow\hat{n}.\overrightarrow{\nabla}_tB_z=0\Rightarrow\partial_nB_z\big|_{\mathcal{S}}=0\ \text{ from Eq.1.19}\end{aligned}$$

Boundary conditions at the cavity side $\mathcal{S}$ for B_z (E_z straightforward)

$$-i\mu\epsilon\omega\overrightarrow{E}_t=\hat{z}\times\partial_z\overrightarrow{B}_t-\hat{z}\times\overrightarrow{\nabla}_tB_z=\hat{z}\times(\partial_z\overrightarrow{B}_t-\overrightarrow{\nabla}_tB_z)\text{ (from Eq.1.19).} \qquad (1.38)$$

take "$\hat{n}\times$":

$$-i\mu\epsilon\omega\hat{n}\times\overrightarrow{E}_t=\hat{n}\times\left[\hat{z}\times(\partial_z\overrightarrow{B}_t-\overrightarrow{\nabla}_tB_z)\right] \qquad (1.39)$$

l.h.s= 0 since $\hat{n} \times \overrightarrow{E} = 0$ and E_z=0 at the cavity walls.
use $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a}.\overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a}.\overrightarrow{b})\overrightarrow{c}$:

$$\left[\hat{n}.(\partial_z \overrightarrow{B}_t - \overrightarrow{\nabla}_t B_z)\right] \hat{z} - (\hat{n}.\hat{z})(\partial_z \overrightarrow{B}_t - \overrightarrow{\nabla}_t B_z) = 0 \tag{1.40}$$

$\hat{n}.\hat{z} = 0$ because $\hat{n} \perp \hat{z}$ on the cavity side

$$\Rightarrow \partial_z(\hat{n}.\overrightarrow{B}_t) - \hat{n}.\overrightarrow{\nabla}_t B_z = 0 \tag{1.41}$$

$\hat{n}.\overrightarrow{B}_t = \hat{n}.\overrightarrow{B} - (\hat{n}.\hat{z})B_z = 0$

$$\Rightarrow \partial_n B_z = 0 \tag{1.42}$$

1.5 Resonant modes categorization

The boundary conditions $E_z\big|_{\mathcal{S}} = 0$ and $\partial_n B_z\big|_{\mathcal{S}} = 0$ cannot generally be satisfied simultaneously. Consequently the fields divide themselves into two distinct categories:

- Transverse Magnetic (TM): $B_z = 0$ everywhere; $E_z\big|_{\mathcal{S}} = 0$.
- Transverse Electric (TE): $E_z = 0$ everywhere; $\partial_n B_z\big|_{\mathcal{S}} = 0$.

1.5.1 Wave equation in cylindrical coordinate and its solution

Consider the case of a resonant cavity with end plates located at $z = 0$ and $z = L$ in cylindrical coordinate (r, ϕ, z). The boundaries condition at the end plates imposes $\overrightarrow{E}_t = 0$ $\Rightarrow \overrightarrow{E}_t \propto \sin(kz)$ with $k = \frac{p\pi}{L}$ $(p \in \mathbb{N})$.
Let $\gamma^2 \equiv \mu\epsilon\omega^2 - \left(\frac{p\pi}{L}\right)^2$, then the longitudinal field are found from:

$$(\nabla_t^2 + \gamma^2)\Psi(r, \phi) = 0 \tag{1.43}$$

where $\Psi \equiv E_z$ (TM) or B_z (TE)

The transverse field are then given by:

$$\text{TM:} \quad \begin{bmatrix} \overrightarrow{E}_t(r,\phi) \\ \overrightarrow{B}_t(r,\phi) \end{bmatrix} = \frac{i}{\gamma^2} \begin{bmatrix} \pm k\overrightarrow{\nabla}_t \\ \mu\epsilon\omega\hat{z} \times \overrightarrow{\nabla}_t \end{bmatrix} \Psi(r,\phi), \tag{1.44}$$

$$\text{TE:} \quad \begin{bmatrix} \overrightarrow{E}_t(r,\phi) \\ \overrightarrow{B}_t(r,\phi) \end{bmatrix} = \frac{i}{\gamma^2} \begin{bmatrix} -\omega\hat{z} \times \overrightarrow{\nabla}_t \\ \pm k\overrightarrow{\nabla}_t \end{bmatrix} \Psi(r,\phi), \tag{1.45}$$

In cylindrical coordinate the wave equation is

$$\left[\frac{1}{r}\partial_r(r\partial_r) + \frac{1}{r^2}\partial_\phi^2 + \gamma^2\right] \Psi(r,\phi) = 0. \tag{1.46}$$

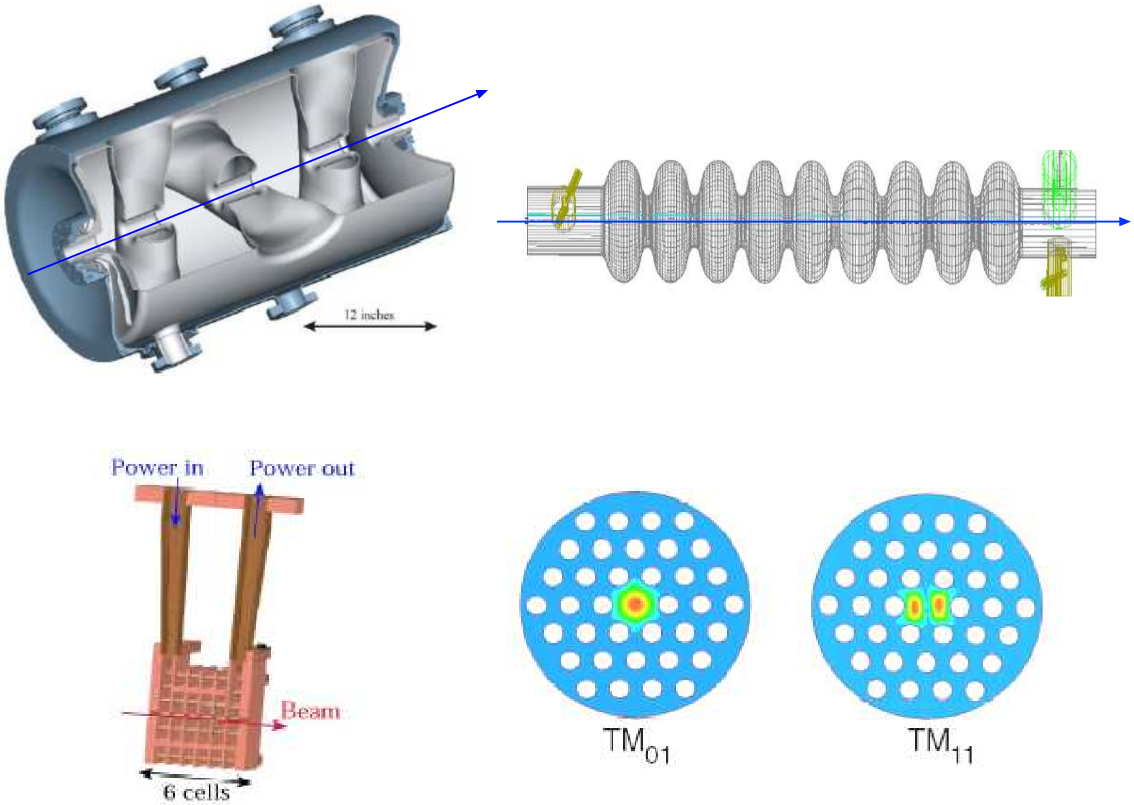

Figure 1.1: example of TM-mode accelerating cavities used in charged particle accelerators.

Assuming an harmonic azimuthal dependence of the form

$$\Psi(r,\phi) = \Psi(r)e^{\pm im\phi}; (m \in \mathbb{N}).$$

the wave equation simplifies to

$$(r^2 d_r^2 + r d_r + r^2\gamma^2 - m^2)\Psi(r) = 0, \tag{1.47}$$

which is a Bessel equation. Such equation has solutions of the the form $\Psi(r) = AJ_m(\gamma_{mn}r) + BN_m(\gamma_{mn}r)$. However $B = 0$ since $\lim_{r\to 0} N_m = -\infty$ which would result in unphysical solutions.

So finally:

$$\Psi(r,\phi) = AJ_m(\gamma_{mn}r)e^{\pm im\phi}; \gamma_{mn} \equiv \frac{x_{mn}}{R} \tag{1.48}$$

where x_{mn} is the n^{th} root of $J_m(x) = 0$, and R cavity radius. γ_{mn} is defined to insure $\Psi \to 0$ as $r \to R$.

1.5.2 Resonant frequencies

We have defined

$$\gamma_{mn}^2 = \mu\epsilon\omega_{mn}^2 - \left(\frac{p\pi}{L}\right)^2 = \left(\frac{x_{mn}}{R}\right)^2 \tag{1.49}$$

expliciting the frequency ω gives

$$\omega_{mn} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2} \tag{1.50}$$

which is the resonant frequency associated to the "*mnp*" mode.

1.6 Transverse magnetic (TM) mode fields

For the TM mode $\Psi \equiv E_z$ so that

$$E_z(r,\phi,z,t) = E_0 J_m(\gamma_{mn}r)e^{\pm i(kz+m\phi)-i\omega_{mn}t}\ ;\ B_z = 0. \tag{1.51}$$

The transverse electric field is

$$\begin{aligned}\overrightarrow{E}_t(r,\phi) &= \pm\frac{ikE_0}{\gamma^2}\left(\partial_r\hat{r} + \frac{1}{r}\partial_\phi\hat{\phi}\right)J_m(\gamma r)e^{\pm im\phi},\\ &= \pm i\frac{kE_0}{\gamma^2}e^{\pm im\phi}\left(\gamma_{mn}J_m'\hat{r} \pm \frac{im}{r}J_m\hat{\phi}\right).\end{aligned} \tag{1.52}$$

Insert $J_m' = \frac{m}{\gamma_{mn}r}J_m - J_{m+1}$:

$$\begin{aligned}\overrightarrow{E}_t(r,\phi,z,t) = \frac{\pm ikE_0}{\gamma^2}\quad &\left\{\left[\tfrac{m}{r}J_m(\gamma_{mn}r) - \gamma_{mn}J_{m+1}(\gamma_{mn}r)\right]\hat{r}\right.\\ &\left.\pm\tfrac{im}{r}J_m(\gamma_{mn}r)\hat{\phi}\right\}e^{\pm im\phi}e^{\pm ikz}e^{-i\omega_{mn}t}.\end{aligned} \tag{1.53}$$

Note that:

- $\overrightarrow{E}_t = 0$ at $z = 0, L \Rightarrow \pm ie^{\pm ikz} \longrightarrow -\sin(kz)$
- "time" is arbitrary: $e^{-i\omega t} \longrightarrow \cos(\omega t)$
- ϕ is also arbitrary $\Rightarrow e^{\pm im\phi} \longrightarrow \cos(m\phi)$ and $\pm ie^{\pm im\phi} \longrightarrow -\sin(m\phi)$.

$\overrightarrow{E}_t$ can be written in the real form:

$$\overrightarrow{E}_t = -\frac{kE_0\sin(kz)}{\gamma_{mn}^2}\left\{\left[\tfrac{m}{r}J_m(\gamma_{mn}r) - \gamma_{mn}J_{m+1}(\gamma_{mn}r)\right]\cos(m\phi)\hat{r} - \tfrac{m}{r}J_m(\gamma_{mn}r)\sin(m\phi)\hat{\phi}\right\}\cos(\omega_{mn}t). \tag{1.54}$$

where $E_0 \in \mathbb{R}$.

likewise,

$$E_z(r,\phi,z,t) = E_0J_m(\gamma_{mn}r)\cos(m\phi)\cos(kz)\cos(\omega_{mn}t) \tag{1.55}$$

The transverse magnetic field is

$$\begin{aligned}\overrightarrow{B}_t(r,\phi) &= \frac{i\mu\epsilon\omega}{\gamma^2}\hat{z}\times\left(\partial_r\hat{r} + \frac{1}{r}\partial_\phi\hat{\phi}\right)\Psi(r,\phi)\\ &= \frac{i\mu\epsilon\omega}{\gamma^2}\left(-\frac{1}{r}\partial_\phi\hat{r} + \partial_r\hat{\phi}\right)\Psi(r,\phi)\\ &= \frac{i\mu\epsilon\omega E_0}{\gamma^2}e^{\pm im\phi}\left[\mp\frac{im}{r}J_m\hat{r} + \left(\frac{m}{r}J_m - \gamma J_{m+1}\right)\hat{\phi}\right].\end{aligned}$$

- $ie^{-i\omega t} \longrightarrow \sin(\omega t)$ $\overrightarrow{B}$ and $\overrightarrow{E}$ are 90° out of phase;
- $e^{\pm im\phi} \longrightarrow \cos(m\phi)$, $\mp ie^{\pm im\phi} \longrightarrow \sin(m\phi)$, and $e^{ikz} \longrightarrow \cos(kz)$.

$$Bz = 0;\ \overrightarrow{B}_t(r,\phi,z,t) = \frac{\mu\epsilon\omega_{mn}E_0\cos(kz)}{\gamma_{mn}^2}\left\{\frac{m}{r}J_m(\gamma_{mn}r)\sin(m\phi)\hat{r} + \left[\frac{m}{r}J_m(\gamma_{mn}r) - \gamma_{mn}J_{m+1}(\gamma_{mn}r)\right]\cos(m\phi)\hat{\phi}\right\}\sin(\omega_{mn}t) \tag{1.56}$$

Let's define the shorthand notations:
$k_p \equiv \frac{p\pi}{L}$, $J_{mn}(r) \equiv J_m(\gamma_{mn}r)$, $\tilde{J}_{mn}(r) \equiv \frac{m}{r}J_m(\gamma_{mn}r) - \gamma_{mn}J_{m+1}(\gamma_{mn}r)$,

$$\begin{bmatrix} s_p(z) \\ c_p(z) \end{bmatrix} = \begin{bmatrix} \sin(k_pz) \\ \cos(k_pz) \end{bmatrix}, \begin{bmatrix} s_m(\phi) \\ c_m(\phi) \end{bmatrix} = \begin{bmatrix} \sin(m\phi) \\ \cos(m\phi) \end{bmatrix}, \begin{bmatrix} s_{mnp}(t) \\ c_{mnp}(t) \end{bmatrix} = \begin{bmatrix} \sin(\omega_{mnp}t) \\ \cos(\omega_{mnp}t) \end{bmatrix}.$$

The TM-fields are:

$$\begin{aligned}
E_r^{TM}(r,\phi,z,t) &= -E_0\frac{k_p}{\gamma_{mn}^2}\tilde{J}_{mn}(r)c_m(\phi)s_p(z)c_{mnp}(t), && (1.57)\\
E_\phi^{TM}(r,\phi,z,t) &= E_0\frac{k_p}{\gamma_{mn}^2}\frac{m}{r}J_{mn}(r)s_m(\phi)s_p(z)c_{mnp}(t), && (1.58)\\
E_z^{TM}(r,\phi,z,t) &= E_0J_{mn}(r)c_m(\phi)c_p(z)c_{mnp}(t), && (1.59)\\
B_r^{TM}(r,\phi,z,t) &= E_0\frac{\mu\epsilon\omega_{mn}}{\gamma_{mn}^2}\frac{m}{r}J_{mn}(r)s_m(\phi)c_p(z)s_{mnp}(t), && (1.60)\\
B_\phi^{TM}(r,\phi,z,t) &= E_0\frac{\mu\epsilon\omega_{mn}}{\gamma_{mn}^2}\tilde{J}_{mn}(r)c_m(\phi)c_p(z)s_{mnp}(t), && (1.61)\\
B_z^{TM}(r,\phi,z,t) &= 0. && (1.62)
\end{aligned}$$

1.7 Transverse Electric (TE) modes

The TE-fields now follow at once by inspection:

$$\begin{aligned}
E_r^{TE}(r,\phi,z,t) &= -B_0\frac{\omega_{mn}}{\gamma_{mn}^2}\frac{m}{r}J_{mn}(r)s_m(\phi)s_p(z)c_{mnp}(t), && (1.63)\\
E_\phi^{TE}(r,\phi,z,t) &= -B_0\frac{\omega_{mn}}{\gamma_{mn}^2}\tilde{J}_{mn}(r)c_m(\phi)s_p(z)c_{mnp}(t), && (1.64)\\
E_z^{TE}(r,\phi,z,t) &= 0, && (1.65)\\
B_r^{TE}(r,\phi,z,t) &= -B_0\frac{k_p}{\gamma_{mn}^2}\tilde{J}_{mn}(r)c_m(\phi)c_p(z)s_{mnp}(t), && (1.66)\\
B_\phi^{TM}(r,\phi,z,t) &= B_0\frac{k_p}{\gamma_{mn}^2}\frac{m}{r}J_{mn}(r)s_m(\phi)c_p(z)s_{mnp}(t), && (1.67)\\
B_z^{TE}(r,\phi,z,t) &= -B_0J_{mn}(r)c_m(\phi)s_p(z)s_{mnp}(t), && (1.68)
\end{aligned}$$

with $\partial_r B_z(r=R)=0 \Rightarrow \gamma_{mn}=x'_{mn}/R$; x'_{mn} root of $J_{mn}(x)=0$.

1.8 summary field associated to TE and TM modes

The Tables and summarize the equations for the electromagnetic field components and resonant frequencies associated to respectively TM and TE modes.

1.9 Physical insight

See Figure 1.2.

1.10 Geometry considerations

Comment on choice of modes and R/L:

$$\omega_{mnp}=\frac{1}{\sqrt{\mu\epsilon}}\frac{1}{R}\sqrt{(x'_{mn})^2+(p\pi)^2(R/L)^2}.$$

$$\omega_{mnp}^{TM} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}; \begin{cases} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \\ p = 0, 1, 2, \ldots \end{cases} \to \begin{cases} x_{0n} = 2.405, 5.520, 8.564, \ldots \\ x_{1n} = 3.832, 7.016, 10.714, \ldots \\ x_{2n} = 5.136, 8.417, 11.620, \ldots \end{cases} .$$

$$\begin{aligned}
E_r(r,\phi,z,t) &= -E_0\left(\frac{p\pi}{x_{mn}}\frac{R}{L}\right)\left[\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right) - J_{m+1}\left(x_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TM}t) \\
E_\phi(r,\phi,z,t) &= E_0\left(\frac{p\pi}{x_{mn}}\frac{R}{L}\right)\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right)\sin(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TM}t) \\
E_z(r,\phi,z,t) &= E_0 J_m\left(x_{mn}\frac{r}{R}\right)\cos(m\phi)\cos\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TM}t) \\
B_r(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}\left(\frac{R}{x_{mn}}\sqrt{\mu\epsilon}\,\omega_{mnp}^{TM}\right)\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right)\sin(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TM}t) \\
B_\phi(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}\left(\frac{R}{x_{mn}}\sqrt{\mu\epsilon}\,\omega_{mnp}^{TM}\right)\left[\left(\frac{m}{x_{mn}}\frac{R}{r}\right)J_m\left(x_{mn}\frac{r}{R}\right) - J_{m+1}\left(x_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TM}t) \\
B_z(r,\phi,z,t) &= 0
\end{aligned}$$

Table 1.1: Summary of resonant frequencies and electromagnetic field components associated to the TM_{mnp} mode in a cylindric cavity of length L and radius R).

$$\omega_{mnp}^{TE} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{x'_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}; \begin{cases} m = 0, 1, 2, \ldots \\ n = 1, 2, 3, \ldots \\ p = 1, 2, 3, \ldots \end{cases} \to \begin{cases} x'_{0n} = 3.832, 7.016, 10.714, \ldots \\ x'_{1n} = 1.841, 5.331, 8.536, \ldots \\ x'_{2n} = 3.054, 6.706, 9.970, \ldots \end{cases} .$$

$$\begin{aligned}
E_r(r,\phi,z,t) &= E_0\left(\frac{R}{x'_{mn}}\sqrt{\mu\epsilon}\,\omega_{mnp}^{TE}\right)\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right)\sin(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TE}t) \\
E_\phi(r,\phi,z,t) &= E_0\left(\frac{R}{x'_{mn}}\sqrt{\mu\epsilon}\,\omega_{mnp}^{TE}\right)\left[\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right) - J_{m+1}\left(x'_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\sin\left(p\pi\frac{z}{L}\right)\cos(\omega_{mnp}^{TE}t) \\
E_z(r,\phi,z,t) &= 0 \\
B_r(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}\left(\frac{p\pi}{x'_{mn}}\frac{R}{L}\right)\left[\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right) - J_{m+1}\left(x'_{mn}\frac{r}{R}\right)\right]\cos(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TE}t) \\
B_\phi(r,\phi,z,t) &= -E_0\sqrt{\mu\epsilon}\left(\frac{p\pi}{x'_{mn}}\frac{R}{L}\right)[\left(\frac{m}{x'_{mn}}\frac{R}{r}\right)J_m\left(x'_{mn}\frac{r}{R}\right)\sin(m\phi)\cos\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TE}t) \\
B_z(r,\phi,z,t) &= E_0\sqrt{\mu\epsilon}J_m\left(x'_{mn}\frac{r}{R}\right)\cos(m\phi)\sin\left(p\pi\frac{z}{L}\right)\sin(\omega_{mnp}^{TE}t)
\end{aligned}$$

Table 1.2: Summary of resonant frequencies and electromagnetic field components associated to the TE_{mnp} mode in a cylindric cavity of length L and radius R).

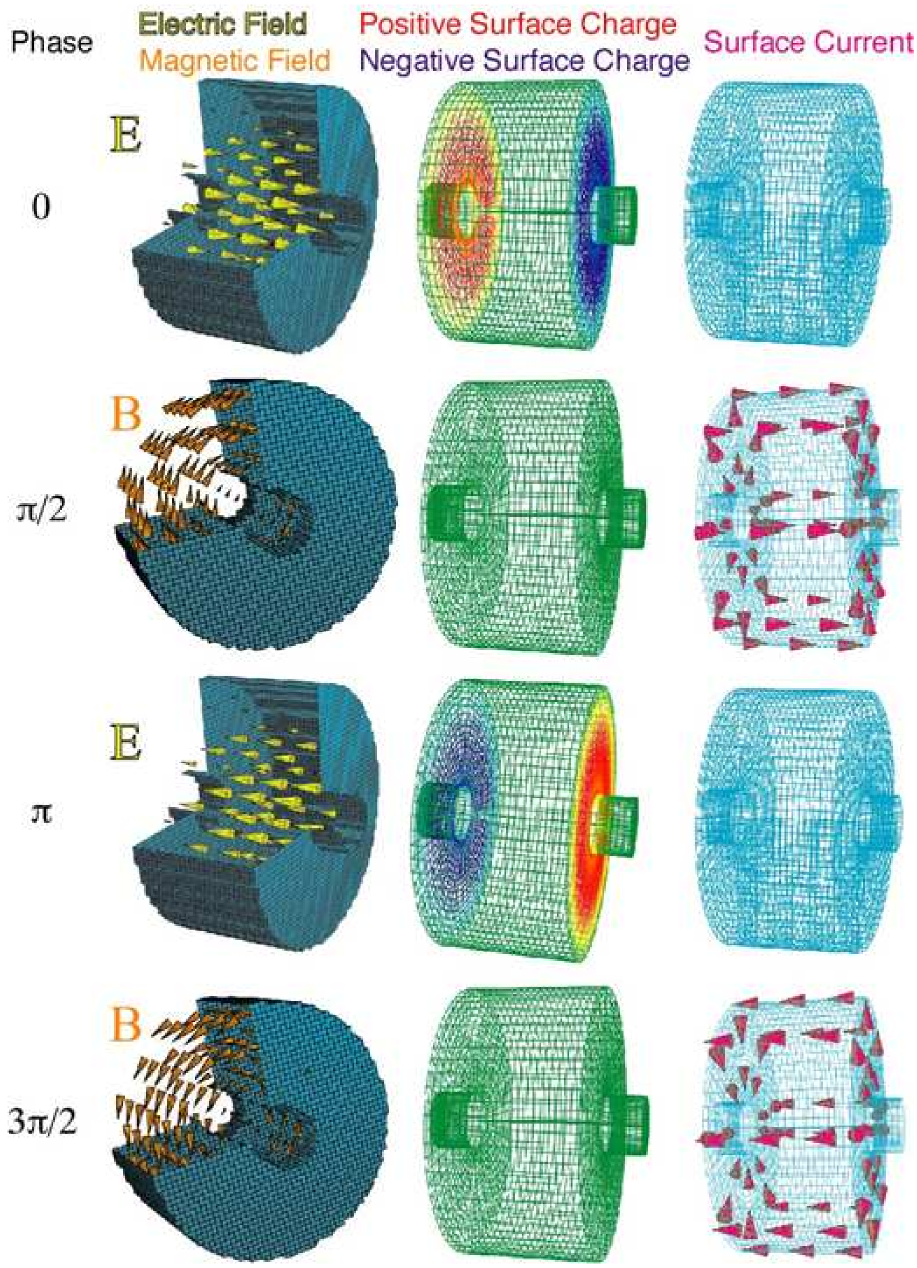

Figure 1.2: Charge and current distribution in a pillbox resonant cavity during an oscillation of the electromagnetic field associated to the $TM_0 10$ accelerating mode [from H. Padamsee, Cornell University].

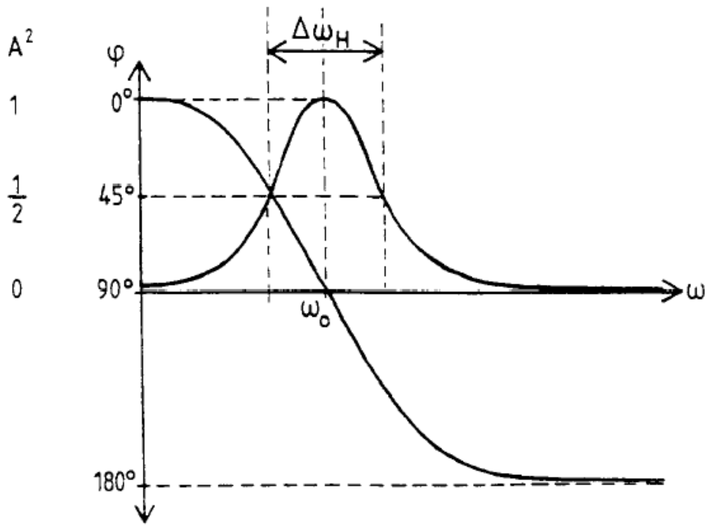

Figure 1.3: graphical interpretation of the quality factor.

Want to choose a frequency such that it is well separated from other resonant frequencies, and such that it is sensitive to both R and L to enable easy tuning. Thus,

- not interested in large p or R: $\lim_{R\to\infty,p\to\infty}\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\frac{p\pi}{L} \Rightarrow R$-independent.
- not interested in large m or n (and/or large L): $\omega_{mnp} \to \frac{1}{\sqrt{\mu\epsilon}}\frac{x_{mn}}{R}$ for large m or n and $R/L \sim 1$ L-independent.

$\Rightarrow$ The most interesting modes in practical applications should be low order modes with $R/L \sim 1$.

1.11 Quality factor

A resonant cavity is a class of harmonic oscillator: it can actually be modeled by an equivalent RLC circuit. Oscillators are usually characterized by a figure-of-merit refer to as quality factor which is define as the ratio of stored to dissipated power per cycle.

$$Q \equiv \frac{\omega U}{P} \tag{1.69}$$

$$\tag{1.70}$$

where U is the stored energy (i.e. the electromagnetic energy that was “injected” in the resonant cavity, and P is the dissipated power. For a normal conducting cavity (as opposed to superconducing cavity), the main source for power dissipation are Ohm losses due the conductor resistance.

Note: having cast things into ”consistent units”, we have in effect inserted into the wave equation:

$$\begin{aligned} \text{TM:} \quad & E_z(r,\phi) = E_0\psi(r,\phi),\ B_z(r,\phi) = 0; \\ \text{TE:} \quad & B_z(r,\phi) = -E_0\sqrt{\mu\epsilon}\psi(r,\phi),\ E_z(r,\phi) = 0; \end{aligned}$$

(note the use of small cap ψ defined as $\Psi = E_0\psi$) so that for both TM and TE-modes

$$E_z^2(r,\phi) + \frac{1}{\mu\epsilon}B_z^2(r,\phi) = E_0^2\psi^2 \tag{1.71}$$

holds and from Eq. 1.44 and 1.45:

$$\text{TM:} \quad \begin{bmatrix} \overrightarrow{E}_t(r,\phi) \\ \overrightarrow{B}_t(r,\phi) \end{bmatrix} = \frac{iE_0}{\gamma^2}\begin{bmatrix} \pm k\overrightarrow{\nabla}_t \\ \mu\epsilon\omega\hat{z}\times\overrightarrow{\nabla}_t \end{bmatrix}\psi(r,\phi);$$

$$\text{TE:} \quad \begin{bmatrix} \overrightarrow{E}_t(r,\phi) \\ \overrightarrow{B}_t(r,\phi) \end{bmatrix} = \frac{i\sqrt{\mu\epsilon}E_0}{\gamma^2}\begin{bmatrix} -\omega\hat{z}\times\overrightarrow{\nabla}_t \\ \pm k\overrightarrow{\nabla}_t \end{bmatrix}\psi(r,\phi);$$

Hence for both TM and TE-modes we have

$$E_t^2 + \frac{1}{\mu\epsilon}B_t^2 = \frac{E_0^2}{\gamma^4}(k^2+\mu\epsilon\omega^2)(\nabla_t\psi)^2. \tag{1.72}$$

The stored energy in the cavity is the volume integral

$$U = \frac{1}{2}\int_V d\overrightarrow{x}^3(ED+BH) = \frac{\epsilon}{2}\int_V d\overrightarrow{x}^3\left(E^2+\frac{B^2}{\mu\epsilon}\right) \tag{1.73}$$

we have $\int dz\sin^2(kz) = L/2$ and $\int dz\cos^2(kz) = L/2(1+\delta_{0p})$. Time averaging of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ gives a factor 1/2. We also note:

$$\begin{aligned}\frac{k^2+\mu\epsilon\omega^2}{\gamma^4} &= \frac{2k^2+(x/R)^2}{(x/R)^4} = \left(\frac{R}{x}\right)^2\left[1+2\left(\frac{kR}{x}\right)^2\right] \\ &= \left(\frac{R}{x}\right)^2\left[1+2\xi^2\right] \text{ where } \ \xi\equiv\frac{p\pi R}{xL}.\end{aligned} \tag{1.74}$$

Note that here we use the shorthand notation x to refer to either x_{mn} or x'_{mn}.

The stored energy in the cavity becomes

$$U = \frac{1+\delta_{0p}}{8}\epsilon E_0^2 L\int_A dA\left\{\left(\frac{R}{x}\right)^2\left[1+2\xi^2\right](\nabla_t\psi)^2+\psi^2\right\}. \tag{1.75}$$

For TE mode $p\neq 0 \Rightarrow \delta_{0p}=0$. Let's consider the integral $\int_A dA(\nabla_t\psi)^2$:

$$\begin{aligned}\int_A dA(\nabla_t\psi)^2 &= \int_A dA\overrightarrow{\nabla}_t.(\psi\overrightarrow{\nabla}_t\psi) - \int_A \psi\nabla_t^2\psi \\ &= \oint_C dl\psi\hat{n}.\overrightarrow{\nabla}_t\psi - \int_A dA\psi\nabla_t^2\psi.\end{aligned} \tag{1.76}$$

but,

$$\oint_C dl\psi\hat{n}.\overrightarrow{\nabla}_t\psi = 0\left\{\begin{array}{c}\psi^{TM}(r=R)=0 \\ \partial_r\psi^{TE}(r=R)=0\end{array}\right. \tag{1.77}$$

we also have, from the wave equation, $\nabla_t^2\psi = -(k^2 - \mu\epsilon\omega^2)\psi = (x/R)^2\psi$.
So

$$\int_A dA(\nabla_t)^2 = \left(\frac{x}{R}\right)^2 \int_A dA\psi^2, \tag{1.78}$$

and the stored energy is

$$U = \frac{\epsilon L}{4} E_0^2(1+\delta_{0p})\left[1+\xi^2\right]\int_A dA\psi^2 \tag{1.79}$$

this is JDJ eq, (8.92). Considering the azimuthal dependence $\psi^2 \propto \cos^2(m\phi)$; one has $\int_0^{2\pi} d\phi\psi^2 \rightarrow \pi(1+\delta_{0m})J^2$ so that

$$U = \frac{\pi}{4}\epsilon L E_0^2(1+\delta_{0m})(1+\delta_{0p})\left[1+\xi^2\right]\int_0^R drr J_m^2\left(x\frac{r}{R}\right). \tag{1.80}$$

Case of TM-modes [1]:

$$\int_0^R drr J_m^2\left(\frac{xr}{R}\right) = \frac{1}{2}R^2 J_{m+1}^2(x_{mn}) \tag{1.81}$$

Let $V \equiv \pi R^2 L$; the stored energy associated to TM-mode is

$$U_{mnp}^{TM} = \frac{1}{8}V\epsilon E_0^2(1+\delta_{0m})(1+\delta_{0p})\left[1+\xi^2\right]J_{m+1}^2(x_{mn}). \tag{1.82}$$

Case of TE-modes:
use the identity:
$\int_0^x d\rho\rho J_m^2(\rho) = \frac{1}{2}x^2\left[J_m^2(x) + J_{m-1}^2(x)\right] - mxJ_m(x)J_{m-1}(x)$

$$\Rightarrow \int_0^R drr J_m^2\left(x\frac{r}{R}\right) = \frac{R^2}{x^2}\left\{\frac{1}{2}x^2[...] - m...\right\}$$

$$= \frac{1}{2}R^2\left[J_m^2(x'_{mn}) + J_{m-1}^2(x'_{mn})\right] - \frac{m}{x'_{mn}}R^2 J_m(x'_{mn})J_{m-1}(x'_{mn}), \tag{1.83}$$

where we used the recursive relation $\frac{m}{x'}J_m(x') = J_{m+1}(x') = J_{m-1}(x')$ [2]
Finally,

$$\begin{aligned}\int_0^R drr J_m^2\left(x\frac{r}{R}\right) &= \frac{1}{2}R^2\left[1+\left(\frac{m}{x'}\right)^2\right]J_m^2(x') - R^2\left(\frac{m}{x'}\right)^2 J_m^2(x') \\ &= \frac{1}{2}R^2\left[1-\left(\frac{m}{x'}\right)^2\right]J_m^2(x'). \end{aligned} \tag{1.84}$$

Thus, with $V \equiv \pi R^2 L$ and $\delta_{0p} = 0$ $(p \neq 0)$ for TE-modes,

$$U_{mnp}^{TE} = \frac{1}{8}V\epsilon E_0^2(1+\delta_{0m})\left[1-\left(\frac{m}{x'}\right)^2\right]\left[1+\xi'^2\right]J_m^2(x'_{mn}). \tag{1.85}$$

[1] the identity $\int_0^1 dx x J_\nu^2(\alpha x) = \frac{1}{2}J_{\nu+1}^2$ if $J_\nu(\alpha) = 0$ was used
[2] Arfken pg. 631 $J_{n-1}(x) = \frac{n}{x}J_n(x) + J'_n(x)$ and $J_{n+1} = \frac{n}{x}J_n(x) - J'_n(x)$.

with $\xi' = \frac{p\pi R}{x'_{mn}L}$.

Let's now turn to the computation of the dissipated power in a cavity:

$$\frac{dP}{dA} = \frac{1}{2}R_s H_{\parallel}^2 = \frac{R_s}{2\mu^2}B_{\parallel}^2 \tag{1.86}$$

with R_s = surface resistance and $_{\parallel}$ means component $\parallel$ to cavity walls. The dissipated power is

$$P = \frac{R_s}{2\mu^2}\left[\int_{side} dAB_{\parallel}^2 + 2\int_{end} dAB_{\parallel}^2\right]. \tag{1.87}$$

Case of TM-modes:

$$\begin{aligned}\text{side:} \qquad \int_{side} dAB_{\parallel}^2 &= \int_0^L dz \int_0^{2\pi} d\phi R B_{\phi}^2(r,\phi,z) \\ &= \frac{L}{2}(1+\delta_{0p})\pi(1+\delta_{0m})\mu\epsilon E_0^2\left[1+\xi^2\right]R\left[\frac{m}{x}J_m(x) - J_{m+1}(x)\right]^2\end{aligned}$$

but $J_m(x) = 0$ so

$$\int_{side} dAB_{\parallel}^2 = \frac{1}{2}\mu\epsilon E_0^2(\pi LR)(1+\delta_{0p})(1+\delta_{0m})\left[1+\xi^2\right]J_{m+1}^2(x) \tag{1.88}$$

$$\begin{aligned}\text{end:} \qquad \int_{end} dAB_{\parallel}^2 &= \int_0^{2\pi} d\phi \int_0^R drrB_t^2 \\ &= \frac{E_0^2\mu^2\epsilon^2\omega^2}{(x/R)^4}\int_A dA(\nabla\psi)^2 = \frac{E_0^2\mu^2\epsilon^2\omega^2}{(x/R)^2}\int_A dA\psi^2 \\ &= \mu\epsilon E_0^2\left[1+\xi^2\right]\frac{\pi}{2}(1+\delta_{0m})R^2 J_{m+1}^2(x)\end{aligned} \tag{1.89}$$

$$\text{end:} \quad \int_{end} dAB_{\parallel}^2 = \frac{1}{2}\mu\epsilon E_0^2(\pi R^2)(1+\delta_{0m})\left[1+\xi^2\right]J_{m+1}^2(x_{mn}), \tag{1.90}$$

where (again) $\xi \equiv \frac{\pi p R}{x_{mn}L}$.

So the total power loss is:

$$P_{mnp}^{TM} = \frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{\pi}{2}R(1+\delta_{0m})\left[1+\xi^2\right]\left[L(1+\delta_{0p})+2R\right]J_{m+1}^2(x_{mn}),$$

or, with $A_s \equiv 2\pi RL$,

$$P_{mnp}^{TM} = \frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{1}{4}A_S(1+\delta_{0m})\left[1+\xi^2\right]\left[1+\delta_{0p}+2\frac{R}{L}\right]J_{m+1}^2(x_{mn}). \tag{1.91}$$

The quality factor is $Q = \frac{\omega U}{P}$ that is:

$$\begin{aligned} Q &= \frac{1}{\sqrt{\mu\epsilon}}\frac{x}{R}\sqrt{1+\xi^2}\times \\ & \frac{\frac{1}{8}V\epsilon E_0^2(1+\delta_{0m})(1+\delta_{0p})(1+\xi^2)J_{m+1}^2}{\frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{1}{4}A_s(1+\delta_{0m})(1+\delta_{0p}+2\frac{R}{L})(1+\xi^2)J_{m+1}^2}\end{aligned} \tag{1.92}$$

after simplification and using $V/A_s = R/2$,

$$QR_s = \sqrt{\frac{\mu}{\epsilon}}(1+\delta_{0p})\frac{x}{R}\frac{R}{2}\frac{\sqrt{1+\xi^2}}{1+\delta_{0p}+2\frac{R}{L}}. \tag{1.93}$$

Re-arranging

$$Q^{TM}_{mnp}R_s = \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}(1+\delta_{0p})\frac{\sqrt{x^2_{mn}+\left(\frac{p\pi R}{L}\right)^2}}{1+\delta_{0p}+2\frac{R}{L}}. \tag{1.94}$$

TE-mode:

$$\begin{aligned} \text{side:} \qquad \int_{side} dAB^2_{\parallel} &= \int_0^L dz \int_0^{2\pi} d\phi R\left[B^2_\phi(R,\phi,z)+B^2_z(R,\phi,z)\right] \\ &= \mu\epsilon E_0^2\frac{L}{2}\pi(1+\delta_{0m})R\left\{\left(\frac{m}{x'}\right)^2 J^2_m(x')\xi'^2 + J^2_m(x')\right\} \end{aligned} \tag{1.95}$$

$$\int_{side} dAB^2_{\parallel} = \mu\epsilon E_0^2\frac{\pi}{2}LR(1+\delta_{0m})\left[1+\left(\frac{Rmp\pi}{x'^2L}\right)^2\right]J^2_m(x') \tag{1.96}$$

$$\begin{aligned} \text{end:} \qquad \int_{end} dAB^2_{\parallel} &= \int_0^{2\pi} d\phi \int_0^R drrB^2_t(r,\phi,0) \\ &= \mu\epsilon E_0^2\xi'^2\left(\frac{R}{x'}\right)^2\int_A dA\left(\vec{\nabla}\psi\right)^2 \\ &= \mu\epsilon E_0^2\xi^2\int_A dA\psi^2. \end{aligned} \tag{1.97}$$

$$\begin{aligned} \int_A dA\psi^2 &= \pi(1+\delta_{0m})\int_0^R drrJ^2_m\left(x'\frac{r}{R}\right) \\ &= \frac{\pi}{2}R^2(1+\delta_{0m})\left[1-\left(\frac{m}{x'}\right)^2\right]J^2_m(x') \end{aligned} \tag{1.98}$$

$$\Rightarrow \int_A dA\psi^2 = \mu\epsilon E_0^2\frac{\pi}{2}R^2(1+\delta_{0m})\xi'^2\left[1+\left(\frac{m}{x'_{mn}}\right)^2\right]J^2_m(x'_{mn}). \tag{1.99}$$

Thus,

$$\begin{aligned} P^{TE}_{mnp} &= \frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{\pi}{2}R(1+\delta_{0m})J^2_m(x'_{mn})\left\{L\left[1+\left(\frac{m}{x'}\right)^2\xi'^2\right]\right. \\ &\quad \left. +2R\left[1-\left(\frac{m}{x'}\right)^2\right]\xi'^2\right\}, \end{aligned} \tag{1.100}$$

or,

$$P^{TE}_{mnp} = \frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{1}{4}A_s(1+\delta_{0m})\left\{1+\left(\frac{m}{x'}\right)^2\xi'^2+ \right.$$
$$\left. +2\frac{R}{L}\left[1-\left(\frac{m}{x'}\right)^2\right]\xi'^2\right\}J_m^2(x'). \tag{1.101}$$

and finally,

$$P^{TE}_{mnp} = \frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{1}{4}A_s(1+\delta_{0m})\left\{1+\left[2\frac{R}{L}+ \right.\right.$$
$$\left.\left. +(1-2\frac{R}{L})\left(\frac{m}{x'}\right)^2\right]\xi'^2\right\}J_m^2(x'_{mn}). \tag{1.102}$$

The quality factor is then

$$\begin{aligned} Q &= \frac{x'}{R}\frac{1}{\sqrt{\mu\epsilon}}\sqrt{1+\xi'^2}\times \\ &\frac{\frac{1}{8}V\epsilon E_0^2(1+\delta_{0m})[1-(m/x')^2](1+\xi'^2)J_m^2}{\frac{R_s}{2\mu^2}\mu\epsilon E_0^2\frac{1}{4}A_s(1+\delta_{0m})\{1+[2R/L+(1-2R/L)(m/x')^2]\xi^2\}J_m^2} \\ &= \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\frac{1}{R_s}\underbrace{\frac{x'[1-(m/x')^2](1+\xi'^2)^{3/2}}{1+[2R/L+2(1-2R/L)(m/x')^2]\xi'^2}}_{q}. \end{aligned} \tag{1.103}$$

$$\begin{aligned} q &= \frac{x'[1-(m/x')^2](1+\xi'^2)^{3/2}}{1+[2R/L+2(1-2R/L)(m/x')^2]\xi'^2} \\ &= \frac{x'[1-(m/x')^2](1+\xi'^2)^{3/2}}{1+2[1-(m/x')^2](R/L)\xi'^2+(m/x')^2\xi'^2} \\ &= \frac{x'[...](1+\xi'^{-2})^{3/2}}{2[...]R/L\xi'^{-1}+(...)^2\xi'^{-1}+\xi'^{-3}} \\ &= \frac{x'[...](1+\xi'^{-2})^{3/2}}{\frac{x'}{p\pi}\frac{L}{R}\{2[...](R/L)+(m/x')^2+\xi'^{-2}\}} \end{aligned} \tag{1.104}$$

Finally one has:

$$Q^{TE}_{mnp}R_s = \frac{p\pi}{2}\sqrt{\frac{\mu}{\epsilon}}\frac{\left[1-\left(\frac{m}{x'_{mn}}\right)^2\right]\left[1+\left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]^{3/2}}{2\left[1-\left(\frac{m}{x'_{mn}}\right)^2\right]+\left(\frac{m}{x'_{mn}}\right)^2\frac{L}{R}+\left(\frac{x'_{mn}}{p\pi}\right)^2\left(\frac{L}{R}\right)^3} \tag{1.105}$$

1.11.1 Comment on JDJ's geometry factor for TE_{mnp} mode – Eq. (8.96)

For the power dissipated on the wall we use:

$$\frac{dP}{dA} = \frac{1}{2} R_s H_{\parallel}^2, \tag{1.106}$$

where the factor 1/2 comes from time-averaging, and R_s, the surface impedance is defined for normal and super-conductor. JDJ uses:

$$\frac{dP}{dA} = \frac{\mu_c \omega \delta}{4} H_{\parallel}^2 \tag{1.107}$$

the factor $\frac{\mu_c \omega \delta}{4}$ is valid for normal conductor only. Compare to infer:

$$R_s \Leftrightarrow \frac{\mu_c \omega \delta}{2} \tag{1.108}$$

JDJ also introduced the geometrical factor G so that:

$$Q = \frac{\mu}{\mu_c} \frac{V}{S\delta} G \tag{1.109}$$

This means

$$QR_s = \frac{\mu}{\mu_c} \frac{V}{S\delta} G \frac{\mu_c \omega \delta}{2} = \frac{\mu}{2} \frac{V}{S} \omega G \tag{1.110}$$

Let $\Gamma \equiv QR_s$ (Γ has units of Ω). One has:

$$G = \frac{2}{\mu} \frac{S}{V} \frac{\Gamma}{\omega}. \tag{1.111}$$

Substitute the earlier results for TE-mode:

$$\begin{aligned} G &= \frac{2}{\mu} \frac{2\pi R L + 2\pi R^2}{\pi R^2 L} \frac{\Gamma}{\frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{x'_{mn}}{R}\right)^2 + \left(\frac{p\pi}{L}\right)^2}} \\ &= \frac{4(1 + \frac{R}{L})}{R} \sqrt{\frac{\epsilon}{\mu}} \frac{L}{p\pi} \frac{\Gamma}{\sqrt{1 + \left(\frac{x'_{mn} L}{p\pi R}\right)^2}} \end{aligned} \tag{1.112}$$

$$
\begin{aligned}
G &= \frac{4(1+R/L)}{R}\sqrt{\frac{\epsilon}{\mu}}\frac{L}{p\pi}\frac{1}{\sqrt{1+\left(\frac{x'}{p\pi}\frac{L}{R}\right)^2}} \\
&\quad \times\frac{p\pi}{2}\sqrt{\frac{\mu}{\epsilon}}\frac{\left[1-\left(\frac{m}{x'_{mn}}\right)^2\right]\left[1+\left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]^{3/2}}{2\left[1-\left(\frac{m}{x'_{mn}}\right)^2\right]+\left(\frac{m}{x'_{mn}}\right)^2\frac{L}{R}+\left(\frac{x'_{mn}}{p\pi}\right)^2\left(\frac{L}{R}\right)^3} \\
&= \left(1+\frac{L}{R}\right)\frac{2\left[1-\left(\frac{m}{x'_{mn}}\right)^2\right]\left[1+\left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]}{2\left[1-\left(\frac{m}{x'_{mn}}\right)^2\right]+\left(\frac{m}{x'_{mn}}\right)^2\frac{L}{R}+\left(\frac{x'_{mn}}{p\pi}\right)^2\left(\frac{L}{R}\right)^3} \\
G &= \left(1+\frac{L}{R}\right)\frac{\left[1+\left(\frac{x'_{mn}L}{p\pi R}\right)^2\right]}{1+\frac{(m/x'_{mn})^2(L/R)+(x'_{mn}/p\pi)^2(L/R)^3}{2\left[1-(m/x'_{mn})^2\right]}}
\end{aligned}
\tag{1.113}
$$

1.11.2 Example of TE_{111}

$x'_{11}=1.841$; $\left(\frac{x'_{11}}{\pi}\right)^2=0.343$; $\frac{(1/x'_{11})^2}{2[1-(1/x'_{11})^2]}=0.209$; $\frac{(x'_{11}/\pi)^2}{2[1-(1/x'_{11})^2]}=0.244$.

$$
\Rightarrow G=\left(1+\frac{L}{R}\right)\frac{1+0.343(L/R)^2}{1+0.209(L/R)+0.244(L/R)^3}
\tag{1.114}
$$

This is JDJ's Eq. (8.97).
The advantage of using G instead of Γ is that $G=\mathcal{O}(1)$ so that for the right circular cylinder:

$$
\begin{aligned}
QR_s &\sim \frac{\mu}{2}\frac{V}{S}\omega=\frac{\mu}{2}\frac{\pi R^2L}{2\pi RL+2\pi R^2}2\pi f \\
&\Rightarrow \Gamma\sim\mu\frac{L}{1+L/R}f \text{ (unit is } \Omega)
\end{aligned}
\tag{1.115}
$$

1.12 Perturbation of cavity wall & Slater's Theorem

Consider a single resonant mode in a cavity. We perturb the cavity wall and estimate the associated change in resonant frequency. This relates to the cavity tuning and also to removing degeneracies between modes.
Consider a volume $\mathcal{V}$ bounded by a surface $\mathcal{S}$, then the the force associated to e.m. field in the volume is related to the Maxwell stress tensor (see JDJ Chapter 6) via:

$$
\overrightarrow{F}=\int_{\mathcal{S}}\mathbb{T}.d\overrightarrow{A}
\tag{1.116}
$$

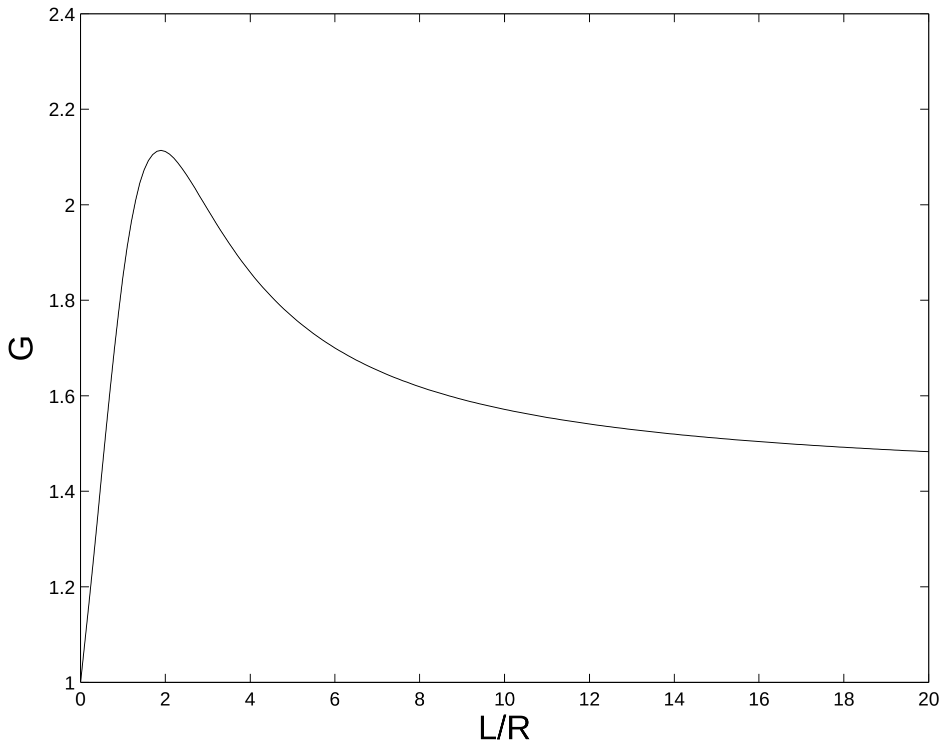

Figure 1.4: G vs. L/R for the TE_{111}-mode.

with $d\overrightarrow{A} = dA\hat{n}$ and

$$\overrightarrow{\mathbb{T}} = \epsilon\left(\overrightarrow{E}\overrightarrow{E} + \frac{1}{\mu\epsilon}\overrightarrow{B}\overrightarrow{B}\right) - \frac{\epsilon}{2}\left(\overrightarrow{E}\overrightarrow{E} + \frac{1}{\mu\epsilon}\overrightarrow{B}\overrightarrow{B}\right)\hat{n}\hat{n}. \tag{1.117}$$

Introducing the displacement $d\overrightarrow{\zeta} = d\zeta\hat{n}$, $\Rightarrow dV = dAd\zeta$;
δU, the work done by the e.m. field against displacement is

$$\delta U = \int dAd\zeta\hat{n}.\overrightarrow{\mathbb{T}}.\hat{n} = \int_{\Delta V} dV\hat{n}.\overrightarrow{\mathbb{T}}.\hat{n} \tag{1.118}$$

Note that $\hat{n}\overrightarrow{\mathbb{T}}\hat{n}$ represents the e.m. pressure on the wall.

$$\hat{n}.\mathbb{T}.\hat{n} = \epsilon E^2 - \frac{\epsilon}{2}(E^2 + \frac{1}{\mu\epsilon}B^2) \tag{1.119}$$

because $\hat{n}.\overrightarrow{B} = \hat{n} \times \overrightarrow{E} = 0$ at the surface of a perfect conductor.

$$\delta U = \int_{\Delta V} \frac{\epsilon}{2}(E^2 - \frac{1}{\mu\epsilon}B^2). \tag{1.120}$$

or, written in terms of time-averaged fields,

$$\delta U = \frac{\epsilon}{4}\int_{\Delta V} dV(E^2 - \frac{1}{\mu\epsilon}B^2). \tag{1.121}$$

In the cavity the "photon number" is conserved, which means U/ω is an invariant. So $\delta U/U = \delta\omega/\omega$, from which

$$\frac{\delta\omega}{\omega} = \frac{\epsilon}{4U}\int_{\Delta V} dV(E^2 - \frac{1}{\mu\epsilon}B^2). \tag{1.122}$$

(E^2 and B^2 are time-averaged).

1.12.1 Example of application: measurement of field profile with a bead pull

Consider TM_{011}, $r = 0$. The only non zero field component is:

$$E_z = E_0\cos(\pi\frac{z}{L})\cos(\omega_{011}^{TM}t). \tag{1.123}$$

Imagine pulling a conducting bead along the z-axis. The volume of the bead is ΔV. Then

$$\begin{aligned}\frac{\delta\omega}{\omega} &\simeq \frac{\epsilon}{4U}\Delta V E_z^2 \ \ \text{(time-averaged)} \\ &\simeq \frac{\epsilon E_0^2\Delta V\cos^2(\pi z/L)}{\frac{\epsilon E_0^2}{8}V2\left[1+\left(\frac{\pi}{x_{01}}\frac{L}{R}\right)^2\right]J_1^2(x_{01})}.\end{aligned} \tag{1.124}$$

$$\Rightarrow\frac{\delta\omega}{\omega} \simeq \frac{\Delta V}{V}\frac{\cos^2(\pi z/l)}{\left[1+\left(\frac{\pi}{x_{01}}\frac{L}{R}\right)^2\right]J_1^2(x_{01})} \tag{1.125}$$

Therefore we can "map" the $\cos^2$-dependence. Same principle for any other mode or superimposition of modes.

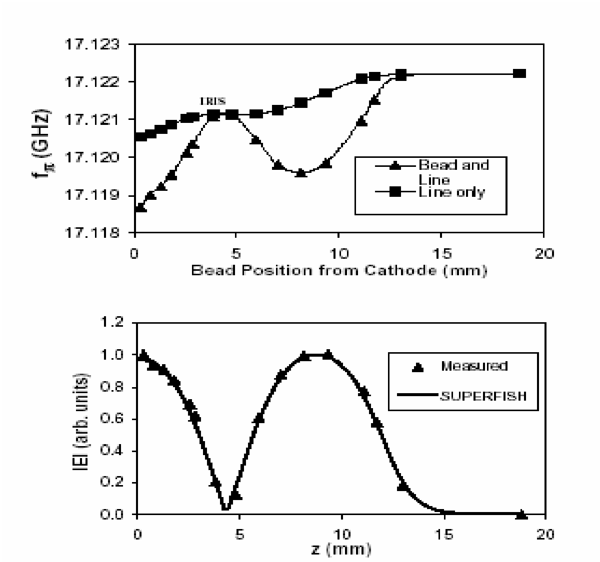

Figure 1.5: Example of bead-pull measurement [taken from W. J. Brown, S. E. Korbly, K. E. Kreischer, I. Mastovsky, and R. J. Temkin from Phys. Rev. ST Accel. Beams 4, 083501 (2001)]
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Chapter 2

Special Relativity & Covariance of Electromagnetism

2.1 EM field of point charge moving at constant velocity

In this Section the em field associated to a point charge moving at contant velocity is derived. We (of course!) start with Maxwell's equations:

$$\overrightarrow{\nabla}.\overrightarrow{D} = \rho, \ \overrightarrow{\nabla}.\overrightarrow{H} = 0,$$
$$\overrightarrow{\nabla} \times \overrightarrow{E} + \partial_t \overrightarrow{B} = 0, \text{ and } \overrightarrow{\nabla} \times \overrightarrow{H} - \partial_t \overrightarrow{D} = \overrightarrow{J}.$$

For now we assume a charge distribution with associated density ρ and current $\overrightarrow{J}$. Writing in terms of electromagnetic potentials, $\overrightarrow{A}$ and Φ gives

$$\begin{aligned}
\overrightarrow{B} &= \overrightarrow{\nabla} \times \overrightarrow{A} \Rightarrow \overrightarrow{\nabla} \times (\overrightarrow{E} + \partial_t \overrightarrow{A}) = 0 \Rightarrow \overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \partial_t \overrightarrow{A} \\
\frac{1}{\mu}\overrightarrow{\nabla} \times \overrightarrow{B} - \epsilon\partial_t \overrightarrow{E} &= \overrightarrow{J} \Rightarrow \overrightarrow{\nabla} \times \overrightarrow{B} - \mu\epsilon\partial_t \overrightarrow{E} = \mu \overrightarrow{J} \\
& \Rightarrow \overrightarrow{\nabla} \times (\overrightarrow{\nabla} \times \overrightarrow{A}) + \mu\epsilon(\overrightarrow{\nabla}\partial_t\Phi + \partial_t^2 \overrightarrow{A}) = \mu \overrightarrow{J}
\end{aligned}$$

Using the identity $\overrightarrow{\nabla} \times (\overrightarrow{\nabla} \times \overrightarrow{A}) = \overrightarrow{\nabla}(\overrightarrow{\nabla}.\overrightarrow{A}) - \nabla^2 \overrightarrow{A}$,

$$-\nabla^2 \overrightarrow{A} + \overrightarrow{\nabla}(\overrightarrow{\nabla}.\overrightarrow{A} + \mu\epsilon\partial_t\Phi) + \mu\epsilon\partial_t^2 \overrightarrow{A} = \mu \overrightarrow{J}$$

but $\overrightarrow{\nabla}.\overrightarrow{A} + \mu\epsilon\partial_t\Phi = 0$ in Lorenz gauge so that

$$\nabla^2 \overrightarrow{A} - \mu\epsilon\partial_t^2 \overrightarrow{A} = -\mu \overrightarrow{J} \quad \text{[JDJ, Eq. (6.16)]} \tag{2.1}$$

$$\overrightarrow{\nabla}.\overrightarrow{D} = \rho \Rightarrow -\nabla^2\Phi - \partial_t \overrightarrow{\nabla}.\overrightarrow{A} = \frac{\rho}{\epsilon}$$

$$\nabla^2 \overrightarrow{\Phi} - \mu\epsilon\partial_t^2\Phi = -\frac{\rho}{\epsilon} \quad \text{[JDJ, Eq. (6.15)]} \tag{2.2}$$

For a source moving at constant velocity, $\overrightarrow{v}$: $\rho = \rho(\overrightarrow{x} - \overrightarrow{v}t)$ and $\overrightarrow{J} = \overrightarrow{v}\rho(\overrightarrow{x} - \overrightarrow{v}t)$. We then have to solve a set of inhomogeneous d'Alembert equations: $\Box f = g(\overrightarrow{x} - \overrightarrow{v}t)$.

Consider the case $\overrightarrow{v} = v\hat{z} \Rightarrow f(\overrightarrow{x} - \overrightarrow{v}t) = (x, y, z - vt) = f(x, y, \zeta)$ with $\zeta \equiv z - vt$. Then

$$\partial_z f \rightarrow \frac{\partial\zeta}{\partial z}\partial_\zeta f = \partial_\zeta f \tag{2.3}$$

$$\partial_t f \rightarrow \frac{\partial\zeta}{\partial t}\partial_\zeta f = -v\partial_\zeta f \tag{2.4}$$

$$\Rightarrow \Box f \rightarrow \left(\partial_x^2 + \partial_y^2 + \partial_\zeta^2 - \mu\epsilon v^2\partial_\zeta^2\right) f = \left(\partial_x^2 + \partial_y^2 + \gamma^{-2}\partial_\zeta^2\right) f. \tag{2.5}$$

with $\gamma \equiv \frac{1}{\sqrt{1-\mu\epsilon v^2}}$. Let $z' = \gamma\zeta \Rightarrow \partial_\zeta = \frac{\partial z'}{\partial\zeta}\partial_{z'} = \gamma\partial_{z'}$:

$$\left(\partial_x^2 + \partial_y^2 + \partial_{z'}^2\right) f(x, y, \gamma^{-1}z') = g(x, y, \gamma^{-1}z'). \tag{2.6}$$

We now specialize to a *point* charge and explicit $\rho(\overrightarrow{x} - \overrightarrow{v}t) \rightarrow \delta(x)\delta(y)\delta(\gamma^{-1}z') = \gamma\delta(x)\delta(y)\delta(z') = \gamma\delta(\overrightarrow{x}')$.

As a results, $\overrightarrow{A} \rightarrow A\hat{z}$ $(A_x = A_y = 0)$;

$$\nabla_{x'}^2 A = -\gamma\mu q v\delta(\overrightarrow{x'}), \;\; \nabla_{x'}^2 \Phi = -\gamma\frac{q}{\epsilon}\delta(\overrightarrow{x'}). \tag{2.7}$$

Which is solved by inspection:

$$\nabla_{x'}^2\left(\frac{1}{|\overrightarrow{x'}|}\right) = -4\pi\delta(\overrightarrow{x'}) \Rightarrow \left\{ \begin{array}{l} A = \frac{\gamma\mu}{4\pi}\frac{qv}{R}, \\ \Phi = \frac{\gamma}{4\pi\epsilon}\frac{q}{R}, \end{array} \right. \tag{2.8}$$

where $R \equiv \sqrt{x^2 + y^2 + \gamma^2(z - vt)^2}$.
Now we can calculate $\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \partial_t\overrightarrow{A}$:

$$\begin{aligned} \overrightarrow{E} &= -\frac{\gamma q}{4\pi\epsilon}(\overrightarrow{\nabla} + \mu\epsilon v\partial_t\hat{z})\frac{1}{R} \\ &= \frac{\gamma q}{4\pi\epsilon R^3}\left[x\hat{x} + y\hat{y} + \gamma^2(z - vt)(1 - \mu\epsilon v^2)\hat{z}\right] \end{aligned} \tag{2.9}$$

$$\overrightarrow{E} = \frac{\gamma q}{4\pi\epsilon R^3}\left[x\hat{x} + y\hat{y} + (z - vt)\hat{z}\right] \tag{2.10}$$

Conversion to spherical coordinates $(r, \psi\theta)$:
$x^2 + y^2 = r^2\sin^2\theta$, $z - vt = r\cos\theta$.

$$\begin{aligned} \Rightarrow R^2 &= r^2(\sin^2\theta + \gamma^2\cos^2\theta) \\ &= \gamma^2 r^2\left(1 + \frac{1-\gamma^2}{\gamma^2}\sin^2\theta\right) = \gamma^2 r^2(1 - \mu\epsilon v^2\sin^2\theta), \end{aligned}$$

$$
\begin{aligned}
\Rightarrow E &= \frac{\gamma q}{4\pi\epsilon} \frac{r}{\gamma^3 r^3 (1-\mu\epsilon v^2 \sin^2\theta)^{3/2}} \\
&= \frac{q}{4\pi\epsilon r^2} \frac{1-\mu\epsilon v^2}{(1-\mu\epsilon v^2 \sin^2\theta)^{3/2}}. \qquad (2.11)
\end{aligned}
$$

Note: In vacuum, take $\mu\epsilon \to \mu_0\epsilon_0 = c^{-2}$, and then

$$
\overrightarrow{E} = \frac{q}{4\pi\epsilon r^2} \frac{\overrightarrow{r}}{\gamma^2 (1-\beta^2 \sin^2\theta)^{3/2}}. \quad \text{[JDJ, Eq. (11.154)]} \qquad (2.12)
$$

Note that $E(\pi/2)/E(0) = \gamma^3 \Rightarrow$ field lines are "squashed" orthogonal to the direction of motion.
Also we can find $\overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}$:

$$
\begin{aligned}
\overrightarrow{A} = \mu\epsilon\Phi\overrightarrow{v} \quad &\Rightarrow \quad \overrightarrow{B} = \mu\epsilon\overrightarrow{\nabla} \times (\Phi\overrightarrow{v}) = \mu\epsilon[\overrightarrow{\nabla}\Phi \times \overrightarrow{v} + \Phi\overrightarrow{\nabla} \times \overrightarrow{v}] \\
&\Rightarrow \quad \overrightarrow{B} = \mu\epsilon\overrightarrow{\nabla}\Phi \times \overrightarrow{v}.
\end{aligned}
$$

$$
\overrightarrow{v} \times \overrightarrow{E} = -\overrightarrow{v} \times (\overrightarrow{\nabla}\Phi + \partial_t \overrightarrow{A}) = \overrightarrow{\nabla}\Phi \times \overrightarrow{v}.
$$

$$
\overrightarrow{B} = \mu\epsilon\overrightarrow{v} \times \overrightarrow{E}, \text{ or } \overrightarrow{B} = \frac{\mu}{4\pi} \frac{\gamma q}{R^3} \overrightarrow{v}(x\hat{y} - y\hat{x}). \qquad (2.13)
$$

Further reductions [toward JDJ Eq. (11.152)]:

$$
\overrightarrow{E} = \frac{q}{4\pi\epsilon_0 r^2} \frac{\hat{r}}{\gamma^2 (1-\beta^2 \sin^2\theta)^{3/2}} \qquad (2.14)
$$

$\sin\theta = \frac{b}{r} = \frac{b}{\sqrt{b^2+v^2t^2}}$.

$$
1-\beta^2\sin^2\theta = 1 - \frac{\beta^2 b^2}{b^2+(vt)^2} = \frac{b^2+v^2t^2-\beta^2b^2}{b^2+v^2t^2} = \frac{(1-\beta^2)b^2+v^2t^2}{b^2+v^2t^2}
$$

$$
1-\beta^2\sin^2\theta = \frac{b^2+\gamma^2v^2t^2}{\gamma^2r^2} \Rightarrow \gamma r\sqrt{1-\beta^2\sin^2\theta} = \sqrt{b^2+\gamma^2v^2t^2}
$$

Finally

$$
\overrightarrow{E} = \frac{q}{4\pi\epsilon_0} \frac{\gamma\overrightarrow{r}}{(b^2+\gamma^2v^2t^2)^{3/2}} \Rightarrow \overrightarrow{E}_\perp = \frac{q}{4\pi\epsilon_0} \frac{\gamma b\hat{x}}{(b^2+\gamma^2v^2t^2)^{3/2}}. \qquad (2.15)
$$

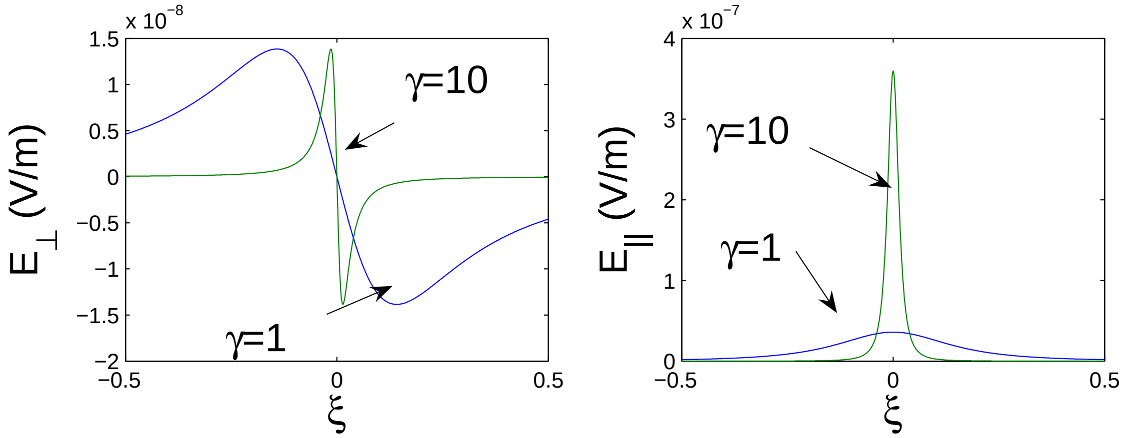

Figure 2.1: transverse (left) and longitudinal (right) electric fields associated to an electron moving at rest and at constant velocity with $\gamma = 10$.

2.1.1 Space-charge effects

Consider a charge q_0 comoving with another charge q. Let' assume (simplistic) that both charge move at the same velocity $\overrightarrow{v}$. The force imparted to q_0 by q is

$$\begin{aligned} F &= q_0(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}) \\ &= q_0[\overrightarrow{E} + \mu\epsilon\overrightarrow{v} \times (\overrightarrow{v} \times \overrightarrow{E})] \end{aligned}$$

$$\begin{aligned} \Rightarrow \overrightarrow{F} &= q_0\left[(1-\mu\epsilon v^2)\overrightarrow{E} + \mu\epsilon v^2 E_z\hat{z}\right] \\ &= q_0\left(\frac{1}{\gamma^2}\overrightarrow{E} + \frac{\gamma^2-1}{\gamma^2}E_z\hat{z}\right) = q_0\left[\frac{1}{\gamma^2}(\overrightarrow{E} - E_z\hat{z}) + E_z\hat{z}\right] \end{aligned}$$

$$\Rightarrow \overrightarrow{F} = q_0\left[\frac{1}{\gamma^2}\overrightarrow{E}_\perp + \overrightarrow{E}_\parallel\right] \tag{2.16}$$

The self-magnetic field of q cancels its self-electric field to within a factor $1/\gamma^2$. This effect is an important one when dealing with charged particle accelerators aimed at producing and accelerating high brightness beams (a lot of charge in a very small phase space volume). The cancelation of the transverse force shown in Eq. 2.16 calls for rapidly accelerating a beam in order to preserve its brightness.

2.2 Special relativity

The squashing of the E-field of a moving charge, as it corresponds to the equation of motion, is suggestive of the Lorentz contraction, and thus indicative that electrodynamics is invariant under Lorentz transformations.

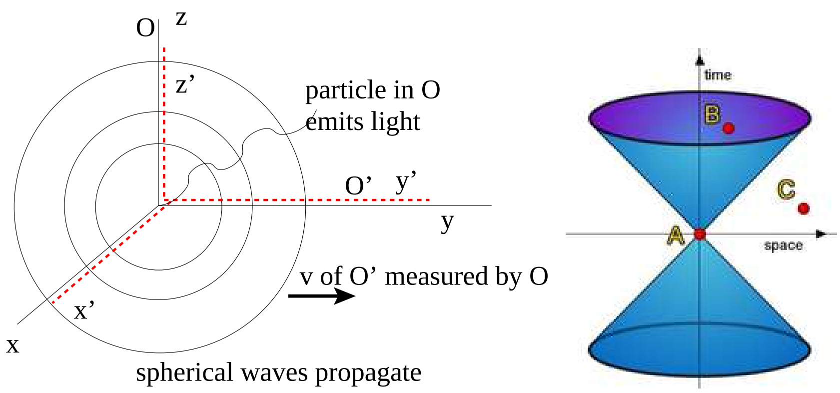

Figure 2.2: left: two inertial frame ($\mathcal{O}$ is the lab frame). right: light cone, $[AB]$ is time-like $[AC]$ is space-like.

2.2.1 Proper time and its invariance

Consider a spherical waves propagating such that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = c^2$; see Fig. 2.2. If c is the same in all inertial reference frames (postulate), then

$$\left(\frac{dx'}{dt'}\right)^2 + \left(\frac{dy'}{dt'}\right)^2 + \left(\frac{dz'}{dt'}\right)^2 = c^2$$

So, we write:

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 \quad \text{for photons.} \tag{2.17}$$

This holds true in any inertial coordinate system. More generally we can define the proper time:

$$d\tau^2 \equiv dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2). \tag{2.18}$$

In SR, the proper time is an invariant – all inertial observers measure the same $d\tau$. Note that:

$$d\tau^2 = dt^2(1 - \beta^2) = \frac{1}{\gamma^2}dt^2; \tag{2.19}$$

$\overrightarrow{\beta} \equiv \frac{1}{c}\overrightarrow{v}$; $\overrightarrow{v}$ = velocity measured in lab frame ($\mathcal{O}$), dt = period between "ticks" of clock in lab frame.
When $\overrightarrow{v} = 0$, $d\tau = dt \Rightarrow d\tau$ = period between "ticks" of clock comoving with $\mathcal{O}'$. Every inertial observer measure the same value for this time interval: it is a scalar – a fixed physical quantity!

If δt represents the period between ticks of $\mathcal{O}'$'s clock, then $\mathcal{O}$ sees it ticks with period:

$$dt = \gamma\delta t \tag{2.20}$$

This is "time dilatation": $\mathcal{O}$ thinks $\mathcal{O}'$'s clock runs slow.

2.2.2 Minkowski metric and Lorentz transformations

Let $x^0 \equiv ct$, $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$ [so $\overrightarrow{x}^i \equiv \overrightarrow{X}$ (i=1,2,3)]. Then we can write:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{2.21}$$

with $\alpha, \beta = 0, 1, 2, 3$ and $g_{\alpha\beta}$ is the Minkowski metric [1]:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.23}$$

standard convention: Use Greek indices to represent sums from 0-3 and Latin indices for sum from 1-3.

The Lorentz transformation matrix from stationary observer $\mathcal{O}$ to moving observer $\mathcal{O}'$ is the "boost matrix" [JDJ, Eq.(11.98)] ($\Lambda^\alpha_\gamma \Lambda^\beta_\delta g_{\alpha\beta} = g_{\gamma\delta}$:

$$\Lambda^\nu_\mu = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + \left(\frac{\beta_x}{\beta}\right)^2 (\gamma - 1) & \frac{\beta_x\beta_y}{\beta^2}(\gamma - 1) & \frac{\beta_x\beta_z}{\beta^2}(\gamma - 1) \\ -\gamma\beta_y & \frac{\beta_x\beta_y}{\beta^2}(\gamma - 1) & 1 + \left(\frac{\beta_y}{\beta}\right)^2 (\gamma - 1) & \frac{\beta_y\beta_z}{\beta^2}(\gamma - 1) \\ -\gamma\beta_z & \frac{\beta_x\beta_z}{\beta^2}(\gamma - 1) & \frac{\beta_y\beta_z}{\beta^2}(\gamma - 1) & 1 + \left(\frac{\beta_z}{\beta}\right)^2 (\gamma - 1) \end{pmatrix}, \tag{2.24}$$

provided the coordinates of $\mathcal{O}$ and $\mathcal{O}'$ are aligned. The the Lorentz transformation from $\mathcal{O}$ and $\mathcal{O}'$ is:

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta. \tag{2.25}$$

Note $\Lambda^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}$. If the coordinate axes are not aligned then the transformation is the product of Λ^α_β and a rotation matrix.

The principle of SR is : All laws of physics must be invariant under Lorentz transformations. "Invariant" $\leftrightarrow$ Laws retain the same mathematical forms and numerical constants (scalars) retain the same value.

2.3 Particle dynamics in SR

Define the "4-velocity": $u^\alpha \equiv \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds}$:

$$u^0 = c\frac{dt}{d\tau} = \gamma c \ \text{ and } u^i = \frac{1}{c}\frac{dx^i}{d\tau} = c\frac{dt}{d\tau}\frac{dx^i}{dt} = c\gamma\beta^i \tag{2.26}$$

[1] we should note that this is not the only definition: in the literature, especially on general relativity, $g_{\alpha\beta}$ is often defined with opposite sign

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.22}$$

Then

$$u_\alpha u^\alpha = g_{\alpha\beta} u^\beta u^\alpha = \gamma^2 - \gamma^2\beta^2 = c^2 \tag{2.27}$$

is an invariant.

Moreover since $d\tau$ is an invariant and x^α conforms to Lorentz transformation, then

$$u'^\alpha = \Lambda^\alpha_\beta u^\beta \tag{2.28}$$

$\Rightarrow u^\alpha$ satisfies the Principle of SR.
Define the 4-momentum of a particle:

$$P_\alpha \equiv m u^\alpha \tag{2.29}$$

$\Rightarrow P^0 = \gamma m c = E/c$, $P^i = p^i$; $E =$ total energy, $p^i =$ ordinary 3-momentum, $m =$ particle's rest mass. Then

$$P_\alpha P^\alpha = m^2 u_\alpha u^\alpha = m^2 c^2 = E/c^2 \tag{2.30}$$

is an invariant. The fundamental dynamical law for particle interactions in SR is that 4-momentum is conserved in any Lorentz frame.

Note that

$$P'^\alpha = \Lambda^\alpha_\beta P^\beta \tag{2.31}$$

also one has:

$$P_\alpha P^\alpha = g_{\alpha\beta} P^\beta P^\alpha = E^2/c^2 - p^2 \tag{2.32}$$

$$E^2/c^2 - p^2 = (mc)^2 \tag{2.33}$$

$$\Rightarrow \quad E = \sqrt{(pc)^2 + (mc^2)^2}.$$

The kinetic energy of a particle is $T = E - mc^2$:

$$T = \sqrt{(pc)^2 + (mc^2)^2} - mc^2 \tag{2.34}$$

2.3.1 Example 1: $n + n \longrightarrow n + n + n + \bar{n}$

Consider the reaction (one neutron at rest)

$$n + n \longrightarrow n + n + n + \bar{n}$$

What is the minimum required energy for the incoming n that will enable the reaction to proceed?
At threshold the four neutron are at rest in the lab frame, so that the 4-momentum conservation requires:

$$P_1^\alpha + P_2^\alpha = P_f^\alpha \tag{2.35}$$

$$\Rightarrow \quad (P_1^\alpha + P_2^\alpha)(P_{1\alpha} + P_{2\alpha}) = P_f^\alpha P_{f\alpha} = 16(m_n c)^2$$

$$P_1^\alpha P_{1\alpha} + 2P_1^\alpha P_{2\alpha} + P_2^\alpha P_{2\alpha} = 2(m_n c)^2 + 2P_1^\alpha P_{2\alpha}$$

$$\Rightarrow P_1^\alpha P_{2\alpha} = 7(m_n c)^2. \tag{2.36}$$

$P_1^\alpha P_{2\alpha} = g_{\alpha\beta} P_1^\alpha P_2^\beta = g_{00} P_1^0 P_2^0 = m_n c \frac{E}{c}$

$$E = 7 m_n c^2. \tag{2.37}$$

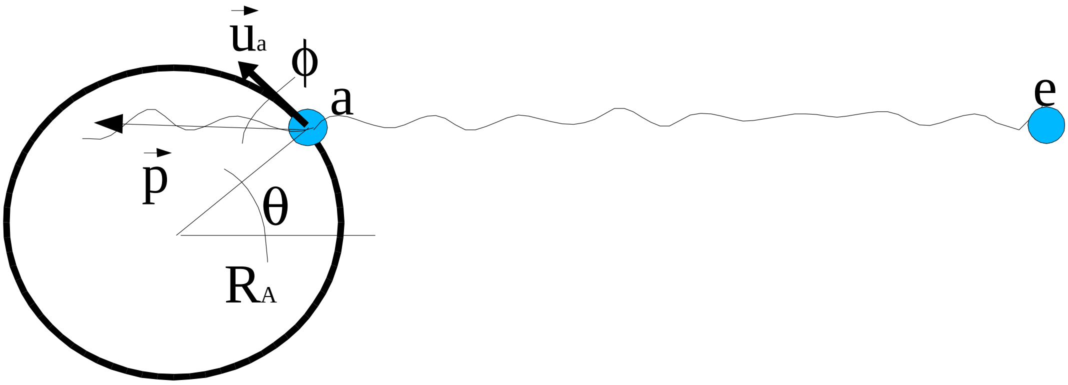

Figure 2.3: ????

2.3.2 Photon emission and absorption:

Let $u^\alpha_{e,a}$ = 4-velocity of emitter, absorber, respectively. $E_{e,a}$ = photon energy measured by emitter, absorber, respectively.
P^α = 4-momentum of photon.

Then look at

$$\begin{aligned} P_\alpha u^\alpha &= g_{\alpha\beta} P^\beta u^\alpha \\ &= P^0 u^0 - P^i u^i = cP^0 = E. \end{aligned}$$

1st term $u^0 = c$, 2nd term $u^i = 0$ in either emitter's or absorber's frame.

So $E = p_\alpha u^\alpha$ is the photon energy measured by an observer with 4-velocity u^α. The expression is the same in any frame, including accelerating frame! So:

$$E_e = P_\alpha u_e^\alpha \text{ and, } E_a = P_\alpha u_a^\alpha$$

Example: "Absorber" is rotating with angular velocity Ω on a circle of radius R_A. Emitter is stationary – Let's find E_a/E_e
In emitter's frame: $c^2 d\tau = g_{\alpha\beta} dx^\alpha dx^\beta$, the emitter is stationary so $u_e^\alpha = (c, 0, 0, 0)$.
In absorber frame:

$$\begin{aligned} c^2 (d\tau)^2 &= g_{\alpha\beta} dx^\alpha dx^\beta \\ &= c^2 dt^2 - v^2 dt^2 = c^2 dt^2 - R_A^2 d\phi^2 \\ d\tau^2 &= dt^2 - \frac{R_A^2}{c^2} d\phi^2 \end{aligned} \tag{2.38}$$

From Eq.????? we have:

$$\begin{aligned} \frac{E_a}{E_e} &= \frac{P_\alpha u_a^\alpha}{P_\alpha u_e^\alpha} = \frac{P_0 u_a^0 - \overrightarrow{p}\,\overrightarrow{u_a^i}}{P_0 c} \\ &= \frac{P_0 u_\alpha^0 - |\overrightarrow{p}||\overrightarrow{u_a^i}| \cos\theta}{P_0 c} \end{aligned} \tag{2.39}$$

SI	G
$\vec{\nabla}.\vec{D} = \rho$	$\vec{\nabla}.\vec{D} = 4\pi\rho$
$\vec{\nabla} \times \vec{H} - \partial_t \vec{D} = \vec{J}$	$\vec{\nabla} \times \vec{H} - \frac{1}{c}\partial_t \vec{D} = \frac{4\pi}{c}\vec{J}$
$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0$	$\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0$
$\vec{\nabla}\vec{B} = 0$	$\vec{\nabla}\vec{B} = 0$
$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = 0$	$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$

(2.44)

Table 2.1: Maxwell's equations in CGS and SI units.

But $\cos\theta = \sin\phi$, and for photons $P_\alpha P^\alpha = (P^0)^2 - |\vec{P}|^2 = 0 \Rightarrow |\vec{P}| = P^0$. Thus

$$\frac{E_a}{E_e} = \frac{u_a^0 - |\vec{u}_a|\sin\phi}{c}. \tag{2.40}$$

But,

$$|\vec{u}_a| = R_A\frac{d\phi}{d\tau} = \frac{R_A\Omega}{\sqrt{1-(R_A\Omega/c)^2}};\ u_a^0 = \frac{c}{\sqrt{1-(R_A\Omega/c)^2}} \tag{2.41}$$

$$\frac{E_a}{E_e} = \frac{\lambda_e}{\lambda_a} \Rightarrow \frac{\lambda_e}{\lambda_a} = \frac{1-(R_A\Omega/c)\sin\phi}{\sqrt{1-(R_A\Omega/c)^2}} \tag{2.42}$$

Doppler shift ($\phi = 90°$):

$$\frac{\lambda_e}{\lambda_a} = \frac{1-(R_A\Omega/c)}{\sqrt{1-(R_A\Omega/c)^2}} = \sqrt{\frac{1-(R_A\Omega/c)}{1+(R_A\Omega/c)}} \tag{2.43}$$

2.4 Covariance of Electrodynamics

2.4.1 CGS versus SI units

We wish to proceed in keeping with Jackson's notation, which involves switching from SI units to Gaussian units. Conversions:
$\frac{\vec{E}^G}{\sqrt{4\pi\epsilon_0}} = \vec{E}^{SI}$; $\sqrt{\frac{\epsilon_0}{4\pi}}\vec{D}^G = \vec{D}^{SI}$; $\sqrt{4\pi\epsilon_0}\rho^G(\vec{J}^G, q^G) = \rho^{SI}(\vec{J}^{SI}, q^{SI})$; $\sqrt{\frac{\mu_0}{4\pi}}\vec{B}^G = \vec{B}^{SI}$; $\frac{\vec{H}^G}{\sqrt{4\pi\mu_0}} = \vec{H}^{SI}$; $\epsilon_0\epsilon^G = \epsilon^{SI}$; $\mu_0\mu^G = \mu^{SI}$; $c = (\mu_0\epsilon_0)^{-1/2}$.

As one check, look at the Lorentz force:

$$\begin{aligned}
\vec{F}^G &= q^G(\vec{E}^G + \frac{1}{c}\vec{v} \times \vec{B}^G) \\
\Rightarrow \vec{F}^{SI} &= \frac{q^{SI}}{\sqrt{4\pi\epsilon_0}}\left[\sqrt{4\pi\epsilon_0}\vec{E}^{SI} + \sqrt{\mu_0\epsilon_0}\vec{v} \times \sqrt{\frac{4\pi}{\mu_0}}\vec{B}^{SI}\right] \\
&= q^{SI}(\vec{E}^{SI} + \vec{v} \times \vec{B}^{SI}).
\end{aligned}$$

The conversion from "Maxwell G" to "Maxwell SI" works the same way. So we do have a prescription to go from Gaussian results to SI results and vice versa.

2.4.2 Current density as a 4-vector

Consider a system of particles with positions $\overrightarrow{x}_n(t)$ and charges q_n. The current and charge densities are:

$$\begin{aligned}\overrightarrow{J}(\overrightarrow{x},t) &= \sum_n q_n \delta^3(\overrightarrow{x} - \overrightarrow{x}_n(t))\dot{\overrightarrow{x_n}}(t),\\ \rho(\overrightarrow{x},t) &= \sum_n q_n \delta^3(\overrightarrow{x} - \overrightarrow{x}_n(t))\end{aligned}$$

Note that for any smooth function $f(\overrightarrow{x})$, δ^3 acts as:

$$\int_{-\infty}^{\infty} f(\overrightarrow{x})\delta^3(\overrightarrow{x} - \overrightarrow{y}) = f(\overrightarrow{y}) \tag{2.45}$$

if we define $J^0 \equiv c\rho$ and $J^i(\overrightarrow{x}) = \sum_n q_n \delta^3(x^i - x^i_n(t))d_t x^i_n(t)$, then using δ^4 function we can write:

$$J^\alpha(x) = \int \sum_n q_n \delta^4(x^\alpha - x^\alpha_n(t))dx^0\, \frac{dx^\alpha_n(t)}{dt} \tag{2.46}$$

J^α is a function of $x^\alpha \rightarrow$ it is a Lorentz invariant; J^α is a 4-vector. $J^\alpha \equiv (c\rho, \overrightarrow{J})$. Also note $J^\alpha \equiv \rho u^\alpha$

2.4.3 Equation of charge continuity

$$\begin{aligned}\overrightarrow{\nabla}.\overrightarrow{J}(\overrightarrow{x},t) &= \sum_n q_n \frac{\partial}{\partial x^i}\delta^3(\overrightarrow{x} - \overrightarrow{x}_n(t))\frac{dx^i_n(t)}{dt}\\ &= -\sum_n q_n \frac{\partial}{\partial x^i_n}\delta^3(\overrightarrow{x} - \overrightarrow{x}_n(t))\frac{dx^i_n(t)}{dt}\\ &= -\sum_n q_n \partial_t \delta^3(\overrightarrow{x} - \overrightarrow{x}_n(t))\\ &= -\partial_t \rho(\overrightarrow{x},t) = -\partial_0[c\rho(\overrightarrow{x},t)].\end{aligned} \tag{2.47}$$

So the equation of charge continuity writes as $\partial^\alpha J_\alpha = 0$

2.4.4 4-gradient

In the previous slide we use the operator ∂_α. It is defined as

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha}. \tag{2.48}$$

This operator transforms as:

$$\partial'_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu_\mu \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu_\mu \frac{\partial_\nu}{.} \tag{2.49}$$

Note that $\partial_\mu = (\partial_0, \overrightarrow{\nabla})$.
We can "upper" the indice and define

$$\partial^\mu = g^{\mu\nu}\partial_\nu = (\partial_0, -\overrightarrow{\nabla}) \tag{2.50}$$

Finally we can define the d'Alembertian: $\Box \equiv \partial^\alpha \partial_\alpha$.

2.4.5 Potential as a 4-vector

$$A^\alpha \equiv (\phi, \overrightarrow{A}) \tag{2.51}$$

Lorentz Gauge then write $\partial_\alpha A^\alpha = 0$. We also have

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha, \tag{2.52}$$

or in SI units

$$\Box A^\alpha = \mu_0 J^\alpha, \qquad \text{[SI]} \tag{2.53}$$

this is the equation we wrote when deriving the field induced by a charge moving at constant velocity.

2.5 Covariant form of Maxwell equations

Returning to Maxwell Equation
Define the matrix $F^{\alpha\beta} \equiv \partial^\alpha A^\beta - \partial^\beta A^\alpha = g^{\alpha\delta}\partial_\delta A^\beta - g^{\beta\delta}\partial_\delta A^\alpha$:

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.54}$$

Look at:
$\partial_\alpha F^{\alpha\beta} = \partial_0 F^{0\beta} + \partial_1 F^{1\beta} + \partial_2 F^{2\beta} + \partial_3 F^{3\beta}$:

$$\begin{aligned} \partial_\alpha F^{\alpha 0} &= \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} \\ &= \partial_i E^i = \overrightarrow{\nabla}.\overrightarrow{E} = 4\pi\rho = \frac{4\pi}{c} J^0. \end{aligned} \tag{2.55}$$

Similarly,

$$\begin{aligned}
\partial_\alpha F^{\alpha 1} &= \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} \\
&= \frac{1}{c}\partial_t(-E_x) + \partial_x(0) + \partial_y(-B_z) - \partial_z(B_y) = -\frac{1}{c}\partial_t(E_x) + [\overrightarrow{\nabla} \times \overrightarrow{B}]_x \\
&= [\overrightarrow{\nabla} \times \overrightarrow{B}]_x - \frac{1}{c}\partial_t E_x = \frac{4\pi}{c} J^1 \qquad (2.56)
\end{aligned}$$

...The same for component 2, and 3. So we cast these equations under:

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta, \qquad (2.57)$$

This corresponds to the inhomogeneous Maxwell's equations. In SI units $F^{\alpha\beta}$ is obtained by replacing $\overrightarrow{E}$ by $\overrightarrow{E}/c$.

How do we get the homogenous Maxwell's equations?
Let's introduce the Levi-Civita (rank 4) tensor as:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if} \alpha,\beta,\gamma,\delta \text{ are } \textbf{even} \text{ permutation of 0,1,2,3} \\ -1 & \text{if} \alpha,\beta,\gamma,\delta \text{ are } \textbf{odd} \text{ permutation of 0,1,2,3} \\ 0 & \text{otherwise} \end{cases}, \qquad (2.58)$$

and consider the quantity $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\delta\gamma}$; with $F_{\delta\gamma} = g_{\gamma\alpha} g_{\delta\beta} F^{\alpha\beta}$.

$$F_{\gamma\delta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \qquad (2.59)$$

$F_{\gamma\delta}$ is obtained from $F^{\alpha\beta}$ by doing the change $\overrightarrow{E} \to -\overrightarrow{E}$. Now consider the component "0" of the 4-vector $\epsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta}$:

$$\begin{aligned}
\epsilon^{0\beta\gamma\delta}\partial_\beta F_{\gamma\delta} &= \epsilon^{0123}\partial_1 F_{23} + \epsilon^{0132}\partial_1 F_{32} + \\
&\quad \epsilon^{0213}\partial_2 F_{13} + \epsilon^{0231}\partial_2 F_{31} + \epsilon^{0312}\partial_3 F_{12} + \epsilon^{0321}\partial_3 F_{21} \\
&= \partial_1 F_{23} - \partial_1 F_{32} - \partial_2 F_{13} + \partial_2 F_{31} + \partial_3 F_{12} - \partial_3 F_{21} \\
&= \partial_x(-B_x) - \partial_x(B_x) - \partial_y(B_y) + \partial_y(-B_y) + \partial_z(-B_z) - \partial_z(B_z) \\
&= -2\overrightarrow{\nabla}.\overrightarrow{B}(=0) \qquad (2.60)
\end{aligned}$$

now let's compute the "1" component

$$\begin{aligned}
\epsilon^{1\beta\gamma\delta}\partial_\beta F_{\gamma\delta} &= \epsilon^{1023}\partial_0 F_{23} + \epsilon^{1032}\partial_0 F_{32} + \epsilon^{1302}\partial_3 F_{02} + \epsilon^{1320}\partial_3 F_{20} \\
&\quad + \epsilon^{1203}\partial_2 F_{03} + \epsilon^{1230}\partial_2 F_{30} \\
&= -\partial_0 F_{23} + \partial_0 F_{32} - \partial_3 F_{02} + \partial_3 F_{20} + \partial_2 F_{03} - \partial_2 F_{30} \\
&= 2(D_0 F_{32} + \partial_2 F_{03} + \partial_3 F_{20}) \\
&= 2\left(\frac{1}{c}\partial_t B_x - \partial_z E_y + \partial_y E_z\right) \\
&= 2\left[(\overrightarrow{\nabla} \times \overrightarrow{E})_x + \frac{1}{c}\partial_t B_x\right](=0) \qquad (2.61)
\end{aligned}$$

It is common to define the **dual tensor** of $F_{\gamma\delta}$ as $\mathcal{F}^{\alpha\beta} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$. With such a definition the homogeneous Maxwell equations can be casted in the expression:

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0. \tag{2.62}$$

Note: $\mathcal{F}_{\alpha\beta} = F_{\alpha\beta}(\overrightarrow{E} \to \overrightarrow{B}, \overrightarrow{B} \to -\overrightarrow{E})$.

To include $\overrightarrow{H}$ and $\overrightarrow{D}$, one defines the tensor $G^{\alpha\beta} = F^{\alpha\beta}(\overrightarrow{E} \to \overrightarrow{D}, \overrightarrow{B} \to \overrightarrow{H})$, and then Maxwell's equations write:

$$\partial_\alpha G^{\alpha\beta} = \frac{4\pi}{c}J^\beta, \text{ and } \partial_\alpha \mathcal{F}^{\alpha\beta} = 0. \tag{2.63}$$

Due to covariance of $F^{\alpha\beta}$, it is a tensor, the calculation of em field from one Lorentz frame to another is made easy. Just consider:

$$F'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x'^\beta}{\partial x^\delta}F^{\gamma\delta}, \tag{2.64}$$

or in matrix notation

$$F' = \tilde{\Lambda}F\Lambda = \Lambda F \Lambda \tag{2.65}$$

Example: Consider a boost along the $\hat{z}$-axis, then

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{2.66}$$

Plug the F matrix associated to $F^{\gamma\delta}$ in Eq. 2.65, the matrix multiplication yields:

$$F'^{\gamma\delta} = \begin{pmatrix} 0 & \gamma(E_x - \beta B_y) & \gamma(E_y + \beta B_x) & E_z \\ -\gamma(E_x - \beta B_y) & 0 & B_z & -\gamma(B_y - \beta E_x) \\ -\gamma(E_y + \beta B_x) & -B_z & 0 & \gamma(B_x + \beta E_y) \\ -E_z & \gamma(B_y - \beta E_x) & -\gamma(B_x + \beta E_y) & 0 \end{pmatrix} \tag{2.67}$$

by inspection we obtain the same equation as [JDJ, Eq. (11.148)].

2.6 Fundamental Invariant of the electromagnetic field tensor

This section i adapted from a paper by Muñoz [2]. The scalar quantities

$$F^{\mu\nu}F_{\mu\nu} = 2(E^2 - B^2), \text{ and } F^{\mu\nu}\mathcal{F}_{\mu\nu} = 4\overrightarrow{E}.\overrightarrow{B}, \tag{2.68}$$

are invariants. Usually one redefines these two invariants as:

$$\mathcal{I}_1 \equiv -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(B^2 - E^2), \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4}F^{\mu\nu}\mathcal{F}_{\mu\nu} = -\overrightarrow{E}.\overrightarrow{B}. \tag{2.69}$$

[2] G. Muñoz, Am. J. Phys. **65** (5), May 1997

Note that these invariants may be rewritten as:

$$\mathcal{I}_1 \equiv -\frac{1}{4}\text{tr}(F^2) \text{ and } \mathcal{I}_2 \equiv -\frac{1}{4}\text{tr}(F\mathcal{F}), \tag{2.70}$$

where $F \equiv F_\mu^\nu = F^{\mu\alpha} g_{\alpha\nu}$ and $\mathcal{F} \equiv \mathcal{F}_\mu^\nu = \mathcal{F}^{\mu\alpha} g_{\alpha\nu}$.
Finally note the identities:

$$F\mathcal{F} = \mathcal{F}F = -\mathcal{I}_2 I, \text{ and } F^2 - \mathcal{F}^2 = -2\mathcal{I}_1 I \tag{2.71}$$

Eigenvalues of F (for later!):
Look for eigenvalue λ associated to eigenvector Ψ:

$$F\Psi = \lambda\Psi \Rightarrow \mathcal{F}F\Psi = \lambda\mathcal{F}\Psi \Rightarrow \mathcal{F}\Psi = -\frac{\mathcal{I}_2}{\lambda}\Psi. \tag{2.72}$$

$$(F^2 - \mathcal{F}^2)\Psi = -2I\mathcal{I}_1\Psi = [\lambda^2 - (\mathcal{I}_2/\lambda)^2]\Psi, \tag{2.73}$$

So characteristic polynomial is: $\lambda^4 + 2\mathcal{I}_1\lambda^2 - \mathcal{I}_2^2 = 0$.
Solutions are:

$$\lambda_\pm = \sqrt{\sqrt{\mathcal{I}_1^2 + \mathcal{I}_2^2} \pm \mathcal{I}_1} \tag{2.74}$$

$\lambda_1 = -\lambda_2 = \lambda_-$, $\lambda_3 = -\lambda_4 = i\lambda_+$.

2.7 Equation of motion

The equation describing the dynamics of a relativistics particle of mass m and charge q moving under the influence of em field $F_{\alpha\beta}$ is:

$$\frac{du^\alpha}{d\tau} = \frac{q}{mc} F_\beta^\alpha u^\beta. \tag{2.75}$$

with $u^\alpha = (\gamma c, \gamma\overrightarrow{v})$. Note that this is equivalent to introducing the "quadri-force"

$$f^\mu = F^{\mu\nu} u_\nu. \tag{2.76}$$

Chapter 3

Particle Dynamics Electromagnetic Fields

3.1 Lagrangian & Hamiltonian formulation

Classical mechanics, Given $\overrightarrow{x}(x^1, x^2, x^3)$ in K and $\overrightarrow{\dot{x}}$ system is characterized by a Lagrangian: $\mathcal{L}(x^i, \dot{x}^i, t)$. The action

$$\mathcal{A} \equiv \int_{t_1}^{t_2} \mathcal{L}(x^i, \dot{x}^i, t)dt \tag{3.1}$$

is a functional of $\overrightarrow{x}(t)$, $\forall \overrightarrow{x}(t)$ defined for $t \in [t_1, t_2]$.
The least action principle states that $\mathcal{A}$ is a stationary function for any small variation $\delta\overrightarrow{x}(t)$ verifying $\delta\overrightarrow{x}(t_1) = \delta\overrightarrow{x}(t_2) = 0$.
The equation of motion then follow from Euler-Lagrange equations:

$$\begin{aligned} P^i &= \frac{\partial \mathcal{L}}{\partial \dot{x}^i} \\ \frac{dP^i}{dt} &= \frac{\partial \mathcal{L}}{\partial x^i} \end{aligned}$$

3.1.1 Case of a free relativistic particle

Equation of motion must be referential-invariant $\Rightarrow$ the least action princple $\delta\mathcal{A} = 0$ must have the same form in different referential $\Rightarrow$ $\mathcal{A}$ must be a scalar invariant.
$\mathcal{A}$ is a sum of infinitesimal elements along a universe line $x^i(t)$
$\Rightarrow$ $\mathcal{L}dt$ associated to a small displacement must be a scalar invariant.
$\Rightarrow \mathcal{L}dt = \alpha ds = \alpha\sqrt{1 - \frac{V^2}{c^2}}dt$ also,

$$\lim_{V \ll c} \mathcal{L} = \frac{1}{2}mV^2 + \text{const} = \alpha\left(1 - \frac{V^2}{c^2} + \mathcal{O}((V/c)^4)\right) \tag{3.2}$$

so $\alpha = -mc$, and the relativistic Lagrangian of a free particle is

$$\mathcal{L}_{free} = -mc^2\sqrt{1 - \frac{V^2}{c^2}} = -\frac{mc}{\gamma}\sqrt{u^\alpha u_\alpha}, \tag{3.3}$$

where $u^\alpha = (\gamma c, \gamma \overrightarrow{v})$ is the four-velocity. One can check:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\overrightarrow{x}}} = \frac{d}{dt}(m\gamma \dot{\overrightarrow{x}}) = 0$$

3.1.2 Lagrangian of a relativistic particle in e.m. field

The Lagrangian now takes the form $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$, where $\mathcal{L}_{int}$ is the interaction potential. In the nonrelativistic limit $\mathcal{L}_{int}^{NR} = -e\Phi = -eA^0$ so let's try

$$\begin{aligned}
\mathcal{L}_{int} &= -\frac{e}{\gamma c} u_\alpha A^\alpha \\
&= -\frac{e}{\gamma c} g_{\alpha\beta} u^\beta A^\alpha \\
&= -\frac{e}{\gamma c}\left(\gamma c \Phi - \gamma \overrightarrow{V}.\overrightarrow{A}\right) \\
\mathcal{L}_{int} &= -e\Phi + e\overrightarrow{\beta}\overrightarrow{A}.
\end{aligned} \tag{3.4}$$

The total Lagrangian is

$$\mathcal{L} = -mc^2\sqrt{1-\frac{V^2}{c^2}} + \frac{e}{c}\overrightarrow{V}.\overrightarrow{A}(\overrightarrow{x}) - e\Phi(\overrightarrow{x}). \tag{3.5}$$

Let's check this gives the equation of motion, by calculating

$$\frac{\partial \mathcal{L}}{\partial \overrightarrow{x}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\overrightarrow{x}}} = 0 \tag{3.6}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \overrightarrow{V}} &= \frac{d}{dt}\left(\gamma m \overrightarrow{V} + \frac{e}{c}\overrightarrow{A}\right) \\
&= \frac{d(\gamma m \overrightarrow{V})}{dt} + \frac{e}{c}\left(\frac{\partial \overrightarrow{A}}{\partial t} + \frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i}\overrightarrow{A}\right) = \frac{d(\gamma m \overrightarrow{V})}{dt} + \frac{e}{c}\left(\frac{\partial \overrightarrow{A}}{\partial t} + (\overrightarrow{V}.\overrightarrow{\nabla})\overrightarrow{A}\right)
\end{aligned}$$

$$\frac{\partial}{\partial \overrightarrow{x}}\mathcal{L} = \frac{e}{c}\overrightarrow{\nabla}.(\overrightarrow{V}.\overrightarrow{A}) - e\overrightarrow{\nabla}\Phi = \frac{e}{c}\left[(\overrightarrow{V}.\overrightarrow{\nabla})\overrightarrow{A} + \overrightarrow{V}\times(\overrightarrow{\nabla}\times\overrightarrow{A})\right] - e\overrightarrow{\nabla}\Phi$$

With $\overrightarrow{B} = \overrightarrow{\nabla}\times\overrightarrow{A}$, one finally has:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \overrightarrow{V}} - \frac{\partial \mathcal{L}}{\partial \overrightarrow{x}} = \frac{d}{dt}(\gamma m \overrightarrow{V}) + \frac{e}{c}\frac{\partial \overrightarrow{A}}{\partial t} + e\overrightarrow{\nabla}\Phi - \frac{e}{c}\overrightarrow{V}\times\overrightarrow{B} = 0$$

which gives the Lorentz force equation (in Gauss units!):

$$\frac{d}{dt}(\gamma m \overrightarrow{V}) = e\overrightarrow{E} + \frac{e}{c}(\overrightarrow{V}\times\overrightarrow{B}) \tag{3.7}$$

Let's check the Lagrangian verifies the "least action principle"
The total Lagrangian can be written

$$\mathcal{L} = -\frac{mc}{\gamma}\sqrt{u^\alpha u_\alpha} - \frac{q}{\gamma c}u_\alpha A^\alpha(x^\beta). \tag{3.8}$$

define $\tilde{\mathcal{L}} \equiv \gamma\mathcal{L}$. The action is $\mathcal{A} = \int_{\tau_1}^{\tau_2} d\tau\tilde{\mathcal{L}}$.

least action principle $\delta\mathcal{A} = 0$.

$$\delta\mathcal{A} = \delta\left[\int_{\tau_1}^{\tau_2} d\tau\tilde{\mathcal{L}}\right] = \int_{\tau_1}^{\tau_2} d\tau\delta\tilde{\mathcal{L}} \tag{3.9}$$

$$-\delta\tilde{\mathcal{L}} = mc\frac{1}{2}\frac{1}{\sqrt{u^\alpha u_\alpha}}\left[\frac{\partial(u^\alpha u_\alpha)}{\partial u^\beta}\right]\delta u^\beta + qA_\alpha\delta u^\alpha + qu^\alpha\frac{\partial A_\alpha}{\partial x^\beta}\delta x^\beta. \tag{3.10}$$

One has $\delta u^\alpha = \delta(\frac{\partial x^\alpha}{\partial\tau}) = \frac{\partial}{\partial\tau}(\delta x^\alpha)$, and

$$\frac{\partial(u^\alpha u_\alpha)}{\partial u^\beta} = g_{\alpha\gamma}\frac{\partial(u^\alpha u^\gamma)}{\partial u^\beta} = g_{\alpha\gamma}\left(\delta^\alpha_\beta u^\gamma + \delta^\gamma_\beta u^\alpha\right) = 2u_\beta \tag{3.11}$$

using $\delta\frac{dx^\alpha}{d\tau} = \frac{d(\delta x^\alpha)}{d\tau}$ (commutation of δ and d operators), one gets:

$$-c\delta\tilde{\mathcal{L}} = (mcu_\beta + qA_\beta)\frac{d(\delta x^\beta)}{d\tau} + qu^\alpha\partial_\beta A_\alpha\delta x^\beta. \tag{3.12}$$

Evaluating the integral by part and noting that $\delta x^\beta(\tau_1) = \delta x^\beta(\tau_2) = 0$ gives:

$$\delta\mathcal{A} = -\int_{\tau_1}^{\tau_2} d\tau\left[-mc\frac{du_\beta}{d\tau} - q(\partial_\alpha A_\beta)u^\alpha + qu^\alpha\partial_\beta A_\alpha\right]\delta x^\beta, \tag{3.13}$$

and $\delta\mathcal{A} = 0 \Rightarrow [...] = 0$ (linear independence argument) gives the equation of motion $m\frac{d}{d\tau}u_\beta = \frac{q}{c}F_{\alpha\beta}u^\alpha$

The canonical momentum $\overrightarrow{P}$ conjugate to $\overrightarrow{x}$ is, by definition,

$$\begin{aligned}\overrightarrow{P} &= \frac{\partial\mathcal{L}}{\partial\overrightarrow{V}} = \gamma m\overrightarrow{\dot{x}} + \frac{e}{c}\overrightarrow{A}\\ \overrightarrow{P} &= \overrightarrow{p} + \frac{e}{c}\overrightarrow{A}\end{aligned} \tag{3.14}$$

and the hamiltonian is defined as:

$$\mathcal{H} \equiv \overrightarrow{P}.\overrightarrow{V} - \mathcal{L} \tag{3.15}$$

3.2 Relativistic Hamiltonian

We use $\overrightarrow{P} = \gamma m \overrightarrow{v} + \frac{e}{c}\overrightarrow{A}$ and calculate $\mathcal{H}$ then express $\mathcal{H}$ only as a function of $\overrightarrow{P}$ and $\overrightarrow{x}$. On can do the algebra (namely explicit $\overrightarrow{v}$ as a function $\overrightarrow{P}$ and replace in the expression of $\mathcal{H}$.

$$\begin{aligned}\mathcal{H} &= \overrightarrow{v}.\left(\gamma m \overrightarrow{v} + \frac{e}{c}\overrightarrow{A}\right) + \gamma mc^2\frac{1}{\gamma} + e\Phi - \frac{e}{c}\overrightarrow{A}\,\overrightarrow{v} &(3.16)\\ &= \gamma m v^2 + \frac{mc^2}{\gamma} + e\Phi = \gamma mc^2 + e\Phi. &(3.17)\end{aligned}$$

We note that the relation between $\overrightarrow{P} - \frac{e}{c}\overrightarrow{A}$ and $\mathcal{H} - e\Phi$ is the same as between $\mathcal{H}$ and $\overrightarrow{p}$ for the case of zero-field so we have:

$$(\mathcal{H} - e\Phi)^2 = \left(\overrightarrow{P} - \frac{e}{c}\overrightarrow{A}\right)^2 c^2 + m^2c^4 \tag{3.18}$$

So finally,

$$\mathcal{H} = \sqrt{\left(\overrightarrow{P}c - e\overrightarrow{A}\right)^2 + m^2c^4} + e\Phi \tag{3.19}$$

3.3 Motion of a particle in a constant uniform E-field

Let $\mathcal{E}$ be the total energy: $\mathcal{E} = \sqrt{(pc)^2 + (mc^2)^2} = \gamma mc^2 \Rightarrow \gamma = \frac{\mathcal{E}}{mc^2}$. Thus,

$$\overrightarrow{p} = \gamma m \overrightarrow{v} = \frac{\mathcal{E}}{c^2}\overrightarrow{v} \Rightarrow \overrightarrow{v} = \frac{c^2}{\mathcal{E}}\overrightarrow{P}. \tag{3.20}$$

Let's consider the case of a particle of charge q interacting with the field $\overrightarrow{E} = E\hat{x}$, and with initial conditions $p(t=0) = p_0\hat{y}$. Lorentz Force gives:

$$\dot{p}_x = qE, \text{ and, } \dot{p}_y = 0 \tag{3.21}$$

which yields:

$$p_x = qEt, \text{ and, } p_y = p_0 \tag{3.22}$$

and $p^2 = (qEt)^2 + p_0^2$.

So the total energy at time t is:

$$\mathcal{E}^2(t) = c^2\left[(qEt)^2 + p_0^2\right] + m^2c^4 = (cqEt)^2 + \mathcal{E}_0 \tag{3.23}$$

where $\mathcal{E}_0 \equiv \mathcal{E}(t=0)$. The velocity is:

$$v_x = \frac{dx}{dt} = c\frac{cqEt}{\sqrt{(cqEt)^2 + \mathcal{E}_0^2}} \tag{3.24}$$

note that $\lim_{t\to\infty} = c$. Performing a time integration yields:

$$x(t) = \frac{1}{qE}\sqrt{(cqEt)^2 + \mathcal{E}_0^2}. \tag{3.25}$$

For y-axis we have:

$$\frac{dy}{dt} = \frac{c^2 p_0}{\sqrt{(cqEt)^2 + \mathcal{E}_0^2}}, \tag{3.26}$$

and $\lim_{t\to\infty} \frac{dy}{dt} = 0$.

A time integration gives:

$$y = \frac{p_0 c}{qE}\sinh^{-1}\left(\frac{cqEt}{\mathcal{E}_0}\right). \tag{3.27}$$

remember: $\int_0^{\xi} \frac{d\tilde{\xi}}{\tilde{\xi}^2+1} = \sinh^{-1}(\xi)$. Expliciting t as a function of y:

$$cqEt = \sinh\left(\frac{qEy}{p_0 c}\right), \tag{3.28}$$

and substituting in x, we have the trajectory equation in (x, y) plane:

$$\begin{aligned} x &= \frac{\mathcal{E}_0}{qE}\sqrt{\sinh^2\left(\frac{qEy}{p_0 c}\right) + 1} \\ &= \frac{\mathcal{E}_0}{qE}\cosh\left(\frac{qEy}{p_0 c}\right). \end{aligned} \tag{3.29}$$

The nonrelativistic limit ($v \ll c$) is given by setting $\mathcal{E}_0 = mc^2$, $p_0 = mv_0$:

$$x = \frac{mc^2}{qE}\cosh\left(\frac{qEy}{mv_0 c}\right) \simeq \frac{qE}{2mv_0^2}y^2 + \text{const.} \tag{3.30}$$

the familiar parabola. The Taylor's expansion $\cosh(x) = 1 + x^2/2! + \mathcal{O}(x^4)$ was used.

3.4 Motion of a particle in a constant uniform B-field

Lorentz force gives (CGS!): $\dot{\overrightarrow{p}} = \frac{q}{c}\overrightarrow{v} \times \overrightarrow{B}$; $\overrightarrow{p} = \frac{\mathcal{E}}{c^2}\overrightarrow{v} \Rightarrow \dot{\overrightarrow{v}} = \frac{cq}{\mathcal{E}}\overrightarrow{v} \times \overrightarrow{B}$.
$\overrightarrow{B}$ changes the direction of $\overrightarrow{v}$ but not its magnitude so W, and γ are constants. Consider for simplicity $\overrightarrow{B} = B\hat{z}$, then

$$\overrightarrow{v} \times \overrightarrow{B} = v_y B\hat{x} - v_x B\hat{y},$$

which gives

$$\begin{aligned} \dot{v}_x &= \frac{cqB}{\mathcal{E}}v_y, \\ \dot{v}_y &= -\frac{cqB}{\mathcal{E}}v_x, \\ \dot{v}_z &= 0. \end{aligned} \tag{3.31}$$

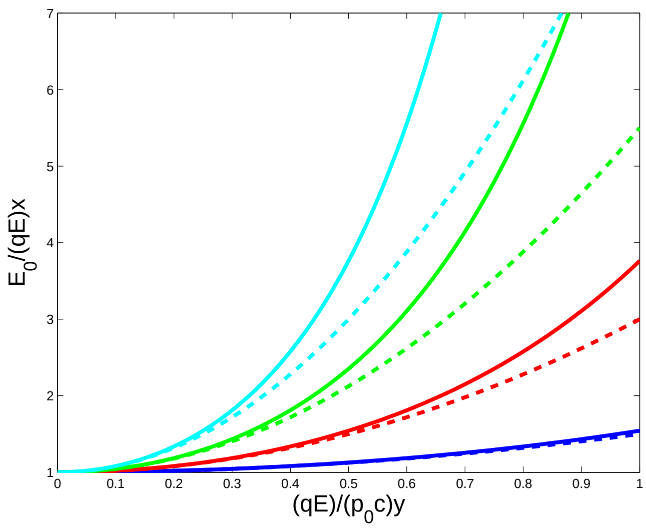

Figure 3.1: Trajectories (in normalized coordinate) in uniform constant E-field: $\hat{x} = \cosh(\kappa\hat{y})$, with $\kappa = 1, 2, 3, 4$. dashed are corresponding parabolic approximation $\hat{x} = 1 + \frac{1}{2}(\kappa\hat{y})^2$.

So we have to solve a system of coupled ODE of the form:

$$\dot{v}_x = \omega v_y,\ \dot{v}_y = -\omega v_x,\ \dot{v}_z \ = \ 0. \tag{3.32}$$

where $\omega \equiv \frac{cqB}{\mathcal{E}}$. Let's cast the transverse equation of motions:

$$\frac{d}{dt}(v_x + iv_y) = -i\omega(v_x + iv_y), \tag{3.33}$$

the solution is of the form $v_x + iv_y = v_\perp e^{-i(\omega t+\alpha)}$. Let $v_\parallel = v_z$. With these notations we can write:

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} v_\perp \cos(\omega t + \alpha) \\ -v_\perp \sin(\omega t + \alpha) \\ v_\parallel \end{pmatrix};\ \text{with } v_\perp = \sqrt{v_x^2 + v_y^2},\ \text{and,} \tag{3.34}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + R\sin(\omega t + \alpha) \\ y_0 + R\cos(\omega t + \alpha) \\ z_0 + v_\parallel t \end{pmatrix};\ \text{with } R \equiv \frac{v_\perp}{\omega} = \frac{v_\parallel \mathcal{E}}{cqB}. \tag{3.35}$$

So the trajectory is a helix whose axis is along $\hat{z}$, with radius R. The frequency ω is the rotation frequency of the trajectory when projected in a plan orthogonal to the helix axis.

R is called the gyroradius, $R = \frac{v_\parallel \mathcal{E}}{cqB} = \frac{p_\perp c}{qB}$. $\omega = \frac{qcB}{\mathcal{E}} = \frac{qcB}{\gamma mc^2} \Rightarrow \frac{qB}{\gamma mc}$ is the gyrofrequency $(\frac{v_\perp}{R})$.

The gyroradius and gyrofrequency arise in all calculations involving particle motion in magnetic fields. Note that in SI units:

$$\omega = \frac{qB}{\gamma m},\ \text{and } R = \frac{\gamma m v_\perp}{qB}. \tag{3.36}$$

3.5 Motion of a particle in a constant uniform magnetic and electric field

We now consider the case where both $\vec{E}$ and $\vec{B}$ fields are present in some arbitrary orientation. The idea is to directly solve the equation of motion

$$\frac{du^\alpha}{d\tau} = \frac{q}{mc} F^\alpha_\beta u^\beta; \tag{3.37}$$

the treatment follows Muñoz's paper [1]. Let $\theta \equiv \frac{q\tau}{mc}$, and rewrite the equation of motion in matrix form:

$$\frac{dU}{d\theta} = Fu \quad \text{with solution } u = e^{\theta F} u(0), \tag{3.38}$$

where,

$$e^{\theta F} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} F^n. \tag{3.39}$$

Now, recall the identity (see handout end of part II) $F^2 = \mathcal{F}^2 - 2\mathcal{I}_1 I$. Because of this, every power of F can be written as a linear combination of I, F, $\mathcal{F}$, and F^2, e.g.:

$$\begin{aligned}
F^3 &= FF^2 = F\mathcal{F}^2 - 2\mathcal{I}_1 F = -\mathcal{I}_2\mathcal{F} - 2\mathcal{I}_1 F; \\
F^4 &= -\mathcal{I}_2 F\mathcal{F} - 2\mathcal{I}_1 F^2 = \mathcal{I}_2^2 I - 2\mathcal{I}_1 F^2; \\
F^5 &= \mathcal{I}_2^2 F - 2\mathcal{I}_1 F^3 = (4\mathcal{I}_1^2 + \mathcal{I}_2^2)F + 2\mathcal{I}_1\mathcal{I}_2\mathcal{F}; \\
&\text{etc...}
\end{aligned} \tag{3.40}$$

This means,

$$e^{\theta F} = \alpha I + \beta F + \gamma \mathcal{F} + \delta F^2. \tag{3.41}$$

To find the constants α, β, γ, and δ, consider the following traces (note that the trace of odd power of F and $\mathcal{F}$ are zero:

$$\begin{aligned}
t_0 &\equiv \frac{1}{4}\mathrm{Tr}[e^{\theta F}] = \alpha - \mathcal{I}_1\delta, \\
t_1 &\equiv \frac{1}{4}\mathrm{Tr}[Fe^{\theta F}] = -\mathcal{I}_1\beta - \mathcal{I}_2\gamma, \\
t_2 &\equiv \frac{1}{4}\mathrm{Tr}[F^2 e^{\theta F}] = -\mathcal{I}_1\alpha + (2\mathcal{I}_1^2 + \mathcal{I}_2^2)\delta, \\
t_3 &\equiv \frac{1}{4}\mathrm{Tr}[F^3 e^{\theta F}] = 2(\mathcal{I}_1^2 + \mathcal{I}_2^2)\beta + \mathcal{I}_1\mathcal{I}_2\gamma.
\end{aligned}$$

[1] G. Muñoz, *Am. J. Phys.* **65**, 429 (1997)

Solving this system of equation for α, β, γ, and δ, yields:

$$\begin{aligned}
\alpha &= \tfrac{(2\mathcal{I}_1^2+\mathcal{I}_2^2)t_0+\mathcal{I}_1 t_2}{\mathcal{I}_1^2+\mathcal{I}_2^2}; \quad \beta = \frac{t_3+\mathcal{I}_1 t_1}{\mathcal{I}_1^2+\mathcal{I}_2^2}; \\
\gamma &= -\tfrac{(2\mathcal{I}_1^2+\mathcal{I}_2^2)t_1+\mathcal{I}_1 t_3}{\mathcal{I}_2(\mathcal{I}_1^2+\mathcal{I}_2^2)}; \quad \delta = \frac{t_2+\mathcal{I}_1 t_0}{\mathcal{I}_1^2+\mathcal{I}_2^2}.
\end{aligned}$$

The traces are found upon diagnonalization of $e^{\theta F} \to e^{\theta F'}$:

$$\mathrm{Tr}[e^{\theta F}] = \mathrm{Tr}[e^{\theta F'}] = \sum_{i=1}^{4} e^{\theta\lambda_i}, \tag{3.42}$$

where λ_i are the eigenvalues of F: $\lambda_1 = -\lambda_2 = \lambda_-$, and $\lambda_3 = -\lambda_4 = i\lambda_+$ where $\lambda_\pm = \sqrt{\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2} \pm \mathcal{I}_1}$.
Thus

$$\begin{aligned}
t_0 &= \frac{1}{4}\mathrm{Tr}[e^{\theta F}] = \frac{1}{2}[\cosh(\theta\lambda_-) + \cos(\theta\lambda_+)] \\
t_k &= \frac{1}{4}\mathrm{Tr}[F^k e^{\theta F}] = \frac{\partial^k t_0}{\partial\theta^k}
\end{aligned} \tag{3.43}$$

So

$$\begin{aligned}
t_1 &= \frac{1}{2}[\lambda_- \sinh(\theta\lambda_-) - \lambda_+ \sin(\theta\lambda_+)] \\
t_2 &= \frac{1}{2}[\lambda_-^2 \cosh(\theta\lambda_-) - \lambda_+^2 \cos(\theta\lambda_+)] \\
t_3 &= \frac{1}{2}[\lambda_-^3 \cosh(\theta\lambda_-) + \lambda_+^3 \sin(\theta\lambda_+)]
\end{aligned}$$

Substitute and simplify to finally obtain the values:

$$\begin{aligned}
\alpha &= \tfrac{\lambda_+^2 \cosh(\theta\lambda_-)+\lambda_-^2 \cos(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}; \quad \beta = \frac{\lambda_- \sinh(\theta\lambda_-) + \lambda_+ \sin(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}; \\
\gamma &= \tfrac{|\mathcal{I}_2|}{\mathcal{I}_2}\tfrac{\lambda_- \sin(\theta\lambda_+)-\lambda_+ \sinh(\theta\lambda_-)}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}; \quad \delta = \frac{\cosh(\theta\lambda_-) - \cos(\theta\lambda_+)}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}.
\end{aligned}$$

Substitute into the power expansion for $e^{\theta F}$ to find:

$$\begin{aligned}
u(\theta) &= \frac{1}{2\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}\Big[(\lambda_+^2 I + F^2)\cosh(\theta\lambda_-) + (\lambda_-^2 I - F^2)\cos(\theta\lambda_+) \\
&+ \left(\lambda_- F - \frac{|\mathcal{I}_2|}{\mathcal{I}_2}\lambda_+\mathcal{F}\right)\sinh(\theta\lambda_-) + \left(\lambda_+ F + \frac{|\mathcal{I}_2|}{\mathcal{I}_2}\lambda_-\mathcal{F}\right)\sin(\theta\lambda_+)\Big] u(0).
\end{aligned}$$

Note that $u(\theta) = \frac{2}{mc}\frac{dx}{d\theta}$, so integrate over $\theta \in [0,\theta)$ to get

$$\begin{aligned}
x(\tau) &= x(0) + \frac{mc}{q\mathcal{I}_2}\mathcal{F}u(0) + \frac{mc}{2q\sqrt{\mathcal{I}_1^2+\mathcal{I}_2^2}}\left[\left(F - \frac{\lambda_+^2}{\mathcal{I}_2}\mathcal{F}\right)\cosh(\theta\lambda_-)\right. \\
&- \left.\left(F + \frac{\lambda_-^2}{\mathcal{I}_2}\mathcal{F}\right)\cos(\theta\lambda_+) + \frac{\lambda_+^2 I + F^2}{\lambda_-}\sinh(\theta\lambda_-) + \frac{\lambda_-^2 I - F^2}{\lambda_+}\sin(\theta\lambda_+)\right] u(0).
\end{aligned}$$

which is the final result.

Consider the special case of $\overrightarrow{E} = E\hat{x}$, $\overrightarrow{B} = B\hat{y}$ then $\overrightarrow{E} \perp \overrightarrow{B} \Rightarrow \mathcal{I}_2 = 0$. Taking the limit $\mathcal{I}_2 \to 0$ gives:
$\lambda_- \to 0$; $\lambda_+ \to \sqrt{2\mathcal{I}_1}$, $\cosh(\theta\lambda_-) \to 1$ and $\sinh(\theta\lambda_-)/\lambda_- \to \theta$.
Consider the case $\mathcal{I}_1 = \frac{1}{2}(B^2 - E^2) > 0$ and let's take $x(0) = 0$. Then:

$$\begin{aligned} x(\tau) &= \frac{mc}{q\mathcal{I}_2}\mathcal{F}u(0) + \frac{mc}{2q\mathcal{I}_1}\left[\left(F - \frac{2\mathcal{I}_1}{\mathcal{I}_2}\mathcal{F}\right) - F\cos(\theta\lambda_+)\right. \\ &\quad \left. + (2\mathcal{I}_1 I + F^2)\theta - \frac{1}{\sqrt{2\mathcal{I}_1}}F^2\sin(\theta\sqrt{2\mathcal{I}_1})\right] u(0). \end{aligned} \tag{3.44}$$

Define $\Omega \equiv \frac{q}{mc}\sqrt{2\mathcal{I}_1}$ then

$$\begin{aligned} x(\tau) &= (I + \frac{F^2}{2\mathcal{I}_1}u(0)\tau + \frac{mc}{2q\mathcal{I}_1}(1 - \cos\Omega\tau \\ &\quad \left. - \frac{F}{\sqrt{2\mathcal{I}_1}}\sin\Omega\tau\right) Fu(0) \end{aligned} \tag{3.45}$$

$$Fu(0) = \gamma_0 c \begin{pmatrix} 0 & E & 0 & 0 \\ E & 0 & 0 & -B \\ 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \beta_{0x} \\ \beta_{0y} \\ \beta_{0z} \end{pmatrix} = \gamma_0 c \begin{pmatrix} E \\ E - \beta_{0z}B \\ 0 \\ \beta_{0x}B \end{pmatrix}; F^2 u(0) = \gamma_0 c \begin{pmatrix} E(E - \beta_{0z}B) \\ -2\mathcal{I}_1\beta_{0x} \\ 0 \\ B(E - \beta_{0z}B) \end{pmatrix}$$

and so:

$$\begin{aligned} x &= \frac{\gamma_0 mc^2}{2q\mathcal{I}_1}\left[(E - B\beta_{0z})(1 - \cos\Omega\tau) + \sqrt{2\mathcal{I}_1}\beta_{0x}\sin\Omega\tau\right] \\ y &= \gamma_0 v_{0y}\tau \\ z &= \frac{\gamma_0 cE}{2\mathcal{I}_1}(B - E\beta_{0z})\tau + \frac{\gamma_0 mc^2 B}{2q\mathcal{I}_1}\left[\beta_{0x}(1 - \cos\Omega\tau) - \frac{E - B\beta_{0z}}{\sqrt{2\mathcal{I}_1}}\sin\Omega\tau\right] \\ t &= \frac{\gamma_0 B}{2\mathcal{I}_1}(B - E\beta_{0z})\tau + \frac{\gamma_0 mcE}{2q\mathcal{I}_1}\left[\beta_{0x}(1 - \cos\Omega\tau) - \frac{E - B\beta_{0z}}{\sqrt{2\mathcal{I}_1}}\sin\Omega\tau\right] \end{aligned}$$

Note that the particle has a velocity perpendicular to $\overrightarrow{E}$ and $\overrightarrow{B}$ fields. The so-called $E \times B$ drift. The drift velocity is $v_d = cE/B$.

3.6 Non uniform magnetic field and adiabatic invariance

Suppose the magnetic field is non uniform but changes “slowly” compared to the ”gyroperiod” of the charge particle (charge=q) under its influence. This is a so-called “adiabatic change. The action integral is conserved:

$$J = \oint \overrightarrow{P}_\perp . \overrightarrow{dl} \tag{3.46}$$

$\overrightarrow{dl}$ is the line element along the particle trajectory. Expliciting $P_\perp$:

$$\begin{aligned} J &= \oint(\gamma m\overrightarrow{v}_\perp + \frac{q}{c}\overrightarrow{A}).\overrightarrow{dl} \\ &= (\gamma m\omega_B a)(2\pi a) + \frac{q}{c}\int_S \overrightarrow{B}.\hat{n}dS \end{aligned} \tag{3.47}$$

$$\Rightarrow J = 2\pi\gamma m\omega_B a^2 - \frac{q}{c}\pi B a^2 \tag{3.48}$$

since $\overrightarrow{B}$ is anti-parallel to $\hat{n}$. Also, $\gamma m\omega_B = \frac{q}{c}B$ so that:

$$J = \frac{q}{c}\pi B a^2. \tag{3.49}$$

This means the magnetic flux

$$\Phi_B = \int_S \overrightarrow{B}.\overrightarrow{dS} = \pi B a^2 \tag{3.50}$$

is an adiabatic invariant.

3.6.1 Example of application of adiabatic invariance

??????????????

3.7 Non uniform magnetic field without adiabatic invariance: the solenoid

The B_r component of magnetic field imparts a p_θ to a charge particle coming off a cathode immersed in a B-field; see Figure 3.2. Let $\overrightarrow{B}(z=0) \equiv B_c\hat{z}$.

$$\begin{aligned} F_\theta &= \frac{q}{c}v_z B_r = \frac{dp_\theta}{dt};\ p_\theta(t=0) = p_\theta(z=0) = 0 \\ &\Rightarrow p_\theta = \frac{q}{c}\int_0^\infty B_r v_z dt = \frac{q}{c}\int_0^\infty B_r dz \end{aligned} \tag{3.51}$$

But integrating over the surface, $\mathcal{S}$, of a "Gauss" cylinder gives:

$$\begin{aligned} \int_S \overrightarrow{B}\overrightarrow{dS} = 0 &= -\pi r^2 B_c + 2\pi r\int_0^\infty B_r dz \\ &\Rightarrow \int_0^\infty B_r dz = \frac{1}{2}B_c r. \end{aligned} \tag{3.52}$$

Consequently the charge q picks-up a total angular momentum $p_\theta = \frac{q}{2c}B_c r$. Note that

$$\frac{p_\theta}{p_c} = \frac{1}{2}\frac{qB_c}{p_c c}r = \frac{r}{2\rho} \tag{3.53}$$

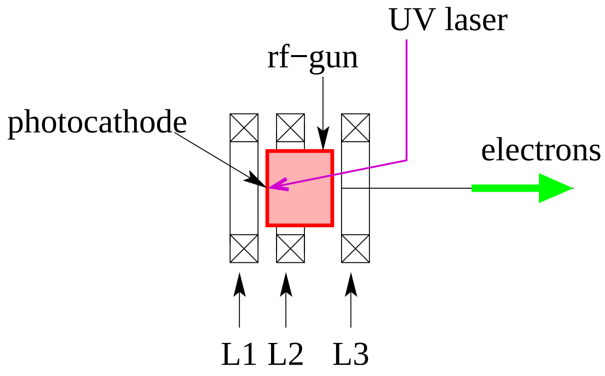

Figure 3.2: Configuration used to generate an angular-momentum-dominated electron beam

where $\rho^{-1} \equiv \frac{qB_c}{p_c c}$. This tells what fraction of initial momentum got converted to angular momentum. ρ is the gyroradius the electron would have had if it was orthogonal to $\overrightarrow{B}_c$. Consequently the angular momentum scales as $L \equiv \overrightarrow{r} \times \overrightarrow{p} \propto r^2$ as confirmed via experiment [2]; see Fig. 3.3. Note that for a particle originating external to the solenoid, $p_\theta = 0$ by symmetry.

[2] Y.-E Sun, P. Piot, K.-J. Kim, N. Barov, S. Lidia, J. Santucci, R. Tikhoplav, and J. Wennerberg, Phys. Rev. ST Accel. Beams **7**, 123501 (2004)

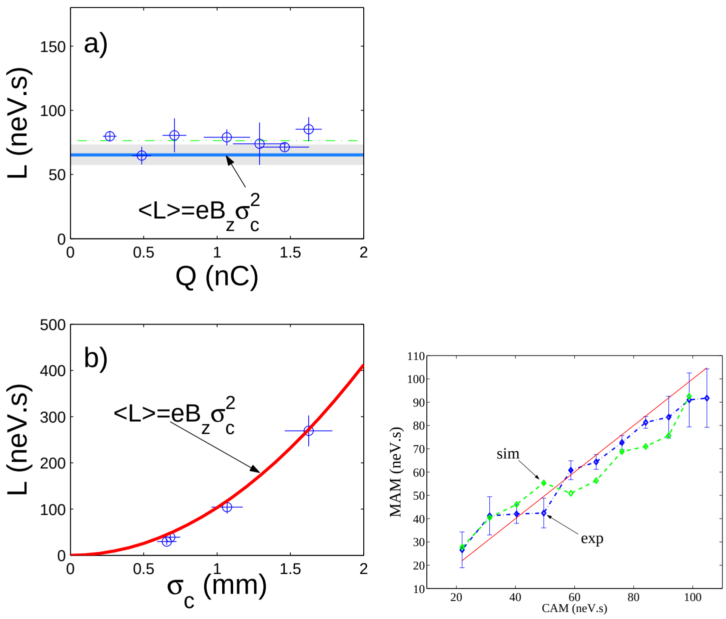

Figure 3.3: Measured averaged angular momentum imparted on an electron beam being photoemitted from a cathode immersed in a solenoidal lens. Left: dependency of the angular momentum versus the laser transverse size ($\sigma_c = r/2$). Right: beam's angular momentum as a function of computed angular momentum [from vector potential given the B_c field (note $B_0 \equiv B_c$]

Chapter 4

Radiation from accelerating charges

4.1 Radiation from accelerating charges

Radiation emitted at time t' reaches the observer (P) at time $t > t'$. It is retarded due to the finite speed of light. Let's first derive the 4-potential due to the moving charge.

4.2 Four-potential produced by a moving charge

Let's start with the inhomogenous Maxwell's equation:

$$\partial_\alpha = F^{\alpha\beta} = \frac{4\pi}{c} j^\beta \tag{4.1}$$

j^β is the 4-current $j^\beta \equiv (c\rho, \overrightarrow{J})$. Use the definition of $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$ and impose the Lorenz Gauge condition $\partial_\alpha A^\alpha = 0$ we have:

$$\partial_\alpha \partial^\alpha A^\beta - \partial_\alpha \partial^\beta A^\alpha = \partial_\alpha \partial^\alpha A^\beta = \frac{4\pi j^\beta}{c} \tag{4.2}$$

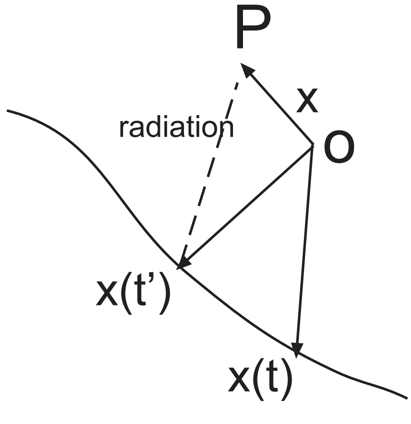

Figure 4.1: geometry associated to the problem of retardation

which can be re-written:

$$\Box A^\beta = \frac{4\pi}{c} j^\beta(x) \tag{4.3}$$

Solution of the latter equation → find Green's function $D(x, x')$ for the equation

$$\Box_x D(x, x') = \delta^{(4)}(x - x') \tag{4.4}$$

where $\delta^{(4)}(x - x') \equiv \delta(x_0 - x'_0)\delta^{(3)}(\overrightarrow{x} - \overrightarrow{x'})$. If free-space (no boundary condition) then $D(x, x') = D(x - x')$. Let $z^\alpha = x^\alpha - x'^\alpha$, $D(x - x') \to D(z)$ and the d'Alembert equation rewrites:

$$\Box_z D(z) = \delta^{(4)}(z).$$

Which can be solved using the Fourier transform method: write

$$\begin{aligned} D(z) &= \frac{1}{(2\pi)^4} \int d^4k \tilde{D}(k) e^{-ikz}, \text{ and,} \\ \delta^4(z) &= \frac{1}{(2\pi)^4} \int d^4k e^{-ikz} \end{aligned} \tag{4.5}$$

Expliciting in the wave equation on finds:

$$\tilde{D}(k) = -\frac{1}{k_\beta k^\beta}$$

where $k^\beta \equiv (k_0, \overrightarrow{\kappa})$ is the four-wavevector, and let $z = (z_0, \overrightarrow{R})$. $k_\beta k^\beta = k_0^2 - \kappa^2$.

so the Green function is given by:

$$\begin{aligned} D(z) &= \frac{1}{(2\pi)^4} \int d^4k (-) \frac{e^{-ikz}}{k_0^2 - \kappa^2} \\ &= -\frac{1}{(2\pi)^4} \int d^3\kappa e^{i\overrightarrow{\kappa}\overrightarrow{R}} \int dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} \end{aligned} \tag{4.6}$$

Consider the integral over k_0. It can be replaced by an integral over a closed contour in the complex space associated to k_0. The integrand has two poles at $k_0 \pm \kappa$ on the real axis. If we consider $z_0 > 0$ the contour need to be closed toward $\mathcal{I}m(k_0) = -\infty$ and the integral is:

$$\begin{aligned} \int_\infty^{+\infty} dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} &= \oint_C dk_0 \frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2} = -2i\pi \sum \text{Res}\left(\frac{e^{-ik_0 z_0}}{k_0^2 - \kappa^2}\right) \\ &= -\frac{2\pi}{\kappa} \sin(kz_0) \end{aligned} \tag{4.7}$$

So D, the retarded Green function, becomes:

$$\begin{aligned} D(z) &= \frac{1}{(2\pi)^3} \int d^3k \frac{\sin(\kappa z_0)}{\kappa} e^{i\overrightarrow{\kappa}.\overrightarrow{R}} \ (z_0 > 0) \\ &= \frac{\Theta(z_0)}{(2\pi)^3} \int d^3k \frac{\sin(\kappa z_0)}{\kappa} e^{i\overrightarrow{\kappa}.\overrightarrow{R}} \end{aligned} \tag{4.8}$$

where $\Theta(x)$ is the Heaviside function. Introducing $d^3\kappa = k^2 d\kappa \sin(\theta) d\theta d\phi$ then we can work out the integral over angle:

$$\begin{aligned}\int \sin\theta d\theta d\phi e^{i\vec{\kappa}\vec{R}} &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta e^{i\kappa z\cos\theta} = 2\pi \left[\frac{e^{i\kappa z\cos\theta}}{-i\kappa z}\right]_0^{\pi} \\ &= 4\pi \frac{\sin(\kappa R)}{\kappa R}. \end{aligned} \tag{4.9}$$

So,

$$D(z) = \frac{\Theta(z_0)}{(2\pi)^3} \int d\kappa \frac{4\pi}{R} \sin(\kappa R)\sin(\kappa z_0) \tag{4.10}$$

$$\begin{aligned} &= \frac{\Theta(z_0)}{2\pi^2 R} \int_0^{\infty} d\kappa \sin(\kappa R)\sin(\kappa z_0) \\ &= -\frac{1}{4\pi R}\frac{1}{2\pi} \int_0^{+\infty} \left[e^{ik(R+z_0)} - e^{ik(R-z_0)} - e^{-ik(R-z_0)} + e^{ik(R+z_0)}\right] \end{aligned}$$

$$= \frac{\Theta(z_0)}{4\pi R}\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[-e^{ik(R+z_0)} - e^{ik(R-z_0)}\right] \tag{4.11}$$

$$= \frac{\Theta(z_0)}{4\pi R}\left[\delta(z_0 - R) + \delta(z_0 + R)\right] = \frac{\Theta(z_0)}{4\pi R}\delta(z_0 - R) \tag{4.12}$$

since the condition $z_0 > 0$ implies $\delta(z_0 + R) = 0$.

$$D(x - x') = \frac{\Theta(z_0)}{4\pi R}\delta(x - x' - R) \tag{4.13}$$

Now use the identity

$$\begin{aligned}\delta[(x - x')^2] &= \delta[(x - x_0)^2 - |x - x'|^2] \\ &= \delta[(x_0 - x_0' - R)(x_0 - x_0' + R)] \\ &= \frac{1}{2R}\left[\delta(x_0 - x_0' - R) + \delta(x_0 - x_0' + R)\right] \end{aligned} \tag{4.14}$$

where we make the use of $\delta[(x - x_1)(x - x_2)] = \frac{\delta(x - x_1) + \delta(x - x_2)}{|x_1 - x_2|}$. The function D becomes:

$$D(x - x') = \frac{1}{2\pi}\Theta(x_0 - x_0')\delta[(x - x')^2]. \tag{4.15}$$

Then the retarded 4-potential is given by the convolution integral:

$$A^{\alpha}(x) = \text{const.} + \frac{4\pi}{c} \int d^4x' D(x - x') J^{\alpha}(x') \tag{4.16}$$

4.3 Liénard-Wiechert Potentials

The 4-potential caused by a charge in motion is:

$$A^\alpha(x) = \frac{4\pi}{c}\int d^4x' D(x-x') j^\alpha(x'), \tag{4.17}$$

The 4-current is (see Part II)

$$j^\alpha(x') = ec\int d\tau v^\alpha(\tau)\delta^{(4)}[x' - r(\tau)] \tag{4.18}$$

τ is the charge's proper time. So expliciting D and j^α the 4-potential takes the form

$$\begin{aligned} A^\alpha(x) &= 2e\int d\tau d^4x'\Theta(x_0 - x'_0)\delta[(x-x')^2]v^\alpha(\tau)\delta^{(4)}[x - r(\tau)] \\ &= 2e\int d\tau\Theta(x_0 - x'_0)v^\alpha(\tau)\delta[(x - r(\tau))^2] \end{aligned} \tag{4.19}$$

$$A^\alpha(x) = 2e\int d\tau\delta(\tau-\tau_0)\Theta(x_0 - x'_0)v^\alpha(\tau)\left|\frac{-1}{2v^\beta(\tau)[x - r(\tau)]_\beta}\right|$$

using the relation

$$\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|\frac{\partial f}{\partial x}|_{x=x_i}}.$$

The four-vector potential finally writes:

$$A^\alpha(x) = \left.\frac{ev^\alpha(\tau)}{v^\beta[x - r(\tau)]_\beta}\right|_{\tau=\tau_0}$$

which can be written in the more familiar form:

$$\Phi(\overrightarrow{x}, t) = \left[\frac{e}{(1-\overrightarrow{\beta}.\hat{n})R}\right]_{ret}, \text{ and, } \overrightarrow{A}(\overrightarrow{x}, t) = \left[\frac{e\overrightarrow{\beta}}{(1-\overrightarrow{\beta}.\hat{n})R}\right]_{ret} \tag{4.20}$$

where *ret* means the quantity in bracket have to be evaluated at the retarded time t' that satisfies the causality condition.

4.4 Field associated to a moving charge

Consider a charge q in motion the Lienard-Witchert potential are given by:

$$\begin{pmatrix} \Phi(\overrightarrow{x}, t) \\ \overrightarrow{A}(\overrightarrow{x}, t) \end{pmatrix} = \left[\frac{q}{(1-\overrightarrow{\beta}.\hat{n})R}\begin{pmatrix} 1 \\ \overrightarrow{\beta} \end{pmatrix}\right]_{ret}. \tag{4.21}$$

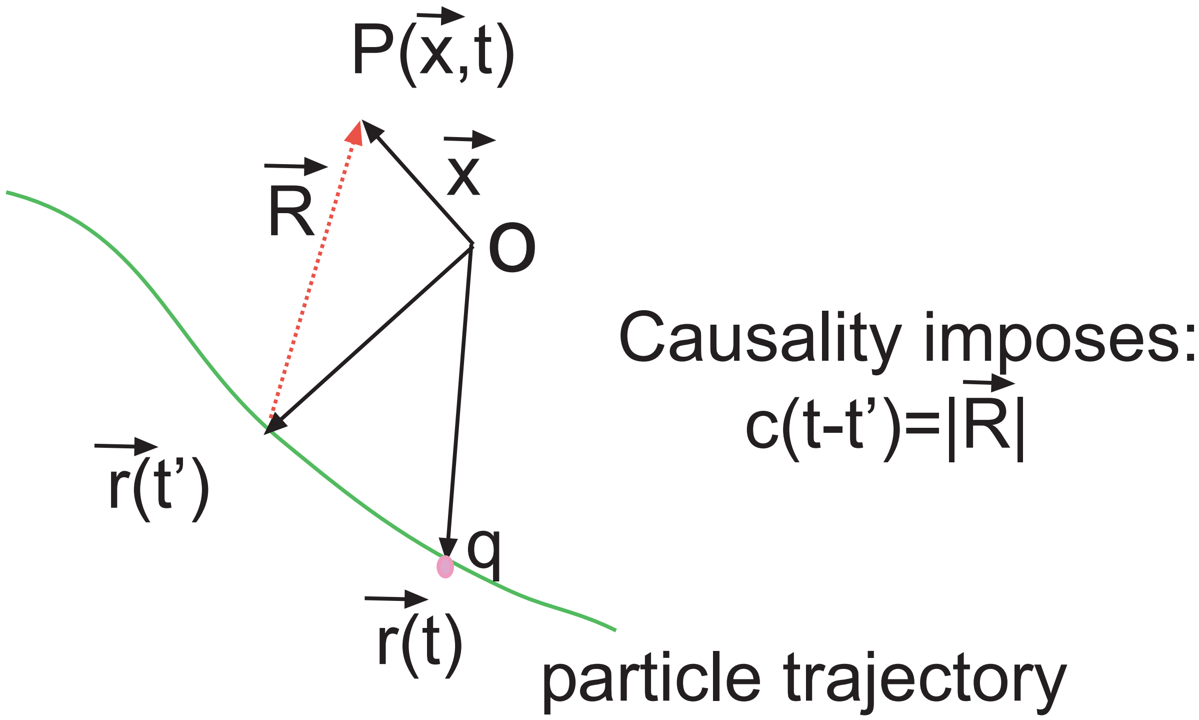

Figure 4.2: Notations and conventions.

The fields are given by $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, but, we need to evaluate the quantities at the retarded time t'. **First** let's express the $\vec{\nabla}$ and $\partial/\partial t$ operators in term of retarded quantities.

$$R(t') = c(t - t') \Rightarrow \frac{\partial R}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right). \tag{4.22}$$

On another hand:

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t}. \tag{4.23}$$

$$\frac{1}{2}\frac{\partial R^2}{\partial t'} = R\frac{\partial R}{\partial t'} = \vec{R}.\frac{\partial \vec{R}}{\partial t'} \text{ , so, } R\frac{\partial R}{\partial t'} = -\vec{v}.\vec{R}. \tag{4.24}$$

Thus

$$\frac{\partial R}{\partial t} = -c\vec{\beta}.\hat{n}\frac{\partial t'}{\partial t} \tag{4.25}$$

From equation 4.22 and 4.25 one gets:

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \vec{\beta}.\hat{n}} \equiv \frac{1}{\kappa} \quad \Rightarrow \frac{\partial}{\partial t} = \frac{1}{\kappa}\frac{\partial}{\partial t'} \tag{4.26}$$

For the operator $\vec{\nabla}$ take:

$$\vec{\nabla}R = \vec{\nabla}[c(t - t')] = -c\vec{\nabla}t' \tag{4.27}$$

if $\overrightarrow{\nabla}_{t'}$ is the gradient operator evaluated at constant t' then

$$\begin{aligned}\overrightarrow{\nabla} R &= \overrightarrow{\nabla}_{t'} R + \frac{\partial R}{\partial t'} \overrightarrow{\nabla} t' \\ &= \hat{n} - c\overrightarrow{\beta}.\hat{n}\overrightarrow{\nabla} t' \end{aligned} \tag{4.28}$$

From equation 4.27 and 4.28 one gets:

$$\overrightarrow{\nabla} t' = \frac{-\hat{n}}{c(1-\overrightarrow{\beta}.\hat{n})} \tag{4.29}$$

So we finally get:

$$\overrightarrow{\nabla} = \overrightarrow{\nabla}_{t'} - \frac{\hat{n}}{c\kappa}\frac{\partial}{\partial t'} \tag{4.30}$$

So the electric field is :

$$\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t} = -\overrightarrow{\nabla}_{t'}\Phi + \frac{\hat{n}}{c\kappa}\frac{\partial \Phi}{\partial t'} - \frac{1}{\kappa c}\frac{\partial \overrightarrow{A}}{\partial t'}, \tag{4.31}$$

with $\Phi = \frac{e}{\kappa R}$.

$$\overrightarrow{\nabla}_{t'}\Phi = \frac{-e}{(\kappa R)^2}[R\overrightarrow{\nabla}_{t'}\kappa + \kappa\overrightarrow{\nabla}_{t'}R] \tag{4.32}$$

$\overrightarrow{\nabla}_{t'}R = \hat{n}$, and

$$\overrightarrow{\nabla}_{t'}\kappa = \overrightarrow{\nabla}_{t'}(1-\overrightarrow{\beta}.\hat{n}) = -\overrightarrow{\nabla}_{t'}(\overrightarrow{\beta}.\hat{n}) = -(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'})\hat{n}. \tag{4.33}$$

$$\begin{aligned}\Rightarrow \overrightarrow{\nabla}_{t'}\kappa &= -(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'})\frac{\overrightarrow{R}}{R} \\ &= -\frac{R(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'})\overrightarrow{R} - \overrightarrow{R}(\overrightarrow{\beta}.\overrightarrow{\nabla}_{t'}}{R^2} \\ &= -\frac{\overrightarrow{\beta} - \hat{n}(\overrightarrow{\beta}.\hat{n})}{R}. \end{aligned} \tag{4.34}$$

So we finally get:

$$\begin{aligned}\overrightarrow{\nabla}_{t'}\Phi &= \frac{-e}{(\kappa R)^2}[-\overrightarrow{\beta} + \hat{n}(\overrightarrow{\beta}.\hat{n}) + (1-\overrightarrow{\beta}.\hat{n})\hat{n}] \\ &= -\frac{e}{(\kappa R)^2}[\hat{n} - \overrightarrow{\beta}] \end{aligned} \tag{4.35}$$

where we have used $\kappa = 1 - \overrightarrow{\beta}.\hat{n}$. Now let's calculate the quantity $\frac{\hat{n}}{c\kappa}\frac{\partial}{\partial t'}\Phi$:

$$\frac{\partial \Phi}{\partial t'} = -e\frac{\partial}{\partial t'}\left(\frac{1}{\kappa R}\right) = \frac{-e}{(\kappa R)^2}[\kappa\dot{R} + R\dot{\kappa}] \tag{4.36}$$

$\dot{R} = -c\overrightarrow{\beta}.\hat{n}$, and $\dot{\kappa} = -\dot{\overrightarrow{\beta}}.\hat{n} - \dot{\hat{n}}.\overrightarrow{\beta}$.

$$\begin{aligned}\dot{\hat{n}} &= \frac{\partial}{\partial t'}\frac{\overrightarrow{R}}{R} = \frac{R\dot{\overrightarrow{R}} - \dot{R}\overrightarrow{R}}{R^2} \\ &= \frac{-\overrightarrow{v} + (\overrightarrow{v}.\hat{n})\hat{n}}{R} = -c\frac{\overrightarrow{\beta} - (\overrightarrow{\beta}.\hat{n})\hat{n}}{R}. \end{aligned} \tag{4.37}$$

Then

$$\begin{aligned}\kappa\dot{R} + \dot{\kappa}R &= -c\overrightarrow{\beta}.\hat{n}\kappa + R\{-\dot{\overrightarrow{\beta}}.\hat{n} + c\overrightarrow{\beta}[\frac{\overrightarrow{\beta} - (\overrightarrow{\beta}.\hat{n})\hat{n}}{R}]\} \\ &= -\dot{\overrightarrow{\beta}}.\overrightarrow{R} + c\beta^2 - c\overrightarrow{\beta}.\hat{n} \end{aligned} \tag{4.38}$$

So

$$\dot{\Phi} = -\frac{e}{(\kappa R)^2}[-\dot{\overrightarrow{\beta}}.\overrightarrow{R} + c\beta^2 - c\overrightarrow{\beta}.\hat{n}]. \tag{4.39}$$

So from Equations 4.35 and 4.39 we have

$$\overrightarrow{\nabla}_{t'}\Phi = -\frac{e}{(\kappa R)^2}\{\hat{n} - \overrightarrow{\beta} + \frac{\hat{n}}{c\kappa}[+\dot{\overrightarrow{\beta}}.\overrightarrow{R} - c\beta^2 + c\overrightarrow{\beta}.\hat{n}]\} \tag{4.40}$$

Now we need to compute $\frac{\partial\overrightarrow{A}}{\partial t}$:

$$\frac{\partial\overrightarrow{A}}{\partial t} = \dot{\overrightarrow{A}}\frac{\partial t'}{\partial t} = \frac{1}{\kappa}\dot{\overrightarrow{A}} \tag{4.41}$$

So

$$\dot{\overrightarrow{A}} = \dot{\overrightarrow{\beta}}\Phi + \overrightarrow{\beta}\dot{\Phi} = +\frac{e}{R\kappa}\dot{\overrightarrow{\beta}}\frac{e\overrightarrow{\beta}}{(R\kappa)^2}[\dot{\overrightarrow{\beta}}\overrightarrow{R} + c\overrightarrow{\beta}\hat{n} - \beta^2 c] \tag{4.42}$$

Finally the E-field is:

$$\begin{aligned}\overrightarrow{E}(t') &= -\overrightarrow{\nabla}\Phi(t') - \frac{1}{c}\frac{\partial\overrightarrow{A}}{\partial t'} \\ &= \frac{e}{(\kappa R)^2\kappa}\left[(\hat{n} - \overrightarrow{\beta})\kappa + \frac{\hat{n}}{c}(\dot{\overrightarrow{\beta}}.\overrightarrow{R} + c\overrightarrow{\beta}.\hat{n} - c\beta^2)\right. \\ &\quad \left. - \frac{\overrightarrow{\beta}}{c}(\dot{\overrightarrow{\beta}}.\overrightarrow{R} + c\overrightarrow{\beta}.\hat{n} - c\beta^2) - \frac{e}{cR\kappa^2}\dot{\overrightarrow{\beta}}\right] \end{aligned} \tag{4.43}$$

After simplification (and using $\overrightarrow{\beta}.\hat{n} = 1 - \kappa$) we end-up with:

$$\begin{aligned}\overrightarrow{E}(t') &= \frac{e}{\kappa^3 R^2}\left[\frac{\hat{n}}{c}\dot{\overrightarrow{\beta}}.\overrightarrow{R} + (1-\beta^2)\hat{n} - \frac{\overrightarrow{\beta}}{c}\dot{\overrightarrow{\beta}}.\overrightarrow{R} + \overrightarrow{\beta} - \overrightarrow{\beta}\beta^2\right] \\ &= \frac{e}{\kappa^3 R^2}\left[(1-\beta^2)(\hat{n} - \overrightarrow{\beta})\right] + \frac{e}{cR\kappa^3}\left[\dot{\overrightarrow{\beta}}.\hat{n}(\hat{n} - \overrightarrow{\beta}) - \dot{\overrightarrow{\beta}}\kappa\right] \end{aligned}$$

So finally the $\overrightarrow{E}$ and $\overrightarrow{B}$ fields are given by:

$$\begin{aligned}\overrightarrow{E}(t') &= \left[\frac{e}{\kappa^3 R^2\gamma^2}(\hat{n}-\overrightarrow{\beta})\right]_{ret} + \left[\frac{e}{\kappa^3 R}\{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]\}\right]_{ret}\\ \overrightarrow{B}(t') &= [n\times\overrightarrow{E}]_{ret}\end{aligned}$$

where the identity $\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}] = \dot{\overrightarrow{\beta}}.\hat{n}(\hat{n}-\overrightarrow{\beta}) - \dot{\overrightarrow{\beta}}(1-\overrightarrow{\beta}.\hat{n})$ was used.

4.5 field of a charge moving at constant velocity

$$\begin{aligned}\overrightarrow{E}(\overrightarrow{x},t) &= q\left[\frac{\hat{n}-\overrightarrow{\beta}}{\gamma^2\kappa^3 R^2}\right]_{ret} + \frac{q}{c}\left[\frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]}{\kappa^3 R}\right]_{ret}, \text{ and}\\ \overrightarrow{B}(\overrightarrow{x},t) &= [\hat{n}\times\overrightarrow{E}]_{ret}.\end{aligned}$$

if $\dot{\beta}=0$, constant velocity then:

$$\overrightarrow{E}(\overrightarrow{x},t) = q\left[\frac{\hat{n}-\overrightarrow{\beta}}{\gamma^2\kappa^3 R^2}\right]_{ret}.$$

from part II, we know that:

$$\overrightarrow{E}_{\perp}(\overrightarrow{x},t) = \frac{\gamma q b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}.$$

$PP' = v(t-t') = \beta R$, $P'Q = PP'\cos(\theta) = \beta R\cos(\theta)$,
$PQ = \beta R\sin(\theta) = \beta R\frac{b}{R} = \beta b$; $QO = R - PP' = (1-\overrightarrow{\beta}.\hat{n})R$
$r^2 = QO^2 + PQ^2 = (1-\overrightarrow{\beta}\hat{n})^2 R^2 = r^2 - \beta^2 b^2 = (vt)^2 + b^2 - \beta^2 b^2$ So,
$r^2 = \gamma^{-2}[b^2+\gamma^2 v^2 t^2] = [\kappa^2 R^2]_{ret}$
and $\hat{x}.(\hat{n}-\overrightarrow{\beta})_{ret} = \sin(\theta) = \frac{b}{R}$ so that

$$E_x = q\left[\frac{\hat{x}.(\hat{n}-\overrightarrow{\beta})}{\gamma^2\kappa^3 R^2}\right]_{ret} = q\frac{b\gamma}{[b^2+\gamma^2 v^2 t^2]^{3/2}}. \tag{4.44}$$

4.6 Radiated Power

$\overrightarrow{S}(\overrightarrow{x},t).\hat{n}$: power crossing a unit area, at time t, of a surface that incircles the radiating particle. $\hat{n}$ is the normal to unit area.
The total energy radiated through the unit area is:

$$W = \int_{-\infty}^{+\infty} dt'\frac{dt}{dt'}\overrightarrow{S}(\overrightarrow{x},t).\hat{n} = \int_{-\infty}^{+\infty} dt'[\kappa\overrightarrow{S}.\hat{n}]_{ret}$$

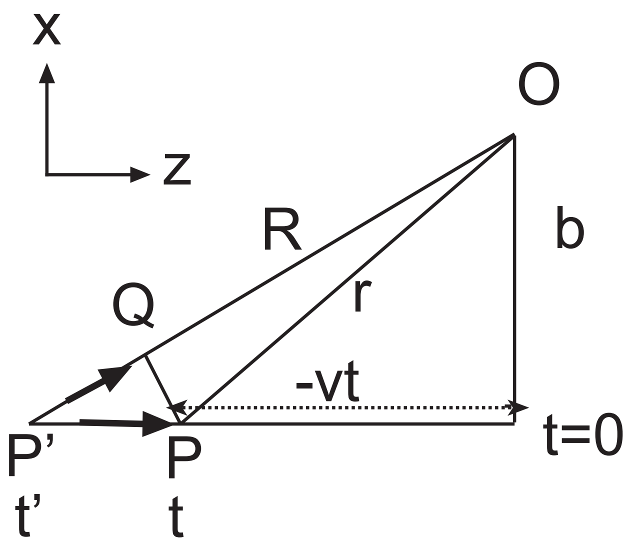

Figure 4.3: ???

So,

$$\frac{dW}{dt} = [\kappa \overrightarrow{S}.\hat{n}]_{ret}$$

This is $dP(t')/dA$ or $1/R^2 dP(t')/d\Omega$ so the instantaneous power radiated at time t' per unit solid angle $d\Omega$ is given by:

$$\frac{dP(t')}{d\Omega} = [\kappa \overrightarrow{S}.\hat{n} R^2]_{ret}$$

From now on, consider only radiation field i.e. R large – this is the "far field" approximation, then

$$\begin{aligned} \overrightarrow{S}.\hat{n} &= \frac{c}{4\pi}[\overrightarrow{E} \times (\hat{n} \times \overrightarrow{E})].\hat{n} \\ &= \frac{c}{4\pi}[E^2 - (\hat{n}.\overrightarrow{E})^2]. \end{aligned}$$

Consider $\hat{n}.\overrightarrow{E}$:

$$\begin{aligned} \hat{n}.\overrightarrow{E} &\propto \hat{n}.\{\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \dot{\overrightarrow{\beta}}]\} \\ &\propto \hat{n}.\{(\hat{n}.\dot{\overrightarrow{\beta}})(\hat{n} - \overrightarrow{\beta}) - [\hat{n}.(\hat{n} - \overrightarrow{\beta})]\dot{\overrightarrow{\beta}}\} \\ &\propto \hat{n}.\{(\hat{n}.\dot{\overrightarrow{\beta}})(\hat{n} - \overrightarrow{\beta}) - (1 - \overrightarrow{\beta}.\hat{n})\dot{\overrightarrow{\beta}}\} \\ &= 0. \end{aligned} \tag{4.45}$$

Hence,

$$\overrightarrow{S}.\hat{n} = \frac{c}{4\pi}E^2 = \frac{q^2}{4\pi c}\left[\frac{|\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \dot{\overrightarrow{\beta}}]|^2}{\kappa^6 R^2}\right]_{ret}$$

$$\frac{dP(t')}{d\Omega} = [\kappa R^2 \overrightarrow{S}.\hat{n}]_{ret} = \frac{q^2}{4\pi c}\left[\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]|^2}{\kappa^5}\right]_{ret}.$$

This is the power radiated per unit solid angle in terms of the charge proper time t'.
If we wants to know $dP(t)/dt$ the power radiated per unit solid angle at the time t it arrives at the enveloping surface, then one must trace back to the associated t' time (retardation).
Note also that $dt = dt'\kappa_{ret} \rightarrow$ if a particle is suddenly (dirac-like) accelerated for a time $\Delta t' = \tau$, a pulse radiation will appear at the observer at time $t = r/c$ and the pulse duration will be $\Delta t = \kappa_{ret}\tau$.
Energy is conserved: total energy radiated=total energy lost by the particle
BUT $\tau\frac{dP(t')}{d\Omega} = \tau\kappa_{ret}\frac{dP(t)}{d\Omega}$; Energy radiated by unit of time = κ_{ret} times the energy lost to far-field per unit time.

4.6.1 Instantaneous rate of radiation

$$P(t') = \frac{q^2}{4\pi c}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin(\theta)\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]|^2}{\kappa^5}$$

$$\begin{aligned}
|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]|^2 &= |(\hat{n}.\dot{\overrightarrow{\beta}})(\hat{n}-\overrightarrow{\beta}) - (1-\overrightarrow{\beta}.\hat{n})\dot{\overrightarrow{\beta}}|^2 \\
&= |(\hat{n}.\dot{\overrightarrow{\beta}})(\hat{n}-\overrightarrow{\beta}) - \kappa\dot{\overrightarrow{\beta}}|^2 \\
&= (\hat{n}.\dot{\overrightarrow{\beta}})^2[1-2\hat{n}.\overrightarrow{\beta}+\beta^2] - 2\kappa\dot{\beta}(\hat{n}.\dot{\overrightarrow{\beta}})(\hat{n}-\overrightarrow{\beta}) + \kappa^2\dot{\beta}^2 \\
&= (\hat{n}.\dot{\overrightarrow{\beta}})^2[1-2\hat{n}.\overrightarrow{\beta}+\beta^2] - 2\kappa(\hat{n}.\dot{\overrightarrow{\beta}})(\hat{n}.\dot{\overrightarrow{\beta}} - \overrightarrow{\beta}.\dot{\beta}) + \kappa^2\dot{\beta}^2
\end{aligned}$$

Using $\overrightarrow{\beta}.\hat{n} = 1-\kappa$ we get:

$$|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]|^2 = -\gamma^{-2}(\hat{n}.\dot{\overrightarrow{\beta}})^2 + 2\kappa(\overrightarrow{\beta}.\dot{\overrightarrow{\beta}})(\hat{n}.\dot{\overrightarrow{\beta}}) + \kappa^2\dot{\beta}^2. \tag{4.46}$$

So,

$$\begin{aligned}
P(t') &= \frac{q^2}{4\pi c}2\pi\int_0^{\pi} d\theta\sin(\theta)\frac{1}{\kappa^5}[\kappa^2\dot{\beta}^2 + 2\kappa(\dot{\overrightarrow{\beta}}.\hat{n})(\overrightarrow{\beta}.\dot{\overrightarrow{\beta}}) - \frac{1}{\gamma^2}(\dot{\overrightarrow{\beta}}.\hat{n})^2] \\
&= \frac{q^2}{2c}\int_0^{\pi} d\theta\sin\theta\left[\frac{\dot{\beta}^2}{\kappa^3} + \frac{2(\overrightarrow{\beta}.\dot{\overrightarrow{\beta}})\dot{\beta}^i n_i}{\kappa^4} - \frac{1}{\gamma^2}\frac{\dot{\beta}^i\dot{\beta}^j n_i n_j}{\kappa^5}\right].
\end{aligned} \tag{4.47}$$

Recall $\kappa \equiv 1-\overrightarrow{\beta}.\hat{n} = 1-\cos\theta$, and let

$$\begin{aligned}
I &\equiv \int_0^{\pi}\frac{\sin\theta d\theta}{(1-\overrightarrow{\beta}.\hat{n})^3} = \int_{-1}^{1} -\frac{du}{(1-\beta u)^3} = \frac{2}{(1-\beta^2)^2} = 2\gamma^4 \\
J_i &\equiv \int_0^{\pi}\frac{n_i\sin\theta d\theta}{(1-\overrightarrow{\beta}.\hat{n})^4} = \frac{1}{3}\frac{\partial I}{\partial\beta^i} = \frac{8}{3}\frac{\beta_i}{(1-\beta^2)^3} = \frac{8}{3}\beta_i\gamma^6, \\
K_{ij} &\equiv \int_0^{\pi}\frac{n_i n_j\sin\theta d\theta}{(1-\overrightarrow{\beta}.\hat{n})^5} = \frac{1}{4}\frac{\partial J_i}{\partial\beta^j} = \frac{2}{3}\frac{\delta_{ij}+\frac{6\beta_i\beta_j}{1-\beta^2}}{(1-\beta^2)^3} = \frac{2}{3}\gamma^6[\delta_{ij}+6\gamma^2\beta_i\beta_j].
\end{aligned}$$

$\Rightarrow P(t') = \frac{q^2}{2c}[\dot{\beta}^2 I + 2(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})\dot{\beta}^i J_i - \frac{1}{\gamma^2}\dot{\beta}^i\dot{\beta}^j K_{ij}]$.

explicit I, J_i, and K_{ij}:

$$\begin{aligned}
P(t') &= \frac{q^2}{2c}\left[2\gamma^2\dot{\beta}^2 + \frac{16}{3}\gamma^6\beta^i\dot{\beta}_i(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}}) - \frac{2}{3}\gamma^4(\delta_{ij}+6\gamma^2\beta_i\beta_j)\dot{\beta}^i\dot{\beta}^j\right] \\
&= \frac{q^2}{2c}\left[2\gamma^4\dot{\beta}^2 + \frac{16}{3}\gamma^6(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})^2 - \frac{2}{3}\gamma^4[\dot{\beta}^2 + 6\gamma^2(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})^2\right] \\
&= \frac{2q^2}{3c}\left[\gamma^4\dot{\beta}^2 + \gamma^6(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})^2\right] \\
&= \frac{2q^2}{3c}\gamma^6\left[(1-\beta^2)\dot{\beta}^2 + (\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})^2\right] = \frac{2q}{3c}\gamma^6[\dot{\beta}^2 - \dot{\beta}^2\beta^2(1-\cos^2\Phi) \\
&= \frac{2q^2}{3c}\gamma^6[1-\dot{\beta}^2\beta^2\sin^2\Phi] = \frac{2q^2}{3c}\gamma^6[\dot{\beta}^2 - (\overrightarrow{\beta}\times\overrightarrow{\dot{\beta}})^2] \qquad (4.48)
\end{aligned}$$

This is the relativistic generalization of the Larmor's results (to recover the standard Larmor power consider $\beta \to 0$).

4.6.2 example 1: radiative energy loss from a linear accelerator

In linear accelerator (or "linac"), $\overrightarrow{\dot{\beta}} \parallel \overrightarrow{\beta}$. In order to calculate $P(t')$, we need to evaluate $\dot{\beta}$. From $p = \gamma\beta mc$ we have:

$$\begin{aligned}
\dot{p} &= mc(\dot{\gamma}\beta + \gamma\dot{\beta}) = mc[(\gamma^3\beta\dot{\beta})\beta + \gamma\dot{\beta}] \\
&= \gamma mc\left(\frac{\beta^2}{1-\beta^2}+1\right)\dot{\beta} = \gamma^3 mc\dot{\beta}. \qquad (4.49)
\end{aligned}$$

So

$$P(t') = \frac{2}{3}\frac{q^2}{m^2c^3}\dot{p}^2 \text{ [JDJ Eq. (14.27)]}$$

Since $P \propto m^{-2}$ lighter particle are subject to higher losses. The rate of momentum change is proportional to the particle energy change:$\dot{p} = dE/dz$ (consider particle being accelerated along the $\hat{z}$-direction).

The question is for what energy gain does radiative effects start to influence the dynamics. Let $P_{ext} \equiv [dE/dt]_{ret}$ be the power associated to the external (accelerating force) then the radiative effect are comparable to external force effects when:

$$\frac{P_{rad}}{P_{ext}} = \frac{P(t')}{vdE/dz} = \frac{2}{3}\frac{q^2}{m^2c^3}\left[\frac{1}{v}\frac{dE}{dz}\right]_{ret} \sim 1.$$

Consider e-: typically $v \simeq c$, and $q = e$ then

$$\frac{P_{rad}}{P_{ext}} = \frac{2}{3}\frac{e^2/(mc^2)}{mc^2}\left[\frac{dE}{dz}\right]_{ret}$$

So $P_{rad} \simeq P_{ext}$ if $dE/dz \simeq mc^2/r_e = 0.511/(2.8\times10^{-15}) = 2\times10^{14}$ MeV/m
compare to 100 MeV/m state-of-art conventional accelerator or to 30 Gev/m plasma-based

accelerator [1]; we see that radiative effects have negligible impact on the dynamics of e-beams.

4.6.3 example 2: radiative energy loss in a circular accelerator

In circular accelerator acceleration is centripetal: $\overrightarrow{\dot{\beta}} \perp \overrightarrow{\beta}$ so

$$\dot{\beta}^2 - (\overrightarrow{\beta} \times \overrightarrow{\dot{\beta}})^2 = \dot{\beta}^2(1-\beta^2) = \frac{\dot{\beta}^2}{\gamma^2}$$

So the radiated power is:

$$P(t') = \frac{2}{3}\frac{q^2 c}{R^2}(\beta\gamma)^4 = \frac{2}{3}\frac{q^2 c}{R^2}\beta^4 \left[\frac{E}{mc^2}\right]^4,$$

where E is the total energy. The revolution period is $T = 2\pi R/(\beta c)$, and $P = \frac{\Delta E}{T}$. So the radiative loss per turn is:

$$\Delta E = PT = \frac{2}{3}\frac{q^2 c}{R^2}\beta^4 \left[\frac{E}{mc^2}\right]^4 \frac{2\pi R}{\beta c}$$

that is:

$$\Delta E = \frac{4\pi}{3}\frac{q^2}{R}\beta^3 \left[\frac{E}{mc^2}\right]^4 \text{ [JDJ Eq. (14.32)]}$$

Consider an e- synchrotron accelerator, the energy loss per turn and per electon is:

$$\Delta E \simeq \frac{4\pi}{3}\frac{e^2}{R}\left(\frac{E}{mc^2}\right)^4.$$

Take $E = 1$ TeV, $R = 2$ km we then have:

$$\Delta E \text{ [eV]} = \frac{1}{3\epsilon_0}\frac{e}{R}\left(\frac{E}{mc^2}\right)^4 = 44.2 \text{ TeV !!}$$

For protons however we gain a factor $(m_e/m_p)^4 = 1/1836^4$ so

$$\Delta E_{proton} \simeq 4 \text{ eV}$$

High energy physics circular accelerator use proton (or ions) reasons for Tevatron at FNAL or LHC at CERN. One can however use e-/e+ storage ring as a copious source of radiation for use or for "cooling" =radiation damping in the internation linear collider proposal.

4.7 Field lines examples

Figures ?? and ?? shown examples of field line associated to radiation field as an electron is accelerated. These Figures were generated with a free software [2].

[1] W. Leeman, *et al.*, *Nature Phys.* **2**, 696-699 (October 2006), also *The Economist*, September 28th, 2006
[2] software available from Shintake-san's homepage SCSS-FEL: http://www-xfel.spring8.or.jp

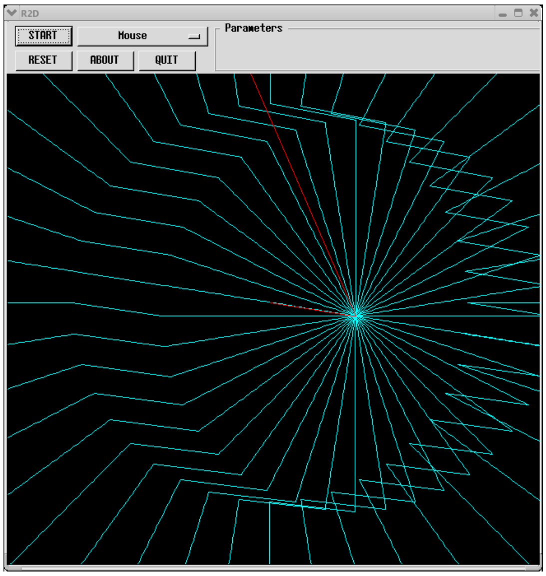

Figure 4.4: Field line associated to a linearly moving charge.

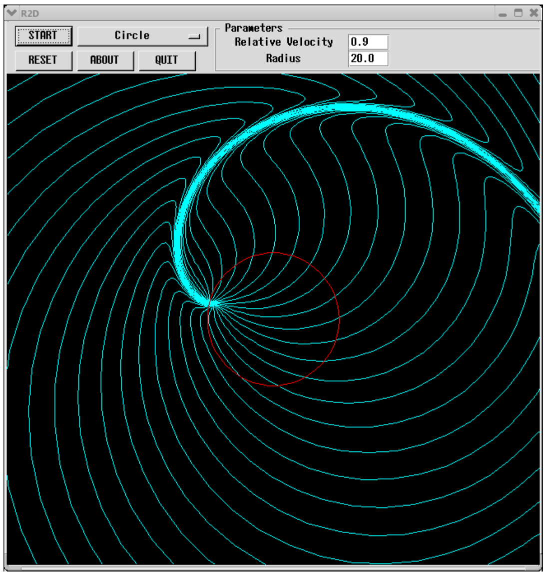

Figure 4.5: Field line associated to a moving charge in circular motion.

4.8 Angular Distribution of radiation emitted by an accelerated charge

$$
\begin{aligned}
\frac{dP(t')}{d\Omega} &= \frac{q^2}{4\pi c}\frac{|\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\overrightarrow{\dot{\beta}}]|^2}{\kappa^5} \\
&= \frac{q^2}{4\pi c}\frac{\kappa^2\dot{\beta}^2+2\kappa(\overrightarrow{\dot{\beta}}.\hat{n})(\overrightarrow{\beta}.\overrightarrow{\dot{\beta}})-\gamma^{-2}(\overrightarrow{\dot{\beta}}.\hat{n})^2}{\kappa^5} \qquad (4.50)
\end{aligned}
$$

where we have used Eq.4.46.

4.8.1 Case of linear motion

$\overrightarrow{\beta}.\hat{n}=\beta\cos\theta$, $\overrightarrow{\dot{\beta}}.\hat{n}=\dot{\beta}\cos\theta$, $\kappa=1-\overrightarrow{\beta}.\hat{n}=1-\beta\cos\theta$, and numerator of $dP(t')/d\Omega$ is:

$$
\begin{aligned}
& \dot{\beta}^2[\kappa^2+2\kappa\beta\cos\theta-(1-\beta^2)\cos^2\theta] \\
=& \dot{\beta}^2[(\kappa^2+2\kappa\beta\cos\theta+\beta^2\cos^2\theta)-\cos^2\theta] \\
=& \dot{\beta}^2[(\kappa+\beta\cos\theta)^2-\cos^2\theta]=\dot{\beta}^2\sin^2\theta. \qquad (4.51)
\end{aligned}
$$

$$
\frac{dP(t')}{d\Omega}=\frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\ \text{[JDJ, Eq.(14.39)]} \qquad (4.52)
$$

The location of peak intensity are given by:

$$
\begin{aligned}
0 &= \frac{d}{d\theta}\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right) \\
&= \frac{\sin(\theta)\left(2\cos(\theta)+3\beta(\cos(\theta))^2-5\beta\right)}{(1-\beta\cos\theta)^4} \qquad (4.53)
\end{aligned}
$$

whose solutions are:

$$
[\cos\theta]_\pm=\frac{1}{3\beta}[-1\pm(1+15\beta^2)^{1/2}] \qquad (4.54)
$$

Only $[\cos\theta]_+$ is viable since we must have $|\cos(\theta)|<1$. So finally

$$
\theta_\pm=\pm\arccos\left[\frac{1}{3\beta}[-1+(1+15\beta^2)^{1/2}]\right]\xrightarrow{\beta\to1}\pm\frac{1}{2\gamma} \qquad (4.55)
$$

these are locations of maximum in power.

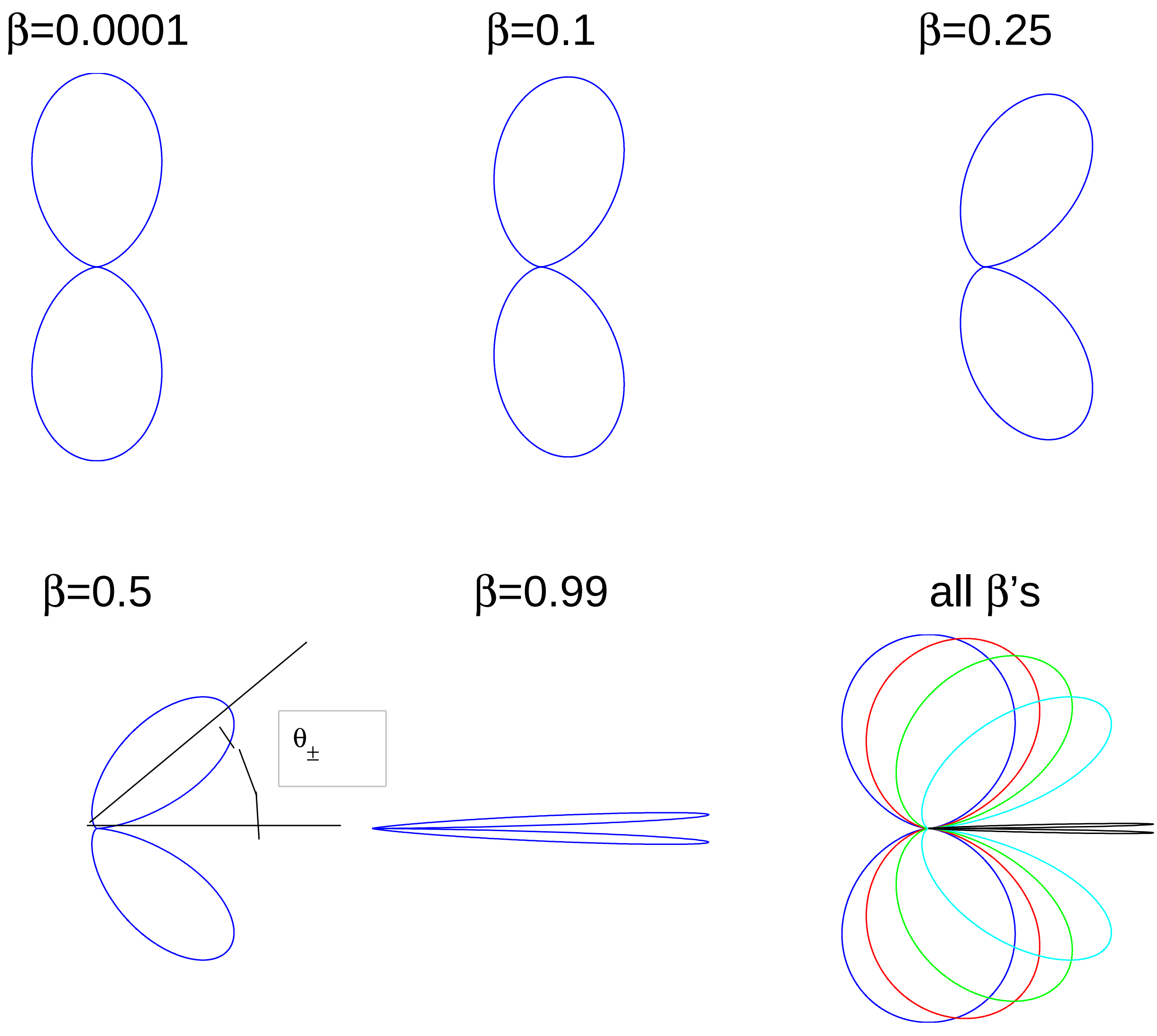

Figure 4.6:

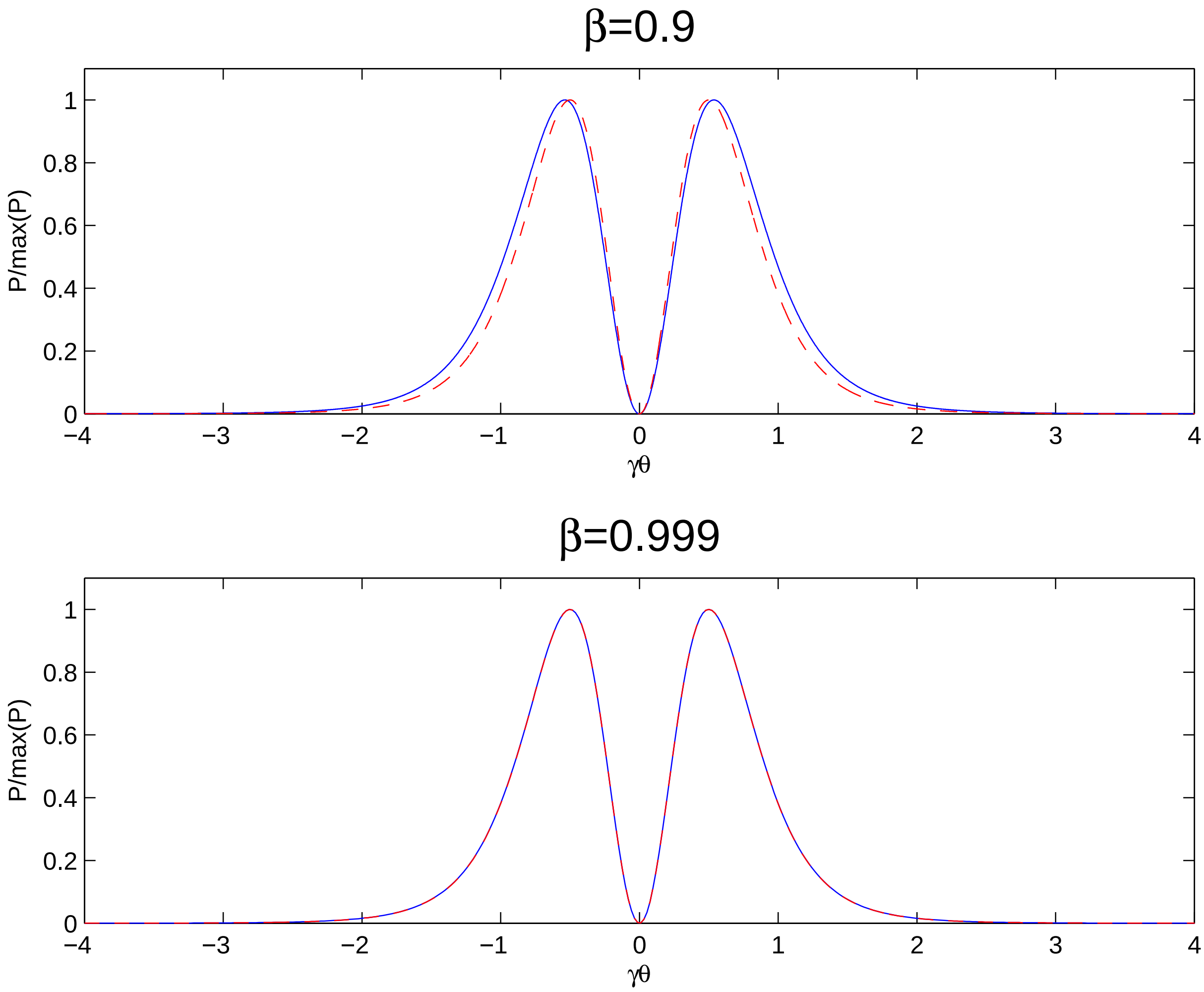

Figure 4.7:

4.8.2 Angular distribution for the case of linear motion

Ultra-relativistic limit: as the $\beta \to 1$ the intensity angular distribution is contained within small angle (so $\theta \ll 1$). The power angular distribution then becomes:

$$\begin{aligned} \frac{dP(t')}{d\Omega} &\simeq \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{\theta^2}{(1-\beta(1-\frac{\theta^2}{2}))^5} = \frac{q^2\dot{\beta}^2}{4\pi c^2}\frac{32\theta^2}{2(1-\beta)+\beta\theta^2))^5} \\ &\simeq \frac{8}{\pi}\frac{\dot{\beta}^2}{c^2}\frac{\gamma^{10}\theta^2}{(1+\gamma^2\theta^2)^5} \text{ [JDJ, Eq.(14.41)].} \end{aligned} \tag{4.56}$$

Comparison of exact θ-dependence (solid line) with ultra-relativistics approximation (dash line) for two cases of β:

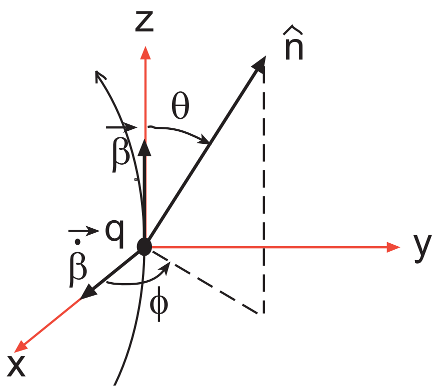

Figure 4.8: snapshot of motion taken at time t'.

4.8.3 Case of circular motion

$$\begin{aligned}\hat{z} &= \cos\theta\hat{n} - \sin\theta\hat{\theta} \\ \hat{x} &= \sin\theta\cos\phi\hat{n} + \cos\theta\sin\phi\hat{\theta} - \sin\phi\hat{\phi}\end{aligned}$$

Thus $\overrightarrow{\beta}.\hat{n} = \beta\cos\theta$, $\overrightarrow{\beta}.\overrightarrow{\dot{\beta}} = 0$, and $\overrightarrow{\dot{\beta}}.\hat{n} = \dot{\beta}\sin\theta\cos\phi$

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi c^2}\frac{\dot{\beta}^2}{(1-\beta\cos\theta)^3}\left[1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2(1-\beta\cos\theta)^2}\right] \text{ [JDJ Eq.(14.44)]} \tag{4.57}$$

Unlike linear motion, the power angular distribution peaks at $\theta = 0$. Considering the ultra-relativistic limit ($\beta \to 1$, $\theta \ll 1$).

$$\frac{dP(t')}{d\Omega} = \frac{8}{\pi}\frac{q^2}{c^2}\frac{\dot{\beta}^2}{(1+\gamma^2\theta^2)^3}\gamma^6\left[1 - \frac{4\gamma^2\theta^2\cos^2\phi}{(1+\gamma^2\theta^2)^2}\right] \text{ [JDJ Eq.(14.44)]} \tag{4.58}$$

A part from a different in the intensity distributions for linear and circular motion, there is also a difference in total radiated power:

$$\begin{aligned}P_{Linear} &= \frac{2}{3}q^2 m^2 c^3 \dot{p}^2 \\ P_{Circular} &= \frac{2}{3}q^2 c\gamma^4\dot{\beta}^2 = \frac{2}{3}q^2 m^2 c^3 \gamma^2 \dot{p}^2\end{aligned}$$

Thus

$$\frac{P_{Circular}}{P_{Linear}} = \gamma^2$$

For a given applied force, there is γ^2 times more radiation energy if the force is applied perpendicular to the charge's velocity that is applied parallel to the velocity.

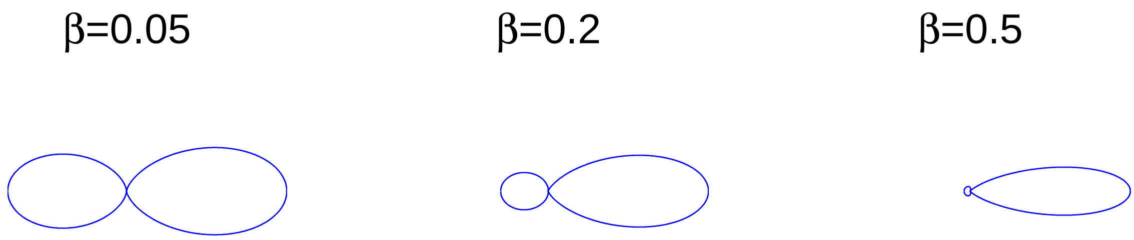

Figure 4.9: Distribution evaluated in the plan $\phi = 0$.

4.9 Radiation Spectrum

Go in the observer's frame:

$$\begin{aligned} \frac{dP(t)}{d\Omega} &= \frac{1}{\kappa(t')}\frac{dP(t')}{d\Omega} \\ &= \frac{q^2}{4\pi c}\left[\frac{|\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]|^2}{\kappa^6}\right]_{ret} \equiv |\vec{A}(t)|^2 \end{aligned} \tag{4.59}$$

wherein

$$\vec{A}(t) = \sqrt{\frac{c}{4\pi}}[R\vec{E}]_{ret} \tag{4.60}$$

to obtain the power spectrum of the radiation we need to work in the frequency domain, so decompose $\vec{A}$ as:

$$\vec{A}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dt\,\vec{A}(t)e^{i\omega t}, \tag{4.61}$$

and reciprocally:

$$\vec{A}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega\,\vec{A}(\omega)e^{-i\omega t}, \tag{4.62}$$

From Parseval's theorem the total energy radiated per $d\Omega$ is

$$\frac{dW}{d\Omega} = \int_{-\infty}^{+\infty} dt|\vec{A}(t)|^2 = \int_{-\infty}^{+\infty} d\omega|\vec{A}(\omega)|^2 \tag{4.63}$$

If $\vec{A}(t)\in\mathbb{R}$, then $\vec{A}^*(\omega) = \vec{A}(\omega)$ and:

$$\frac{dW}{d\Omega} = 2\int_0^{\infty} d\omega|\vec{A}(\omega)|^2 \tag{4.64}$$

So the radiation spectrum per unit of solid angle is:

$$\frac{d^2I(\hat{n},\omega)}{d\Omega d\omega} = 2|A(\omega)|^2 \tag{4.65}$$

Thus we need to evaluate $\overrightarrow{A}(\omega)$

$$\overrightarrow{A}(t) = \frac{q}{\sqrt{4\pi c}}\left[\frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]}{\kappa^3}\right]_{ret} \tag{4.66}$$

and so,

$$\overrightarrow{A}(\omega) = \frac{q}{2\pi\sqrt{2c}}\int_{-\infty}^{+\infty} dt\left[\frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]}{\kappa^3}\right]_{ret} e^{i\omega t} \tag{4.67}$$

since the quantity [...] must be evaluated at the retarded time, let $dt = \kappa(t')dt'$ and $t = t' + \frac{R(t')}{c}$ then the integral becomes:

$$\overrightarrow{A}(\omega) = \frac{q}{2\pi\sqrt{2c}}\int_{-\infty}^{+\infty} dt'\frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]}{\kappa^2}e^{i\omega(t'+\frac{R(t')}{c})} \tag{4.68}$$

In the far-field regime (large $|\overrightarrow{x}|$) we have: $\hat{n} = \frac{\overrightarrow{x}-\overrightarrow{r}(t')}{|\overrightarrow{x}-\overrightarrow{r}(t')|} \simeq \hat{x}$ constant in time. And $R = x - \overrightarrow{r}.\hat{n} + \mathcal{O}(1/x)$.

In the far-field regime the argument of the exponential rewrites:

$$\Xi = i\omega[t' + \frac{R(t)}{c}] = i\omega x + i\omega[t' - \frac{\hat{n}.\overrightarrow{r}(t')}{c}] \tag{4.69}$$

we henceforth ignore the term $i\omega x$ since it has no contribution (the final result is $\propto |A(\omega|^2)$ and define

$$\Xi(t') = i\omega[t' - \frac{\hat{n}.\overrightarrow{r}(t')}{c}], \tag{4.70}$$

we have

$$\overrightarrow{A}(\omega) = \frac{q}{2\pi\sqrt{2c}}\int_{-\infty}^{+\infty} dt\frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]}{\kappa^2}e^{\Xi(t)}, \tag{4.71}$$

and the intensity distribution takes the form

$$\frac{d^2I(\hat{n},\omega)}{d\Omega d\omega} = 2A^2(\omega) = \frac{q^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} dt\frac{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]}{\kappa^2}e^{\Xi(t)}\right|^2. \tag{4.72}$$

To follow JDJ, let's show that the vectorial quantity in the integral can be written as a time-derivative, in the far-field approximation. Consider

$$\frac{\hat{n}\times(\hat{n}\times\overrightarrow{\beta})}{\kappa}, \tag{4.73}$$

and let's compute

$$\frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\overrightarrow{\beta})}{\kappa}\right]=\frac{(-\dot{\kappa}\hat{n}+(1-\kappa)\dot{\hat{n}}-\dot{\overrightarrow{\beta}})\kappa-\dot{\kappa}[(1-\kappa)\hat{n}-\overrightarrow{\beta}]}{\kappa^2} \tag{4.74}$$

It is straightforward (see Eq. 4.37) to show that $\dot{\hat{n}}\propto 1/R$ and $\dot{\kappa}=-\dot{\overrightarrow{\beta}}.\hat{n}-\overrightarrow{\beta}.\dot{\hat{n}}=-\dot{\overrightarrow{\beta}}.\hat{n}+\mathcal{O}(1/R)$. So

$$\begin{aligned}\frac{d}{dt}[...] &= \frac{1}{\kappa^2}\left\{[(\dot{\overrightarrow{\beta}}.\hat{n})\hat{n}-0-\dot{\overrightarrow{\beta}}]\kappa+(\dot{\overrightarrow{\beta}}.\hat{n})[(1-\kappa)\hat{n}-\overrightarrow{\beta}]\right\}\\ &= \frac{1}{\kappa^2}\left\{-\dot{\overrightarrow{\beta}}\kappa+(\dot{\overrightarrow{\beta}}.\hat{n})(\hat{n}-\overrightarrow{\beta})\right\}+\mathcal{O}(1/R)=\frac{1}{\kappa^2}\left\{\hat{n}\times[(\hat{n}-\overrightarrow{\beta})\times\dot{\overrightarrow{\beta}}]\right\}.\end{aligned}$$

So the vectorial quantity is a time-derivative and we can write:

$$\begin{aligned}\overrightarrow{A}(\omega) &= \frac{q}{2\pi\sqrt{2c}}\int_{-\infty}^{+\infty}dt\frac{d}{dt}\left[\frac{\hat{n}\times(\hat{n}\times\overrightarrow{\beta})}{\kappa}\right]e^{\Xi(t)}\\ &= \frac{q}{2\pi\sqrt{2c}}\left\{\left|\left[\frac{\hat{n}\times(\hat{n}\times\overrightarrow{\beta})}{\kappa}\right]e^{\Xi(t)}\right|_{-\infty}^{+\infty}-i\omega\int_{-\infty}^{+\infty}dt\left[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})\right]e^{\Xi(t)}\right\}\end{aligned}$$

The first integral is zero (in principle on should introduce a decay term $e^{-\epsilon|t|}$, with $\epsilon>0$, perform the integral and take the limit $\epsilon\rightarrow 0$).We finally have:

$$\frac{d^2I(\hat{n},\omega)}{d\Omega d\omega}=\frac{q^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{+\infty}dt[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})]e^{i\omega[t'-\frac{\hat{n}.\overrightarrow{r}(t)}{c}]}\right|^2 \tag{4.75}$$

Nota: $[\hat{n}\times(\hat{n}\times\overrightarrow{\beta})]=\beta\sin\theta=|\hat{n}\times\overrightarrow{\beta}|$ where $\theta=\angle(\hat{n},\overrightarrow{\beta})$.

4.9.1 Case of circular motion

$$\hat{n} = \sin\theta\hat{y}+\cos\theta\hat{z}, \tag{4.76}$$
$$\overrightarrow{\beta} = \beta[\sin(\omega_0t')\hat{x}+\cos(\omega_0t')\hat{z}], \tag{4.77}$$
$$\hat{\epsilon}_{\|} = \hat{x}, \tag{4.78}$$
$$\hat{\epsilon}_{\perp} = \hat{n}\times\hat{x}=-\sin\theta\hat{z}+\cos\theta\hat{y}. \tag{4.79}$$

$$\begin{aligned}\hat{n}\times(\hat{n}\times\overrightarrow{\beta}) &= (\hat{n}.\overrightarrow{\beta})\hat{n}-\overrightarrow{\beta}\\ &= \beta[c_{\omega_0t}c_\theta\hat{y}+c_{\omega_0t}(c_\theta^2-1)\hat{z}-c_{\omega_0t}\hat{x}\\ &= \beta[-s_{\omega_0t}\hat{\epsilon}_{\|}+c_{\omega_0t}s_\theta\hat{\epsilon}_{\perp}]\end{aligned} \tag{4.80}$$

Let's now consider the argument of the exponential function Ξ. First we note that $\hat{n}.\overrightarrow{r} = r\cos\theta\cos(\pi/2-\omega_0 t') = r\sin(\omega_0 t')\cos\theta$ and

$$\Xi = i\omega(t' - \frac{\hat{n}.\overrightarrow{r}}{c}) = \omega[t' - \frac{r}{c}\sin(\omega_0 t')\cos\theta] \tag{4.81}$$

Also if P catch an impulse of radiation from q: q's radiation is confined in forward direction, θ is small, and the pulse originated near $\omega_0 t \simeq 0$. Under these approximations:

$$\lim_{\theta\ll 1,\omega_0 t\ll 1} \hat{n}\times(\hat{n}\times\overrightarrow{\beta}) = \beta(-\omega_0 t\hat{\epsilon}_{\parallel} + \theta\hat{\epsilon}_{\perp}) \tag{4.82}$$

and,

$$\begin{aligned} \lim_{\theta\ll 1,\omega_0 t\ll 1}\frac{1}{i}\Xi &= \omega\left\{t' - \frac{r}{c}[\omega_0 t' - \frac{1}{6}(\omega_0 t')^3](1-\frac{\theta^2}{2})\right\} \\ &= \omega\left\{(1-\beta)t' + \frac{\beta t'}{2}\theta^2 + \frac{1}{6}\frac{r}{c}(\omega_0 t')^3\right\} \\ &= \frac{\omega t'}{2}(\gamma^{-2}+\beta\theta^2) + \frac{\omega\beta}{6\omega_0}(\omega_0 t')^3. \end{aligned} \tag{4.83}$$

The spectral energy density is:

$$\begin{aligned} \frac{d^2 I}{d\Omega d\omega} &= \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{+\infty} dt\beta(-\omega_0 t\hat{\epsilon}_{\parallel} + \theta\hat{\epsilon}_{\perp})e^{\Xi}\right|^2 \\ &= \left| - A_{\parallel}(\omega)\hat{\epsilon}_{\parallel} + A_{\perp}(\omega)\hat{\epsilon}_{\perp}\right|^2 \end{aligned} \tag{4.84}$$

This displays the two polarization associated to the radiation. Nota: $\parallel$ and $\perp$ polarizations are also respectively refer to as σ and π-polarizations.

where

$$\begin{pmatrix} A_{\parallel} \\ A_{\perp} \end{pmatrix} = \frac{q\omega}{2\pi\sqrt{c}}\int_{-\infty}^{+\infty} dt \begin{pmatrix} \omega_0 t \\ \theta \end{pmatrix} e^{i\frac{\omega}{2}[(\gamma^{-2}+\theta^2)t + \frac{1}{3\omega_0}(\omega_0 t')^3]}. \tag{4.85}$$

let $x = \frac{\omega_0 t}{\sqrt{\gamma^{-2}+\theta^2}}$, $dt = \frac{1}{\omega_0}\sqrt{\gamma^{-2}+\theta^2}dx$; and let $\xi \equiv \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2}$, then

$$\begin{pmatrix} A_{\parallel}(\omega) \\ A_{\perp}(\omega) \end{pmatrix} = \frac{q\omega}{2\pi\sqrt{c}}\int_{-\infty}^{+\infty} dx \begin{pmatrix} (\gamma^{-2}+\theta^2)x\frac{1}{\omega_0} \\ (\gamma^{-2}+\theta^2)^{1/2}\theta\frac{1}{\omega_0} \end{pmatrix} e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]}. \tag{4.86}$$

we have the identity:

$$\int_{-\infty}^{+\infty} dt e^{i(xt+at^3)} = \frac{2\pi}{(2a)^{1/3}}A_i\left(\frac{x}{(3a)^{1/3}}\right),$$

where A_i is the Airy function, Note also that $A_i(x) = \frac{1}{\pi}\sqrt{\frac{1}{3}x}K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$. Thus:

$$\int_{-\infty}^{+\infty} dx e^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]} = \frac{2\pi}{(3\xi/2)^{1/3}}A_i\left[\left(\frac{3\xi}{2}\right)^{2/3}\right] = \frac{2}{\sqrt{3}}K_{1/3}(\xi). \tag{4.87}$$

For the other integral. Note that

$$\int_{-\infty}^{+\infty} dtte^{i(xt+at^3)} = \frac{1}{i}\frac{d}{dx}\int_{-\infty}^{+\infty} te^{i(xt+at^3)}dt = \frac{2\pi}{(2a)^{1/3}} A_i' \left(\frac{x}{(3a)^{1/3}}\right),$$

The prime denote the differentiation w.r.t. total argument of A_i. Inserting $a = \xi/2$, and $x = 3\xi/2$ we get:

$$\int_{-\infty}^{+\infty} xe^{i\frac{3}{2}\xi[x+\frac{1}{3}x^3]}dx = \frac{2\pi}{(3\xi/2)^{1/3}} A_i' \left[\left(\frac{3\xi}{2}\right)^{2/3}\right] = -\frac{1}{i}\frac{2}{\sqrt{3}} K_{2/3}(\xi). \tag{4.88}$$

where we have used: $A_i'(x) = \frac{-1}{\pi}\sqrt{\frac{1}{3}x} K_{2/3}\left(\frac{2}{3}x^{3/2}\right)$. So the spectral intensity per unit of solid angle takes the form:

$$\begin{aligned} \frac{d^2I}{d\Omega d\omega} &= |A_{\|}(\omega)|^2 + |A_{\perp}(\omega)|^2 \\ &= \frac{q^2}{3\pi^2 c}\left(\frac{\omega}{\omega_0}\right)^2 (\gamma^{-2}+\theta^2)^2 \left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2}+\theta^2} K_{1/3}^2(\xi)\right] \end{aligned}$$

or, introducing $\xi = \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2} \equiv \frac{1}{2}\frac{\omega}{\omega_c}[1+\gamma^2\theta^2]^{3/2}$:

$$\frac{d^2I}{d\Omega d\omega} = \frac{3q^2}{\pi^2 c}\xi^2 \frac{1}{\gamma^{-2}+\theta^2}\left[K_{2/3}^2(\xi) + \frac{\theta^2}{\gamma^{-2}+\theta^2} K_{1/3}^2(\xi)\right]$$

High frequency radiation occupies $\theta < \gamma^{-1}$ ($\ll \gamma^{-1}$ for $\omega \gg \omega_c$) and low frequency radiation occupies $\theta > \gamma^{-1}$. It is usual to also define a critical angle as $\theta_c = \frac{1}{\gamma}\left(\frac{2\omega_c}{\omega}\right)$

For low frequency $\omega \ll \omega_c$, the frequency spectrum integrated over the solid angle is:

$$\begin{aligned} \frac{dI}{d\omega} &\simeq 2\pi\theta_c \left[\frac{d^2I}{d\omega d\Omega}\right]_{\theta=0} \\ &= \frac{2\pi}{\gamma}\left(\frac{2\omega_c}{\omega}\right)^{1/3} \frac{3}{\pi^2}\frac{q^2}{c}\gamma^2[\xi^2 K_{2/3}^2(\xi)]_{\theta=0}. \end{aligned} \tag{4.89}$$

$\xi(0) = \frac{\omega}{2\omega_c} \ll 1$ so that

$$[\xi K_{2/3}(\xi)]_{\theta=0}^2 \simeq \left[\frac{\Gamma(2/3)}{2^{1/3}}\right]^2 [\xi(0)]^{2/3} \simeq \left(\frac{\omega}{2\omega_c}\right)^{2/3}. \tag{4.90}$$

So

$$\frac{dI}{d\omega} \simeq \frac{6}{\pi}\frac{q^2}{c}\gamma\left(\frac{\omega}{2\omega_c}\right)^{1/3} = \frac{6}{\pi}\frac{q^2}{c}\gamma\left(\frac{\omega}{3\gamma^3\omega_0}\right)^{1/3} \propto \omega^{1/3} \tag{4.91}$$

for $\omega \ll \omega_c$, so it is very broad γ-independent spectrum.

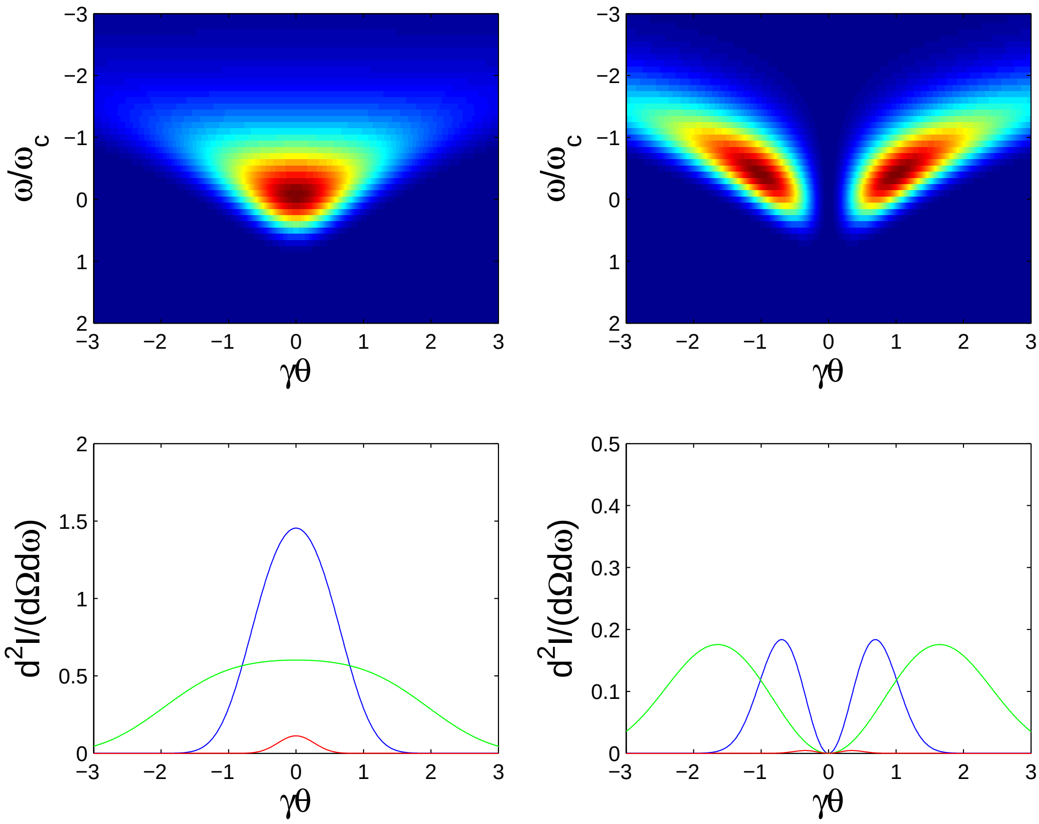

Figure 4.10: ??????

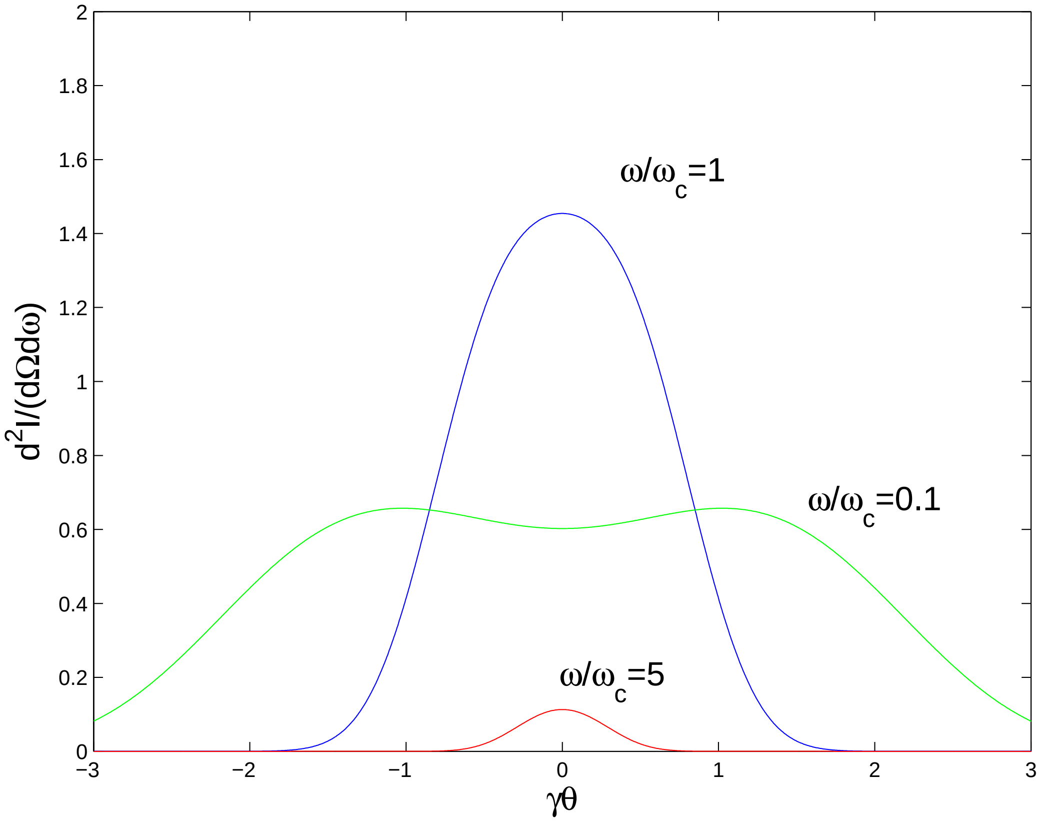

Figure 4.11: ??????

4.9.2 Angular distribution

we need to calculate $\int_0^\infty d\omega \frac{d^2 I}{d\omega d\Omega}$. Do the variable change $\xi = \frac{1}{3}\frac{\omega}{\omega_0}[\gamma^{-2}+\theta^2]^{3/2}$ then:

$$\begin{aligned}\frac{dI}{d\Omega} &= \frac{3q^2}{\pi^2 c}\frac{3\omega_0}{[\gamma^{-2}+\theta^2]^{5/2}}\int_0^\infty \xi^2\left\{K_{2/3}^2(\xi)+\frac{\theta^2}{\gamma^{-2}+\theta^2}K_{1/3}^2(\xi)\right\}d\xi \\ &= \frac{9}{\pi^2}\frac{q^2}{c}\frac{\gamma^5\omega_0}{[1+(\gamma\theta)^2]^{5/2}}\left[\frac{7\pi^2}{144}+\frac{5\pi^2}{144}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)}\right] \qquad (4.92)\end{aligned}$$

where we have used the identity:

$$\int_0^\infty \omega^2 K_\mu^2(a\omega)d\omega = \frac{\pi^2}{32a^3}\frac{1-4\mu^2}{\cos(\pi\mu)}$$

Thus we finally have:

$$\frac{dI}{d\Omega} = \frac{7}{16}\frac{q^2}{c}\frac{\gamma^5\omega_0}{[\gamma^{-2}+\theta^2]^{5/2}}\left[1+\frac{5}{7}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)}\right] \quad \text{[JDJ, Eq.(14.80)]}$$

The total energy radiated is $\Delta W = \int d\Omega \frac{dI}{d\Omega} = 2\pi\int d\theta\frac{dI}{d\theta}$ the integral on θ should be within $[-\pi,\pi]$ however because we did a small angle approximation and since $dI/d\Omega$ is significant only for $\gamma\theta < 1$ we do this integral from $[-\infty,\infty]$:

$$\begin{aligned}\Delta W &= 2\pi\int_0^\infty d\theta\frac{dI}{d\theta} \\ &= \frac{7\pi}{8}\frac{q^2}{c}\gamma^5\omega_0\int_{-\infty}^{+\infty}\left[\frac{1}{(1+\gamma^2\theta^2)^{5/2}}+\frac{5}{7}\frac{\gamma^2\theta^2}{(1+\gamma^2\theta^2)^{7/2}}\right] \\ &= \frac{7\pi}{8}\frac{q^2}{c}\gamma^5\omega_0\left[\frac{4}{3\gamma}+\frac{4}{15\gamma}\right] = \frac{7\pi}{6}\frac{q^2}{c}\gamma^4\omega_0\left[1+\frac{1}{7}\right] \qquad (4.93)\end{aligned}$$

There is 7 times more energy radiated in the $\parallel$-polarization than in the $\perp$-polarization. The total energy radiated is

$$\Delta W = \frac{4\pi}{3}\frac{q^2}{c}\gamma^4\omega_0$$

where $\omega_0 = c/r$.

Let show that the previous result is in agreement with the radiated power associated to circular motion we computed earlier in this chapter.

$$\begin{aligned}\Delta W_{circ} &= P_{circ}\frac{2\pi}{\omega_0} = \left(\frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2\dot{p}^2\right)\frac{2\pi}{\omega_0} \\ &= \frac{2}{3}\frac{q^2}{m^2c^3}\gamma^2(\gamma m r\omega_0^2)^2\frac{2\pi}{\omega_0} = \frac{4\pi}{3}\frac{q^2}{c^3}\gamma^4 r^2\omega_0^3 \\ &= \frac{4\pi}{3}\frac{q^2}{r}\gamma^4. \qquad (4.94)\end{aligned}$$

The $dI/d\omega$ angular-integrated spectrum was derived by Schwinger [3] to be:

$$\frac{dI}{d\omega} = \sqrt{3}\frac{q^2}{c}\gamma\frac{\omega}{\omega_c}\int_{\omega/\omega_c}^{+\infty} dx K_{5/3}(x). \qquad (4.95)$$

[3] *Phys. Rev. Lett.* **75**, 1912 (1949)

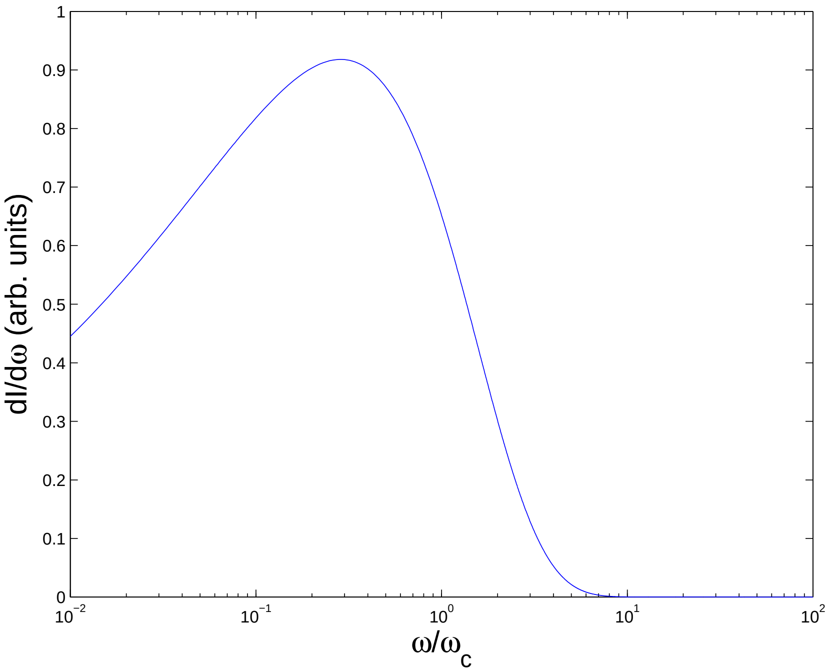

Figure 4.12: Integrated frequency spectrum.

4.9.3 Case of periodic circular motion

The results derived in the previous pages pertains to instantaneous circular motion, for which the spectrum is a continuum. If the motion is periodic, the associated spectrum is discrete. The tool for analyzing this type of motion are the Fourier series. First we note that the period measured by an observed in the far field (T) is the same as the period of the particle motion (T'). We now have to introduce the Fourier series decomposition:

$$\overrightarrow{A}(t) = \sqrt{\frac{c}{4\pi}}[E\overrightarrow{E}]_{ext} = \sum_{n=-\infty}^{n=+\infty} \overrightarrow{A}_n e^{-in\omega_0 t},$$

where,

$$\overrightarrow{A}_n = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} \overrightarrow{A}(t)e^{in\omega_0 t}$$

Following what we did previously we can show:

$$\overrightarrow{A}_n = \sqrt{2\pi}\frac{q}{2\pi\sqrt{2c}}\frac{\omega_0}{2\pi}(-in\omega_0)\int_0^{2\pi/\omega_0} dt\hat{n}\times(\hat{n}\times\overrightarrow{\beta})e^{in\omega_0(t-\hat{n}.\overrightarrow{r})}$$

where the $\sqrt{2\pi}$ come from the difference in normalization factor between the Fourier integral transform and series.
The spectrum is now discrete $\omega = n\omega$ with $n \in \mathbb{N}$.

4.10 Thomson & Compton Scattering

"Scattering" of an e.m. wave by a charged particle (say e-). But e- has no surface $\rightarrow$ radiation is not really scattered. Radiation emitted by the e- as it oscillates in the incoming radiation field is the "scattered" radiation.
In term of photon: photon with wavelength λ strikes a stationary e- and bounce off with wavelength λ'.

$$\begin{aligned} P^\alpha_\gamma + P^\alpha_\gamma &= P^\alpha_{\gamma'} + P^\alpha_{e-'} && (4.96) \\ P^\alpha_{e-'} &= P^\alpha_{e-} + P^\alpha_\gamma - P^\alpha_{\gamma'} && (4.97) \end{aligned}$$

So the norm is:

$$m^2c^2 = (P^\alpha_{e-} + P^\alpha_\gamma - P^\alpha_{\gamma'})(P_{e-,\alpha} + P_{\gamma,\alpha} - P_{\gamma',\alpha}) \tag{4.98}$$

remembering that for a photon $P_\mu P^\mu = 0$ we finally end up with

$$P^\alpha_{\gamma'} P_{e-,\alpha} - P^\alpha_{\gamma'} P_{e-,\alpha} - P^\alpha_{\gamma'} P_{e-,\alpha} = 0 \tag{4.99}$$

we have:

$$\begin{aligned} P_{e-,\alpha} &= mc(E_\gamma/2) - \overrightarrow{p}_{e-}.\overrightarrow{p}_\gamma = mE_\gamma && (4.100) \\ P^\alpha_{\gamma'} P_{\gamma,\alpha} &= \frac{E_\gamma}{c}\frac{E_{\gamma'}}{c} - \overrightarrow{p}_\gamma.\overrightarrow{p}_{\gamma'} = -p_\gamma p_{\gamma'}\cos\theta + \frac{E_\gamma E_{\gamma'}}{c^2} && (4.101) \\ P_{e-,\alpha} P^\alpha_{\gamma'} &= mE_\gamma. && (4.102) \end{aligned}$$

Taking $E_\gamma \equiv \frac{hc}{\lambda}$ (and similarly for γ') we finally obtain:

$$\lambda - \lambda' = \frac{h}{mc}(1 - \cos\theta). \tag{4.103}$$

This is the usual Compton scattering. Thomson scattering is the non relativistic limit of Compton scattering (so take $c \rightarrow \infty$) so $\lambda = \lambda'$.
Cross section for Thomson scattering:
The cross-section is defined as:

$$\sigma \equiv \frac{\text{E radiated/time/solid angle}}{\text{incident flux/unit area/time}}. \tag{4.104}$$

e- is at rest $\overrightarrow{\beta} = 0$, and

$$\begin{aligned} \frac{dP(t')}{d\Omega} &= \frac{e^2}{4\pi c}\frac{\left|\hat{n} \times [(\hat{n} - \overrightarrow{\beta}) \times \dot{\overrightarrow{\beta}}]\right|^2}{\kappa^5} && (4.105) \\ &\xrightarrow{\beta\to 0} \frac{e^2}{4\pi c}\frac{\left|\hat{n} \times [\hat{n} \times \dot{\overrightarrow{\beta}}]\right.}{\kappa^5} = \frac{e^2}{4\pi c}\dot{\beta}^2 \sin^2\Theta && (4.106) \end{aligned}$$

where $\Theta\angle(\hat{n}, \dot{\overrightarrow{\beta}})$. Introducing the acceleration $\overrightarrow{a} \equiv c\dot{\overrightarrow{\beta}}$ we have:

$$\frac{dP(t')}{d\Omega} = \frac{e^2}{4\pi c^3}\sin^2\Theta. \tag{4.107}$$

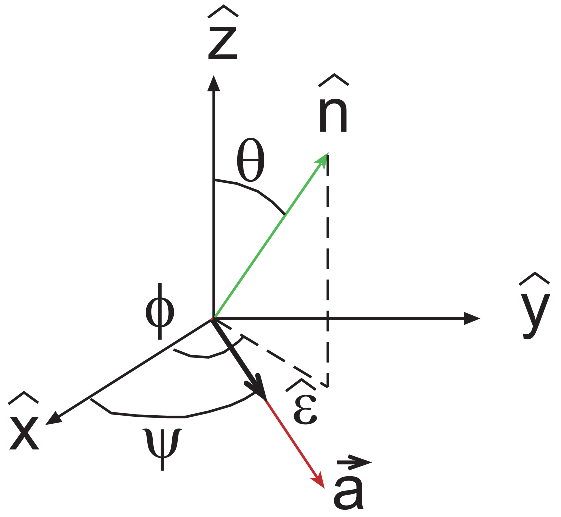

Figure 4.13: Conventions and notations for the Thomson scattering geometry

Also note that in the NR limit $t \to t'$ so $\frac{dP(t)}{d\Omega} = \frac{dP(t')}{d\Omega}$. We now need to find $\overrightarrow{a}$.

Let's consider an incoming plane e.m. wave of the form $\overrightarrow{E}(\overrightarrow{x}, t) = \hat{\epsilon} E_0 e^{i(\overrightarrow{k}.\overrightarrow{x} - \omega t)}$ then we have

$$\overrightarrow{a}(t) = \frac{e}{m} \hat{\epsilon} E_0 e^{i(\overrightarrow{k}.\overrightarrow{x} - \omega t)} \tag{4.108}$$

we only consider the E-field contribution since $\beta = 0$. Let $\overrightarrow{k} = k\hat{z}$.

From the figure we have:

$$\hat{\epsilon} = \cos\psi \hat{x} + \sin\psi \hat{y} \tag{4.109}$$

$$\hat{n} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{4.110}$$

$$\hat{n}.\overrightarrow{a} = a s_\theta (c_\psi c_\phi + s_\psi s_\phi) = a s_\theta c_{\psi - \phi} \tag{4.111}$$

$$= a\sin\theta\cos(\psi - \phi) = a\cos\Theta. \tag{4.112}$$

Thus

$$\sin^2\Theta = 1 - \sin^2\theta\cos^2(\psi - \phi). \tag{4.113}$$

t-averaged emitted power scales as $\langle a^2(t)\rangle_t$.

$$\langle a^2 \sin^2\Theta\rangle_t = \frac{1}{2}\left(\frac{eE_0}{m}\right)^2 [1 - \sin^2\theta\cos^2(\psi - \phi)]. \tag{4.114}$$

If incident radiation is not polarized:

$$\langle \cos^2(\psi - \phi)\sin^2\theta\rangle_\psi = \frac{1}{2}\sin^2\theta. \tag{4.115}$$

So

$$\langle a^2 \sin^2\Theta\rangle_{t,\psi} = \frac{1}{2}\left(\frac{eE_0}{m}\right)^2 [1 - \frac{1}{2}\sin^2\theta] \tag{4.116}$$

So finally the radiated power per unit of solid angle takes the form:

$$\langle\frac{dP}{d\Omega}\rangle_{t,\psi} = \frac{cE_0^2}{16\pi}\left(\frac{e^2}{mc^2}\right)^2[1+\cos^2\theta] = \frac{c}{16\pi}r_e(1+\cos^2\theta). \tag{4.117}$$

The incoming Poyinting flux is:

$$S = \frac{c}{8\pi}\overrightarrow{E}\times\overrightarrow{H}^* \tag{4.118}$$

the time average power per unit area is:

$$\frac{dP}{d\sigma} = S = \frac{c}{8\pi}E_0^2. \tag{4.119}$$

So the cross-section is:

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dP}\frac{dP}{d\Omega} = \frac{\frac{cr_e^2}{16\pi}E_0^2}{\frac{c}{8\pi}E_0^2}[1+\cos^2\theta] \tag{4.120}$$

$$= \frac{1}{2}r_e^2(1+\cos^2\theta) \tag{4.121}$$

This is Thomson scattering formula. The integrated cross-section is:

$$\sigma = \int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\frac{d\sigma}{d\Omega} = \frac{16\pi}{3}\frac{1}{2}r_e^2 \tag{4.122}$$

$$= \frac{8\pi}{3}r_e^2. \tag{4.123}$$

4.10.1 Case of Bounded Electrons

Thomson and Compton scattering apply to a free-electron. Let's now consider a bounded electron whose dynamics is described as a damped oscillator model:

$$\overrightarrow{a} + \Gamma\overrightarrow{v} + \omega_0^2\overrightarrow{x} = \frac{q}{m}\overrightarrow{E}. \tag{4.124}$$

As before consider $\overrightarrow{E} = \hat{\epsilon}E_0 e^{i(\overrightarrow{k}.\overrightarrow{x}-\omega t)}$, Take $\overrightarrow{x} = \overrightarrow{x}_0 e^{-i\omega t}$. We then have:

$$(-\omega^2 - i\omega\Gamma + \omega_0^2)\overrightarrow{x}_0 = \hat{\epsilon}\frac{q}{m}E_0 e^{i\overrightarrow{k}.\overrightarrow{x}} \tag{4.125}$$

assume $\overrightarrow{k}.\overrightarrow{x} = 0$, that is $|x| \ll \lambda$ (e- orbit is small compared to radiation wavelength). Then

$$\overrightarrow{x}_0 \simeq \frac{\frac{e}{m}E_0}{\omega_0^2 - \omega^2 - i\omega\Gamma}\hat{\epsilon}, \tag{4.126}$$

and

$$\overrightarrow{a} = -\omega^2\overrightarrow{x} \Rightarrow |a^2| = \left(\frac{e}{m}E_0\right)^2\frac{\omega^4}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}. \tag{4.127}$$

$$|a^2| = \frac{\left(\frac{e}{m}E_0\right)^2}{\left(\frac{\omega_0}{\omega}\right)^2 + \left[\left(\frac{\Gamma}{\omega}\right)^2 - 1\right]^2}. \tag{4.128}$$

Same as before but modified to include $\overrightarrow{a}$'s denominator. So finally for a bounded e-, we get:

$$\langle\frac{dP}{d\Omega}\rangle = \frac{c}{16\pi} r_e^2 E_0^2 \frac{1 + \cos^2\theta}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + \left(\frac{\Gamma}{\omega}\right)^2}, \tag{4.129}$$

and the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \frac{1 + \cos^2\theta}{\left[\left(\frac{\omega_0}{\omega}\right)^2 - 1\right]^2 + \left(\frac{\Gamma}{\omega}\right)^2}, \tag{4.130}$$

The limit $\omega \gg \omega_0$, $\omega \gg \Gamma$ corresponds to Thomson scattering, while the limit $\omega \ll \omega_0$, $\omega \gg \Gamma$ gives the Rayleigh formula:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \left(\frac{\omega}{\omega_0}\right)^4 [1 + \cos^2\theta] \propto \omega^4 \tag{4.131}$$

So high frequencies are scattered more preferably than low frequencies. This explains why the sky is blue...

Chapter 5

Scattering

5.1 introduction

Two types of scattering:
• e- ($q = -e, m_e = 9.1 \times 10^{-31}$ kg)
$\rightarrow$ high energy loss, small deflection,
• nuclei ($q = Ze, m_n \gg m_e$)
$\rightarrow$ low energy loss, large deflection
There are more e- than nuclei (factor Z) so Z more time e- scattering...

5.2 Energy transfer

Impulse approximation (IA):
- incident particle is not deflected by collision,
- target particle is stationary during collision.
E-field at target is: (we ignore magnetic field since e is stationary (in IA).

$$\overrightarrow{E}(x = -b, y = z = 0, t) = -\gamma q \frac{b\hat{x} + vt\hat{z}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} \tag{5.1}$$

The momentum transfer from q to e is:

$$\begin{aligned} \Delta \overrightarrow{p} &= \int_{-\infty}^{+\infty} dte\overrightarrow{E} = -qe\gamma \int_{-\infty}^{+\infty} dt \frac{b\hat{x} + vt\hat{z}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \qquad (5.2) \\ &= -qe\gamma \int_{-\infty}^{+\infty} dt \frac{b\hat{x}}{(b^2 + \gamma^2 v^2 t^2)^{3/2}} = \frac{-qe}{vb}\hat{x} \int_{-\infty}^{+\infty} \frac{du}{(1 + u^2)^{3/2}}, \\ &= \frac{-qe}{vb}\hat{x} \left[\frac{u}{\sqrt{1 + u^2}} \right]_{-\infty}^{+\infty} = -\frac{2qe}{vb}\hat{x}. \qquad (5.3) \end{aligned}$$

The associated energy change is:

$$\Delta T = \sqrt{(\Delta p_e c)^2 + (mc^2)^2} - mc^2 \simeq \frac{2}{m} \left(\frac{qe}{vb} \right)^2 \tag{5.4}$$

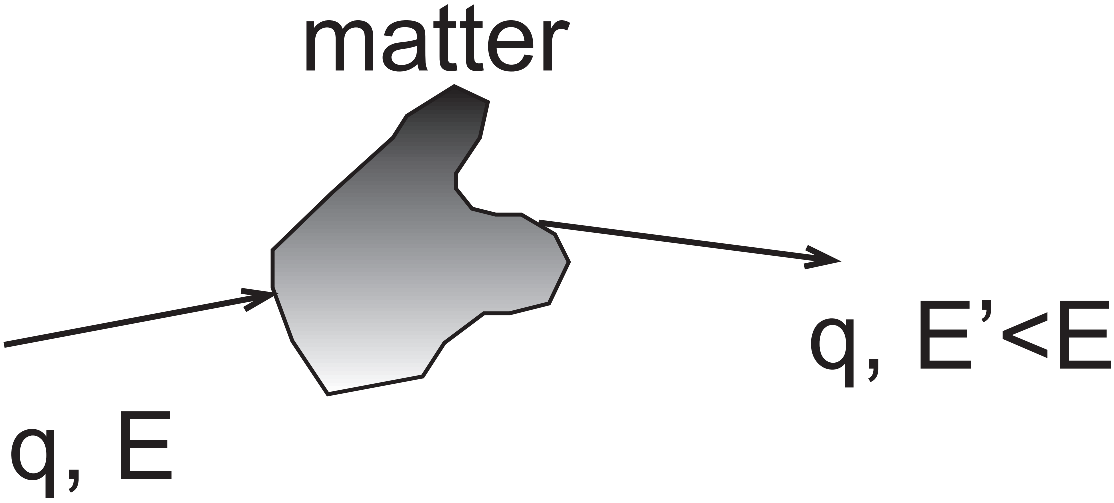

Figure 5.1:

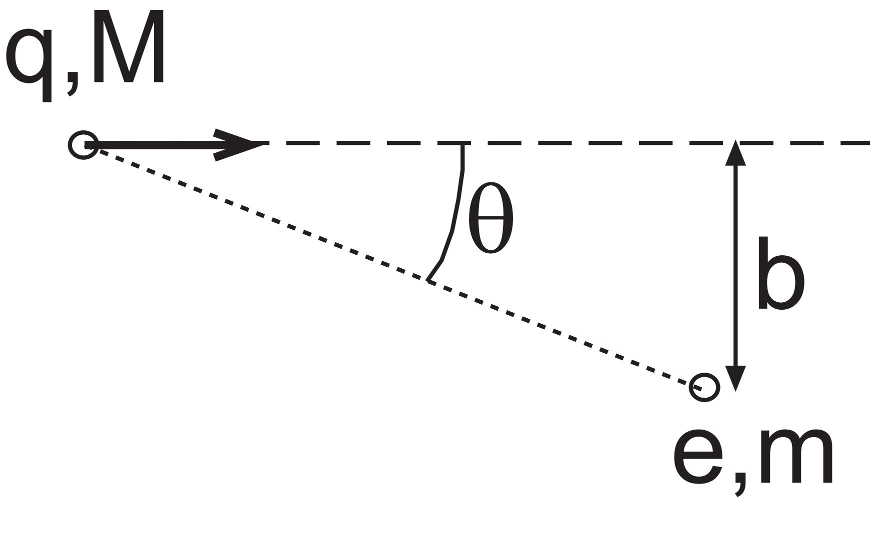

Figure 5.2: ???

where the RHS approximation is written in the non-relat. (NR) limit ($\Delta p \ll mc$). For e-: $\Delta T_e \equiv \Delta T \propto \frac{e^2}{m}$. For nuclei $\Delta T_n \propto \frac{q_n^2}{m_n} \propto \frac{Z^2e^2}{m_n}$. Hence

$$\frac{\Delta T_n}{\Delta T_e} = \left(\frac{q_n}{e}\right)^2 \frac{m_e}{m_n} = Z^2 \frac{m_e}{m_n} = \frac{Z}{1836} \ll 1. \tag{5.5}$$

So we see that e- are much more efficient that nuclei at extracting energy from incident particles. But when is the IA valid? Let's check the assumptions:
1- incident particle travels on straight path:

$$\theta = \frac{\Delta p_e}{\gamma M v} = \frac{2qe}{\gamma M v^2 b} = \frac{2}{\gamma}\frac{qe/b}{Mv^2} = 2\frac{\text{E electrostat.}}{\text{incident Energy}} = 2\frac{\mathcal{V}}{E}. \tag{5.6}$$

So $\theta \ll 1 \Rightarrow \mathcal{V} \ll E$.
2- target remains stationary $\Rightarrow$ recoil distance during collision is $d \ll b$. The interation time is $\tau \sim \frac{b}{\gamma v}$, and corresponding recoil distance is $d \sim \frac{\Delta p_e}{m}\tau$ so

$$d \ll b \Rightarrow \frac{\Delta p_e}{m}\frac{b}{\gamma v} \ll 1 \Rightarrow \frac{2qe/b}{\gamma m v^2} \ll 1. \tag{5.7}$$

this is a stronger condition than the one coming from $\theta \ll 1$ by a factor M/m.

So let's keep the stronger condition (and rewrite it by making the classical radius of e- appearing):

$$\frac{2}{\gamma}\frac{q}{e}\frac{e^2/(mc^2)}{b}\frac{c^2}{v^2} \ll 1 \tag{5.8}$$

So IA is valid when:

$$\frac{2}{\beta\gamma}\frac{q}{e}\frac{r_e}{b} \ll 1 \tag{5.9}$$

The NR limit implies:

$$\begin{aligned}\frac{\Delta p_e}{mc} &\ll 1 \Rightarrow \frac{2qe}{mvcb} \ll 1 \Rightarrow 2\frac{q}{e}\frac{e^2/(mc^2)}{\beta b} \ll 1 \\ &\Rightarrow \frac{2}{\beta}\frac{q}{e}\frac{r_e}{b} \ll 1.\end{aligned} \tag{5.10}$$

Same as condition for IA to be valid but with $\gamma \to 1$.

Now consider the case when the charge q passes through a bulk material (many e-), let n_e be the electron density. We have to add all their respective energy gain to deduce their influence on q's slowing down. We do not need to consider the nuclei to a good approximation. The total number of e- in a cylindrical shell of radius b and thickness db is:

$$N_e = n_e(vdt)(2\pi b db) \tag{5.11}$$

The differential energy loss by the charge q is:

$$\frac{d^2T_q}{dtdb} = -2\pi n_e v b \left[\frac{2}{m}\left(\frac{qe}{bv}\right)^2\right] \tag{5.12}$$

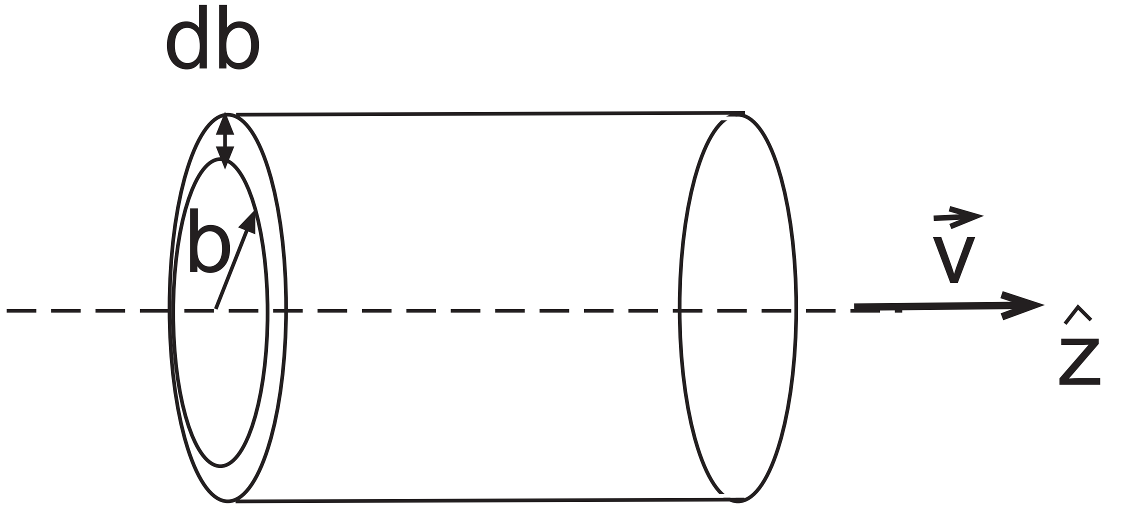

Figure 5.3: ????

(minus comes from energy is lost by q). Integrating over b gives:

$$\begin{aligned} \frac{dT_q}{dt} &= -4\pi n_e \frac{(qe)^2}{mv} \int_{b_{min}}^{b_{max}} \frac{db}{b} && (5.13) \\ &= -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{b_{max}}{b_{max}}. && (5.14) \end{aligned}$$

We cannot integrate from 0 to ∞. When $b \to 0$, IA is no more valid. IA is valid when

$$\frac{2}{\beta^2\gamma} \frac{q}{e} \frac{r_e}{b} \ll 1 \qquad (5.15)$$

$\Rightarrow$ Take $b_{min} \equiv \frac{2}{\beta^2\gamma}\frac{q}{e} r_e = \frac{qe}{\gamma m v^2}$.

Choice of b_{max}: e- are bounded with energy E_e. Their orbits have angular frequency $\omega_e = E_e/h$. We must have the collision time $\tau \ll \omega_0^{-1}$ otherwise target not stationary and IA not applicable. This gives:

$$\tau \sim \frac{b}{\gamma v}; \ \frac{b_{max}}{\gamma v} = \frac{1}{\omega_0} \qquad (5.16)$$
$$\Rightarrow b_{max} = \frac{\gamma v}{\omega_0}.$$

Then

$$\begin{aligned} \frac{dT_q}{dt} &= -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\frac{\gamma v}{\omega_0}}{\frac{qe}{\gamma m v^2}} = -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\gamma v}{\omega_0 b_{min}}, && (5.17) \\ &= -4\pi n_e \frac{(qe)^2}{mv} \ln \frac{\gamma^2 m v^3}{q e \omega_0} && (5.18) \end{aligned}$$

Note that $d\Delta T_q/dt = dE/dt$ (E is total energy of q), and $(1/v)d/dt = d/dz$ so we can write:

$$\frac{dE}{dz} = -4\pi n_e \frac{(qe)^2}{mv^2} \ln \frac{\gamma^2 m v^3}{q e \omega_0} \ \text{[JDJ Eq (13.9)]}. \qquad (5.19)$$

This equation has been derived under the IA. Compare to Bohr's results (1915) more carefully derived:

$$\frac{dE}{dz} = -4\pi n_e \frac{(qe)^2}{mv^2}\left[\ln\frac{1.123\gamma^2 mv^2}{qe\langle\omega\rangle} - \frac{1}{2}\frac{v^2}{c^2}\right], \tag{5.20}$$

where $\langle\omega\rangle$ represents the average angular frequency of the bound electron in target. The agreement between Bohr's and the equation we derived is no bad: the IA seems to contain the essential physics.

5.3 Influence of Dielectric Screening

For particles not too relativistic the observed energy loss is accurately given by the Bohr's formula for all kinds of particles in all types of media. For ultra-relativistic particles observed energy loss less than what predicted with Bohr formula $\Rightarrow$ reduction of energy loss is due to "density" effects.

In dense media, dielectric polarization alter the particle's field compared to free-space

Problem of finding field in the medium can be solved using the Fourier transform. Consider a dielectric medium, $\epsilon = \epsilon(\omega)$, $\mu = 1$. In Gaussian units we have:

$$\Box A^\alpha = \frac{4\pi}{c} J^\alpha \tag{5.21}$$

where $\Box \equiv \partial_\alpha \partial^\alpha = \frac{\epsilon}{c^2} - \partial_t^2 \nabla^2$. $A^\alpha = (\Phi, \overrightarrow{A})$ and $J^\alpha = (\rho c/\epsilon, \overrightarrow{J})$.
Define

$$\begin{aligned}
F(\overrightarrow{x}, t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega F(\overrightarrow{x}, \omega) e^{-i\omega t} \\
F(\overrightarrow{x}, \omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\omega F(\overrightarrow{x}, t) e^{+i\omega t} \\
F(\overrightarrow{x}, \omega) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} d\overrightarrow{k} F(\overrightarrow{k}, \omega) e^{\overrightarrow{k}.\overrightarrow{x}} \\
F(\overrightarrow{k}, \omega) &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} d\overrightarrow{k} F(\overrightarrow{x}, \omega) e^{-\overrightarrow{k}.\overrightarrow{x}}
\end{aligned}$$

two first eqns: Fourier transform in time, two last eqns: Fourier transform in space.
The source of the field is the incident point charge (q) so we have:

$$\begin{aligned}
\rho(\overrightarrow{x}, t) &= q\delta(\overrightarrow{x} - \overrightarrow{v}t) \\
J(\overrightarrow{x}, t) &= \overrightarrow{v}\rho(\overrightarrow{x}, t)
\end{aligned}$$

The time and space Fourier transform is:

$$\begin{aligned}
\rho(\overrightarrow{k}, \omega) &= \frac{q}{(2\pi)^2}\int d\overrightarrow{x}\int dt[q\delta(\overrightarrow{x} - \overrightarrow{v}t)]e^{-i(\overrightarrow{k}.\overrightarrow{x}-\omega t)} \\
&= \frac{q}{(2\pi)^2}\int dt e^{-i(\overrightarrow{k}.\overrightarrow{v}-\omega)t} = \frac{q}{2\pi}\delta(\omega - \overrightarrow{k}.\overrightarrow{v})
\end{aligned} \tag{5.22}$$

So finally:

$$\rho(\overrightarrow{k},\omega) = \frac{q}{2\pi}\delta(\omega - \overrightarrow{k}.\overrightarrow{v}); \text{ and } \overrightarrow{J}(\overrightarrow{k},\omega) = \frac{q\overrightarrow{v}}{2\pi}\delta(\omega - \overrightarrow{k}.\overrightarrow{v}) \tag{5.23}$$

Now transform the wave equation in the Fourier space:
$\square A^\alpha \rightarrow (k^2 - \epsilon(\omega)\frac{\omega^2}{c^2})A^\alpha = \frac{4\pi}{c}J^\alpha$
So that:

$$A^\alpha = \frac{4\pi J^\alpha}{c[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} \tag{5.24}$$

or

$$\begin{aligned} \overrightarrow{A} &= \frac{4\pi\overrightarrow{J}}{c[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} = \frac{\overrightarrow{v}}{c}\epsilon(\omega)\Phi(\overrightarrow{k},\omega) \\ \Phi &= \frac{4\pi\rho}{\epsilon(\omega)[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]} = \frac{2q\delta(\omega - \overrightarrow{k}.\overrightarrow{v})}{\epsilon(\omega)[k^2 - \epsilon(\omega)\frac{\omega^2}{c^2}]}. \end{aligned}$$

The electric field is then given by
$\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \frac{1}{c}\frac{\partial A}{\partial t}$ or $\overrightarrow{E}(\overrightarrow{k},\omega) = i(\frac{\omega}{c^2}\epsilon(\omega)\overrightarrow{v} - \overrightarrow{k})\Phi$ and
$\overrightarrow{B} = \overrightarrow{\nabla}\times\overrightarrow{A} \rightarrow i\overrightarrow{k}\times\overrightarrow{A} = i\frac{\epsilon(\omega)}{c}\overrightarrow{k}\times\overrightarrow{v}\Phi$. Hence

$$\begin{pmatrix} \overrightarrow{E}(\overrightarrow{k},\omega) \\ \overrightarrow{B}(\overrightarrow{k},\omega) \end{pmatrix} = i\begin{pmatrix} \frac{\omega\epsilon(\omega)}{c^2}\overrightarrow{v} - \overrightarrow{k} \\ \frac{\epsilon(\omega)}{c}\overrightarrow{k}\times\overrightarrow{v} \end{pmatrix}\Phi(\overrightarrow{k},\omega) \tag{5.25}$$

We want to find the flow of energy away from the incident particle's trajectory $\Rightarrow$ find the Poynting flux $\Rightarrow$ find $\overrightarrow{E}(\overrightarrow{x},\omega)$ and $\overrightarrow{B}(\overrightarrow{x},\omega)$:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{i}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty} d\overrightarrow{k}[\frac{\omega}{c}\epsilon(\omega)\frac{\overrightarrow{v}}{c} - \overrightarrow{k}]\Phi(\overrightarrow{k},\omega)e^{+i\overrightarrow{k}.\overrightarrow{x}} \tag{5.26}$$

Let's specify the problem: consider $\overrightarrow{x} = b\hat{x}$ and take $\overrightarrow{v} = v\hat{z}$.

$$\begin{aligned} \overrightarrow{E}(\overrightarrow{x},\omega) &= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int\int dk_x dk_y dk_z \left[\frac{\omega}{c}\epsilon(\omega)\frac{\overrightarrow{v}}{c} - \overrightarrow{k}\right]\frac{\delta(\omega - k_z v)}{k^2 - \epsilon\frac{\omega^2}{c^2}}e^{ik_x b} \\ &= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int\int dk_x dk_y dk_z \left[-k_x\hat{x} - k_y\hat{y} + \left(\frac{\omega}{c}\frac{v}{c}\epsilon(\omega) - k_z\right)\hat{z}\right] \\ &\quad \times\frac{\delta(\omega - k_z v)}{k^2 - \epsilon\frac{\omega^2}{c^2}}e^{ik_x b}. \end{aligned}$$

the term in $k_y\hat{y}$ has no contribution to the integral of k_y. Let's integrate over dk_z:

$$\begin{aligned} \overrightarrow{E}(\overrightarrow{x},\omega) &= \frac{i}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\int\int dk_x dk_y \left[-k_x\hat{x} + \frac{\omega}{v}\left(\frac{v^2}{c^2}\epsilon - 1\right)\hat{z}\right] \\ &\quad \times\frac{e^{ik_x b}}{k_x^2 + k_y^2 + \left(\frac{\omega}{v}\right)^2(1 - \epsilon\frac{v^2}{c^2})} \end{aligned} \tag{5.27}$$

Let $\lambda \equiv \frac{\omega}{v}\sqrt{1-\epsilon\frac{v^2}{c^2}}$ and $\mathcal{I} \equiv \int\int dk^2 \frac{e^{ik_x b}}{k_x^2+k_y^2+\lambda^2}$. Then the E-field takes the form:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{1}{(2\pi)^{3/2}}\frac{2q}{\epsilon}\left[-\frac{d\mathcal{I}}{db}\hat{x} + \frac{i\omega}{v}(\epsilon\beta^2-1)\mathcal{I}\hat{z}\right] \quad (5.28)$$

The integration over dk_y gives:

$$\int_{-\infty}^{+\infty} dk_y \frac{1}{k_x^2+k_y^2+\lambda^2} = \frac{\arctan\left(\frac{k_y}{\sqrt{k_x^2+\lambda^2}}\right)}{\sqrt{k_x^2+\lambda^2}}\Bigg|_{-\infty}^{+\infty} = \frac{\pi}{\sqrt{k_x^2+\lambda^2}} \quad (5.29)$$

So

$$\begin{aligned}\mathcal{I} &= \pi\int_{-\infty}^{\infty} dk_x \frac{e^{ik_x b}}{\sqrt{k_x^2+\lambda^2}} = \pi\int_0^{\infty} dk_x \frac{e^{ik_x b}+e^{-ik_x b}}{\sqrt{k_x^2+\lambda^2}} \\ &= 2\pi\int_0^{+\infty} dk_x \frac{\cos(k_x b)}{\sqrt{k_x^2+\lambda^2}} = 2\pi K_0(b\lambda). \end{aligned} \quad (5.30)$$

and $\frac{d\mathcal{I}}{db} = -2\pi\lambda K_1(b\lambda)$.

So finally the E-field re-writes:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \sqrt{\frac{2}{\pi}}\frac{q}{v}\left[\frac{\lambda}{\epsilon}\lambda K_1(b\lambda)\hat{x} - i\frac{\omega}{v}(\epsilon\beta-1)K_0(b\lambda)\hat{z}\right]. \quad (5.31)$$

Now let's find the B-field

$$\begin{aligned}\overrightarrow{B}(\overrightarrow{k},\omega) &= i\frac{\epsilon(\omega)}{c}\overrightarrow{k}\times\overrightarrow{v}\Phi(\overrightarrow{k},\omega) = i\epsilon\frac{v}{c}(-k_x\hat{y}+k_y\hat{x})\Phi(\overrightarrow{k},\omega) \\ &= \frac{i}{(2\pi)^{3/2}}\int\int\int dk_x dk_y dk_z\left[-k_x\hat{y}+k_y\hat{x}\right]\frac{\epsilon v}{c}\frac{2q\delta(\omega-\overrightarrow{k}.\overrightarrow{v})}{\epsilon[k^2-\epsilon\frac{\omega^2}{c^2}]}e^{i\overrightarrow{k}.\overrightarrow{x}} \\ &= -\frac{i}{(2\pi)^{3/2}}2q\frac{v}{c}\hat{y}\int\int\int dk_x dk_y dk_z k_x \frac{\delta(\omega-k_z v)}{k_x^2+k_y^2+k_z^2-\epsilon\frac{\omega^2}{c^2}}e^{i\overrightarrow{k}.\overrightarrow{x}}.\end{aligned}$$

the term in $k_y\hat{x}$ has no contribution to the integral in dk_y. Integrate over dk_z:

$$\overrightarrow{B} = -\frac{i}{(2\pi)^{3/2}}2q\frac{v}{c}\hat{y}\int\int dk_x dk_y k_x \frac{e^{ik_x b}}{k_x^2+k_y^2+\lambda^2} = -\frac{i}{(2\pi)^{3/2}}2q\frac{1}{c}\hat{y}\frac{d\mathcal{I}}{db}.$$

By inspection this is the same as $\hat{x}$-component of $\overrightarrow{E}$ thus

$$\overrightarrow{B} = \sqrt{\frac{2}{\pi}}\frac{q}{c}\lambda K_1(b\lambda)\hat{y}. \quad (5.32)$$

Now that we have $\overrightarrow{E}$ and $\overrightarrow{B}$ we are in position of computing the e.m. field energy flowing out of a cylindrical surface of radius b extending from $-\infty$ to $+\infty$ in z:

$$\frac{d\mathcal{E}_f}{dz} = 2\pi b\int_{-\infty}^{\infty}\overrightarrow{S}.\hat{n}dt = 2\pi b\frac{1}{4\pi}\int_{-\infty}^{+\infty}(\overrightarrow{E}\times\overrightarrow{B}).\hat{n}dt \quad (5.33)$$

we have:

$$\begin{aligned}
(\overrightarrow{E} \times \overrightarrow{B}).\hat{n} &= [(E_x\hat{x} + E_z.\hat{z}) \times B_y\hat{y}].\hat{n} \\
&= (E_xB_y\hat{z} - E_zB_y\hat{x}).\hat{n} = -E_zB_y \qquad (5.34)
\end{aligned}$$

So

$$\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= -\frac{b}{2}\int_{-\infty}^{\infty} E_zB_ydt \\
&= -\frac{b}{4\pi}\int_{-\infty}^{\infty} dt \left[\int_{-\infty}^{\infty} d\omega E_z(\omega)e^{-i\omega t}\right]\left[\int_{-\infty}^{\infty} d\omega' B_y(\omega')e^{-i\omega' t}\right] \\
&= -\frac{b}{2}\int_{-\infty}^{+\infty} E_z(\omega)B_y(-\omega)d\omega = -\frac{b}{2}\int_{-\infty}^{+\infty} E_z(\omega)B_y^*(\omega)d\omega \\
&= \mathcal{R}e\left(-b\int_0^{+\infty} E_z(\omega)B_y^*(\omega)d\omega\right) \qquad (5.35)
\end{aligned}$$

Expliciting E_z and B_y we have:

$$\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= -b\mathcal{R}e\left\{\int_0^{+\infty} d\omega\left[-i\sqrt{\frac{2}{\pi}}\frac{q}{v}\frac{\omega}{v}(1/\epsilon - \beta^2)K_0(\lambda b)\right]\right. \\
&\quad \left.\times\left[\sqrt{\frac{2}{\pi}}\frac{q}{c}\lambda^*K_1(\lambda^* b)\right]\right\} \qquad (5.36)
\end{aligned}$$

$$\begin{aligned}
\frac{d\mathcal{E}_f}{dz} &= \mathcal{R}e\left\{\int_0^{+\infty} d\omega\frac{2}{\pi}\frac{q^2}{v^2}[i\omega(1/\epsilon - \beta^2)\lambda^* b]K_0(\lambda b)K_1(\lambda^* b)\right\} \\
&= \frac{2}{\pi}\frac{q^2}{v^2}\mathcal{R}e\left\{\int_0^{+\infty} d\omega(i\omega\lambda^* b)(1/\epsilon - \beta^2)K_0(\lambda b)K_1(\lambda^* b)\right\} \qquad (5.37)
\end{aligned}$$

The equation was first derived by Fermi. Note that λ or ϵ need to be complex to have $\frac{d\mathcal{E}_f}{dz} \neq 0$.
To proceed with our calculation we now need to introduce a model for $\epsilon(\omega)$. Use the same model as the one used to study Thomson Scattering: we model the bound target electron as a damped harmonic oscillator:

$$\overrightarrow{x}(\omega) = \frac{-\frac{e}{m}\overrightarrow{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\Gamma} \qquad (5.38)$$

The dipole moment is just $-e\overrightarrow{x}$ and the polarization is defined as the dipole moment density that is $-n_e e\overrightarrow{x}$:

$$\begin{aligned}
\overrightarrow{P}(\omega) &= \frac{n_e e^2}{m}\frac{\overrightarrow{E}(\omega)}{\omega_0^2 - \omega^2 - i\omega\Gamma} \\
&= \frac{\epsilon(\omega) - 1}{4\pi}\overrightarrow{E}(\omega). \qquad (5.39)
\end{aligned}$$

So we can write

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\Gamma} \tag{5.40}$$

wherein $\omega_p \equiv \sqrt{4\pi n_e e^2/m}$ is the plasma frequency. Now we just plug this into $\frac{d\mathcal{E}_f}{dz}$ and perform the integral.

Integral not so simple to perform. We follow JDJ's suggestion and use the "narrow resonance approximation"

$$\omega \simeq \omega_0 \Rightarrow b\lambda = b\frac{\omega}{v}\sqrt{1-\epsilon\beta^2} \sim b\frac{\omega_0}{v}\sqrt{1-\epsilon\beta^2} \tag{5.41}$$

So

$$b\frac{\omega_0}{v} \simeq 2\pi\frac{b}{\lambda_e} \tag{5.42}$$

$b\lambda \ll 1$ if $b <$ an atomic radius.Then, using the small argument approximation for the modified bessel function we have: (see JDJ Eq. 3.103)

$$\begin{aligned} b\lambda^* K_1(b\lambda^*) &\sim b\lambda^*\frac{1}{b\lambda^*} \sim 1 \\ K_0(b\lambda) &\sim \ln 2 - \ln(b\lambda) - \gamma = \ln\left(\frac{2e^{-\gamma}}{2\gamma}\right) = \ln\left(\frac{1.123}{b\lambda}\right) \end{aligned} \tag{5.43}$$

$\gamma = 0.577$ Euler constant.

$$\begin{aligned} \frac{d\mathcal{E}_f}{dz} &= \frac{2}{\pi}q^2v^2\mathcal{R}e\left\{\int_0^{+\infty} d\omega i\omega(1/\epsilon - \beta^2)\ln\left(\frac{1.123}{b\lambda}\right)\right\} \\ &\equiv \frac{2}{\pi}q^2v^2\mathcal{R}e(\mathcal{I}) \end{aligned} \tag{5.44}$$

where $\mathcal{I} \equiv \int_0^{+\infty} d\omega i\omega(\frac{\epsilon-1}{\epsilon})\ln\left(\frac{1.123}{b\lambda}\right)$ (we took $\beta = 1$). Explicit $\epsilon(\omega)$ [recall that $b\lambda = \frac{b\omega}{c}\sqrt{1-\epsilon}$]:

$$\begin{aligned} \mathcal{I} &= i\int_0^{+\infty} d\omega\omega\left(\frac{\omega_p^2}{\omega_p^2+\omega_0^2-\omega^2-i\omega\Gamma}\right)\left[\ln\left(\frac{1.123c}{b\omega_p}\right) - \ln\omega + \right. \\ &\left. + \frac{1}{2}\ln(\omega^2-\omega_0^2+i\omega\Gamma)\right] \end{aligned} \tag{5.45}$$

Now let's perform the integration in the complex plane...

Two sources of poles: $-\omega_0^2+\omega^2+i\omega\Gamma = 0$ from the ln(...), and $\omega_p^2+\omega_0^2-\omega^2-i\omega\Gamma = 0$ from denominator of $\frac{1-\epsilon}{\epsilon}$. All the poles are in the lower part of the complex plane. Consider the integral along $\mathcal{C}$. This gives:

$$I_1 + I_2 + I_3 = 0$$

, so $\mathcal{I} = iI_1 = i(-I_2 - I_3)$. The i comes from the fact we drop the i when evaluating the integrals I_n.

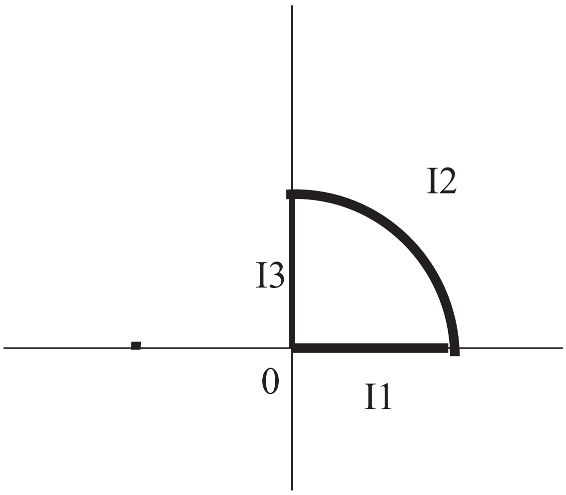

Figure 5.4: ???

Let's evaluate the integrals:

$$I_3 = \int_{+i\infty}^{0} d\omega\omega(...)\ln(...) \tag{5.46}$$

Let $\omega \equiv i\Omega$ with $\Omega \in \mathbb{R}$ then:

$$\begin{aligned} I_3 &= -\int_{\infty}^{0} d\Omega\Omega(...)\ln(...) = \int_{0}^{\infty} d\Omega\Omega\frac{\omega_0^2+\Omega^2+\Omega\Gamma}{\omega_p^2+\omega_0^2+\Omega^2+\Omega\Gamma} \\ &\quad\times\left(\ln\frac{1.123c}{b\omega_p} - \ln i\Omega + \frac{1}{2}\ln(-\Omega^2-\omega_0^2-\Omega\Gamma)\right) \end{aligned} \tag{5.47}$$

the bracket simplifies:

$$(...) = \ln\frac{1.123c}{b\omega_p} - \ln i - \ln\Omega + \frac{1}{2}\ln(-1) + \frac{1}{2}\ln(\Omega^2+\omega_0^2+\Omega\Gamma) \tag{5.48}$$

the $\ln i$ and $1/2\ln(-1)$ cancel each other. So I_3 becomes:

$$\begin{aligned} I_3 &= \int_{0}^{\infty} d\Omega\Omega\frac{\omega_0^2+\Omega^2+\Omega\Gamma}{\omega_p^2+\omega_0^2+\Omega^2+\Omega\Gamma} \\ &\quad\times\left(\ln\frac{1.123c}{b\omega_p} - \ln\Omega + \frac{1}{2}\ln(\Omega^2+\omega_0^2+\Omega\Gamma)\right) \end{aligned} \tag{5.49}$$

So I_3 is real, so iI_3 is pure imaginary and therefore its contribution to $\mathcal{R}e\mathcal{I}$ is zero. Now consider I_2, let $\omega \equiv Re^{i\theta}$, then

$$\begin{aligned} I_2 &= \lim_{R\to\infty}\int_{0}^{\pi/2} id\theta Re^{i\theta}Re^{i\theta}\frac{\omega_p^2}{\omega_p^2+\omega_0^2-R^2e^{2i\theta}-iRe^{i\theta}\Gamma} \\ &\quad\times\left(\ln\frac{1.123c}{b\omega_p} - \ln Re^{i\theta} + \frac{1}{2}\ln(-\omega_0^2+R^2e^{2i\theta}+iRe^{i\theta}\Gamma)\right) \end{aligned} \tag{5.50}$$

Taking the limit $R =\longrightarrow \infty$, we get:

$$I_2 = \int_0^{\pi/2} id\theta\omega_p^2 \ln\frac{1.123c}{b\omega_p} = i\frac{\pi\omega_p^2}{2}\ln\frac{1.123c}{b\omega_p} \quad (5.51)$$

So finally energy loss is:

$$\frac{d\mathcal{E}_f}{dz} = \frac{2}{\pi}q^2v^2\mathcal{R}e(\mathcal{I}) = -\frac{q^2\omega_p^2}{c^2}\ln\frac{1.123c}{b\omega_p}. \quad (5.52)$$

where we have taken $v = c$. On another hand we have derived at the beginning of Part V the energy loss under the impulse approximation to be:

$$\frac{d\mathcal{E}_f}{dz} = -4\pi n_e\frac{(qe)^2}{mv}\ln\frac{\gamma v^2}{\omega_0 b} = -\frac{q^2\omega_p^2}{c^2}\ln\frac{\gamma c}{b\omega_0}. \quad (5.53)$$

note that we have actually derived $\frac{d\mathcal{E}_f}{dt}$, we also took $v = c$ in the latter equation.

The influence of dielectric screening is two-folds:

1- It removes the dependence of energy loss on atomic structure ω_0 is replaced by ω_p which only depend on the density number of e- (and not on their binding energy).

2- It reduces the energy loss from highly relativistics incident charge, the γ in the argument of ln is gone.

5.4 Cerenkov radiation

We now consider density effect in the extreme limit $b\lambda \gg 1$ and look at the energy deposited in the target. The large argument approximation for the modified bessel function gives: $K_0(b\lambda) = K_1(b\lambda) = \sqrt{\frac{\pi}{2}}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}$. So the fields are:

$$\overrightarrow{E}(\overrightarrow{x},\omega) = \frac{q}{v}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}\left(\frac{\lambda}{\epsilon}\hat{x} - i\frac{\omega}{v}(\frac{1}{\epsilon} - \beta^2)\hat{z}\right), \quad (5.54)$$

$$\overrightarrow{B}(\overrightarrow{x},\omega) = \frac{q}{v}\frac{e^{-b\lambda}}{\sqrt{b\lambda}}\hat{y}. \quad (5.55)$$

to get radiation λ or $\epsilon \in \mathbb{C}$. Let's take $\epsilon \in \mathbb{R}$ (no dielectric screening). Then

$$\lambda = \frac{\omega}{v}\sqrt{1 - \epsilon(\omega)\beta^2} \quad (5.56)$$

To have $\lambda \in \mathbb{I}$, $1 - \epsilon\beta^2 < 0 \Rightarrow \epsilon\beta^2 > 1$, this is the Cerenkov condition.

Now replace the field in the expression for $\frac{d\mathcal{E}_f}{dz}$:

$$\frac{d\mathcal{E}_f}{dz} = \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^\infty d\omega(i\omega\lambda^* b)(\frac{1}{\epsilon} - \beta^2)K_0(b\lambda)K_1(b\lambda^*)\right) \quad (5.57)$$

$$= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^\infty d\omega(i\omega\lambda^* b)(\frac{1}{\epsilon} - \beta^2)\frac{e^{-b(\lambda+\lambda^*)}}{b\sqrt{\lambda\lambda^*}}\right) \quad (5.58)$$

$$= \frac{q^2}{v^2}\mathcal{R}e\left(\int_0^\infty i\omega\sqrt{\frac{\lambda^*}{\lambda}}(\frac{1}{\epsilon} - \beta^2)\right) \quad (5.59)$$

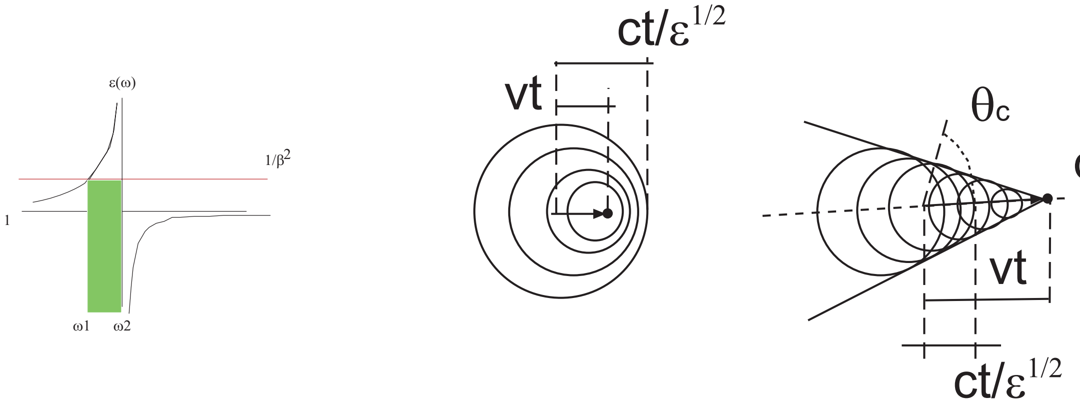

Figure 5.5: ???

but radiation only when $\lambda \in \mathbb{I}$ that is for a frequency band $\omega \in [\omega_l, \omega_0]$.

$$\frac{d\mathcal{E}_f}{dz} = \frac{q^2}{v^2}\int_{\omega_L}^{\omega_0} d\omega\omega\left(1-\frac{1}{\epsilon\beta^2}\right) \tag{5.60}$$

This is Frank-Tamm (1937) equation.

The propagation direction of the wave is given by $\overrightarrow{k}$ and k is perpendicular to $\overrightarrow{E}$ and $\overrightarrow{B}$. So if $\theta_c \angle(\overrightarrow{v}, \overrightarrow{k})$ then

$$\begin{aligned}\cos\theta_c &= \frac{|E_x|}{|E|} = \frac{E_x}{\sqrt{E_x^2+E_z^2}} \\ &= \frac{\frac{\lambda}{\epsilon}}{\left[\left(\frac{\lambda}{\epsilon}\right)^2 - \frac{\omega^2}{v^2}\left(\frac{1}{\epsilon}-\beta^2\right)^2\right]^{1/2}}.\end{aligned} \tag{5.61}$$

introducing $\lambda^2 = (\omega/v)^2(1-\epsilon\beta^2)$, we finally obtain:

$$\cos\theta_c = \frac{1}{\sqrt{1-1+\beta^2\epsilon}} = \frac{1}{\beta\sqrt{\epsilon}} = \frac{c_m}{v} \tag{5.62}$$

wherein $c_m \equiv c/\sqrt{\epsilon}$ is the velocity of light in the medium; $c_m < c$ so $\cos\theta_c < 1$ and $\theta \in \mathbb{R}$.

The shock wave feature should be derivable from the e.m. potential,

$$\left(k^2 - \frac{\omega^2}{c_m^2}\right)\sqrt{\epsilon}\Phi(\overrightarrow{k},\omega) = \frac{4\pi}{\sqrt{\epsilon}}\rho(\overrightarrow{k},\omega)$$

$$\left(k^2 - \frac{\omega^2}{c_m^2}\right)\sqrt{\epsilon}\overrightarrow{\mathcal{A}}(\overrightarrow{k},\omega) = \frac{4\pi}{c_m}\overrightarrow{J}(\overrightarrow{k},\omega)$$

So in the medium, A^α takes the same form as in vacuum, under the renormalization $q \rightarrow q/\sqrt{\epsilon}$, $c \rightarrow c_m$. Using these potential we can directly get the Lienard-Wiechert potentials:

$$\begin{pmatrix}\sqrt{\epsilon}\Phi(\overrightarrow{x},t) \\ \overrightarrow{A}(\overrightarrow{x},t)\end{pmatrix} = \frac{q}{\sqrt{\epsilon}}\frac{1}{[\kappa R]_{ret}}\begin{pmatrix}1 \\ \frac{\overrightarrow{v}}{c_m}\end{pmatrix} \tag{5.63}$$

Let $\overrightarrow{\zeta} = \overrightarrow{x} - \overrightarrow{v}t$, $\overrightarrow{R} = \overrightarrow{x} - \overrightarrow{x}(t') = \overrightarrow{x} - \overrightarrow{v}t'$. So $\overrightarrow{R} = \overrightarrow{x} - \overrightarrow{v}t + \overrightarrow{v}(t-t') = \overrightarrow{\zeta} + v(t-t')$.

So $t - t' = \frac{R(t')}{c_m} = \frac{|\overrightarrow{\zeta} + \overrightarrow{v}(t-t')|}{c_m}$.

$\Rightarrow (t-t')^2 = \frac{1}{c_m^2}[\zeta^2 + 2\overrightarrow{\zeta}.\overrightarrow{v}(t-t') + v^2(t-t')^2]$. $\Rightarrow (v^2 - c_m^2)(t-t')^2 + 2\overrightarrow{\zeta}.\overrightarrow{v}(t-t') + \zeta^2 = 0$; solve to get

$$(t-t')_\pm = \frac{-\overrightarrow{\zeta}.\overrightarrow{v} \pm \sqrt{(\overrightarrow{\zeta}.\overrightarrow{v})^2 - (v^2 - c_m^2)\zeta^2}}{v^2 - c_m^2}. \tag{5.64}$$

For cherenkov radiation $v > c_m$ to obtain $t - t' > 0 \in \mathbb{R}$ we need: $\overrightarrow{\zeta}.\overrightarrow{v} < 0$ and $(\overrightarrow{\zeta}.\overrightarrow{v})^2 > (v^2 - c_m^2)\zeta^2$, which means $\zeta v \cos\theta < 0$ or $\theta > \pi/2$, and $\cos^2\theta > 1 - c_m^2/v^2$. So

$$\theta > \arccos(-\sqrt{1 - (c_m/v)^2}), \tag{5.65}$$

which lies in $[\pi/2, \pi]$.

So potential and fields exist at time t only within a cone which the apex lies at $\zeta = \overrightarrow{x} - \overrightarrow{v}t$ (i.e. the present position of incident charge) and for which the apex angle is $\pi - \arccos(-\sqrt{1 - (c_m/v)^2})$. The 4-potential is $A^\alpha = A^\alpha_- + A^\alpha_+$ where the $\pm$ corresponds to $(t-t')_\pm$. Now,

$$\begin{aligned}
[\kappa R]_{ret} &= |(1 - 1/c_m \overrightarrow{v}.\hat{n})\overrightarrow{R}| = |\overrightarrow{R} - \frac{\hat{n}}{c_m}\overrightarrow{v}.[\overrightarrow{\zeta} + \overrightarrow{v}(t-t')]| \\
&= |\overrightarrow{R} - \frac{\hat{n}}{c_m}\overrightarrow{\zeta}.\overrightarrow{v} - \frac{\hat{n}}{c_m}v^2(t-t')| \\
&= |\hat{n}[c_m(t-t') - \frac{\overrightarrow{\zeta}.\overrightarrow{v}}{c_m} - \frac{v^2}{c_m}(t-t')]| \\
&= \frac{1}{c_m}|(c_m^2 - v^2)(t-t') - \overrightarrow{\zeta}.\overrightarrow{v}|.
\end{aligned} \tag{5.66}$$

Expliciting $(t-t')$ in the latter equation (using 5.64), we get:

$$[\kappa R]_{ret} = \frac{\zeta}{c_m}\sqrt{c_m^2 - v^2\sin^2\theta} = \zeta\sqrt{1 - \frac{v^2}{c_m^2}\sin^2\theta}. \tag{5.67}$$

both for $(t-t')_\pm$. So the potentials are given by:

$$\begin{pmatrix} \sqrt{\epsilon}\Phi(\overrightarrow{x}, t) \\ \overrightarrow{A}(\overrightarrow{x}, t) \end{pmatrix} = \frac{2q}{\sqrt{\epsilon}} \frac{1}{\zeta\sqrt{1 - \frac{v^2}{c_m^2}\sin^2\theta}} \begin{pmatrix} 1 \\ \frac{\overrightarrow{v}}{c_m} \end{pmatrix} \tag{5.68}$$

The potentials have a singularity (a shock front) at $\sin^2\theta = (c_m/v)^2$, which corresponds to the earlier results $\cos^2\theta = 1 - (c_m/v)^2$. Note that when the frequency-dependence of ϵ is introduced the shock wave-front is smeared.

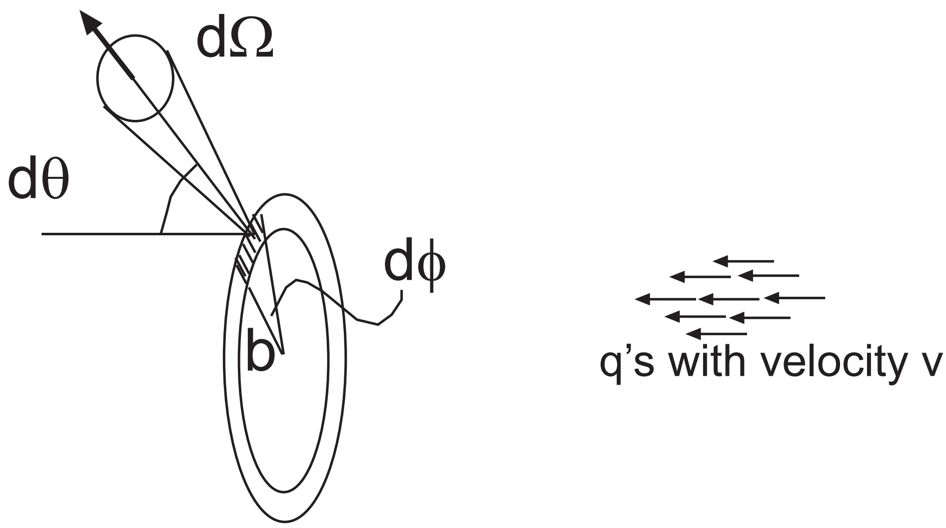

Figure 5.6: ???

5.5 Scattering

Thus far we have only looked at energy loss from charges incident to a target. Now let's look at momentum transfer that is scattering. Let N' be the number of incident particle scattered from $bdbd\phi$ into $d\Omega$ per unit time; we have:

$$d^2N = nvbdbd\phi = N'd\Omega \Rightarrow bdbd\phi = \frac{N'}{nv}d\Omega = \frac{d\sigma}{d\Omega}d\Omega; \Rightarrow \frac{d\sigma}{d\Omega} = \frac{N'}{nv}. \tag{5.69}$$

$$bdb = \frac{d\sigma}{d\Omega}\sin\theta d\theta \Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta} \tag{5.70}$$

Under the impulse approximation we have $\sin\theta \sim \theta \Rightarrow \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$. But

$$|\theta| = \frac{\Delta p}{p} = \frac{2qe}{\gamma bMv^2} \tag{5.71}$$

for target e$^-$ (see beginning of part V). so $b = \frac{2qe}{\gamma\theta Mv^2} \Rightarrow \left|\frac{db}{d\theta}\right| = \frac{2qe}{\gamma\theta^2 Mv^2}$. So finally,

$$\frac{d\sigma}{d\Omega} = \frac{b\theta}{\theta^2}\left|\frac{db}{d\theta}\right| = \left(\frac{2qe}{\gamma Mv^2}\right)^2\frac{1}{\theta^4} \tag{5.72}$$

For target Nuclei:

$$\frac{d\sigma}{d\Omega} = \frac{1}{\theta^4}\left(\frac{2qZe}{\gamma Mv^2}\right)^2 \tag{5.73}$$

this is the small-angle Rutherford formula. scattering by nuclei is Z^2 times stronger than by e-.

There are Z times more e- than nuclei, so the net effect is that nuclei scattering is Z times stronger than e- scattering.

Average deflection angle in a material: To get the mean-square deflection angle, evaluate:

$$\langle\theta^2\rangle = \frac{\int d\Omega\theta^2 d\sigma/d\Omega}{\int d\Omega d\sigma/d\Omega} \simeq \frac{\int d\theta\theta^3 1/\theta^4}{\int d\theta\theta 1/\theta^4} \tag{5.74}$$

$$= \frac{\int_{\theta_{min}}^{\theta_{max}} d\theta 1/\theta}{\int d\theta 1/\theta^3} = \frac{\ln\frac{\theta_{max}}{\theta_{min}}}{\frac{1}{2}(1/\theta_{min}^2 - 1/\theta_{max}^2)} \tag{5.75}$$

So $\langle\theta^2\rangle \simeq 2\theta_{min}^2 \ln\frac{\theta_{max}}{\theta_{min}}$ for a single scattering event. This is just few times θ_{min}^2 which is a small number.

Let's estimate θ_{min} from physical arguments: $b_{max} \simeq a$, the atomic radius because atomic electrons almost completely screen the nucleus if $b > a$. So

$$\theta_{min} = \frac{2qZe}{\gamma b_{max} M v^2} \simeq \frac{2qZe}{\gamma a M v^2} \tag{5.76}$$

$$\sim \frac{e^2}{a m_p c^2} \sim \frac{e^2/(m_e c^2)}{a m_p c^2} \sim \frac{r_e}{1836 a} \ll 1 \tag{5.77}$$

So to achieve a sizeable deflection angle, the incident charge needs either to undergo many small-angle scattering of a few large-angle scattering.

• Case of many small-angle scattering:
Net effect: charge q random-walk through the target $\langle\Theta^2\rangle = N\langle\theta^2\rangle$ and:

$$\Rightarrow \frac{d\langle\Theta^2\rangle}{dz} = n\sigma\langle\theta^2\rangle \simeq 2n\sigma\theta_{min}^2 \ln(\theta_{max}/\theta_{min}) \tag{5.78}$$

The distribution of angle after many small-scattering event (random-walk) is given by:

$$P_{RW}(\theta_p) \propto e^{\frac{-\theta_p^2}{2\langle\Theta_p^2\rangle}}.$$

• Case of few large-angle scattering:
Consider the distribution of scattering angle for a single scattering event:

$$\frac{d\sigma}{d\Omega} d\Omega = \left(\frac{2qZe}{\gamma M v^2}\right) \frac{1}{\theta^4} d\phi\theta d\theta \tag{5.79}$$

In terms of projected angle $\theta_p = \theta\sin\phi$, this takes the form:

$$\frac{d\sigma}{d\Omega} d\Omega = \left(\frac{2qZe}{\gamma M v^2}\right) \frac{1}{\theta_p^3} d\theta_p \sin^2\phi d\phi \tag{5.80}$$

Upon integration over ϕ we find that the distribution scales as:

$$P_1(\theta_p)d\theta_p \propto \frac{d\theta_p}{\theta_p^3} \leftrightarrow P_1(\theta_p) \propto \frac{1}{\theta_p^3}$$

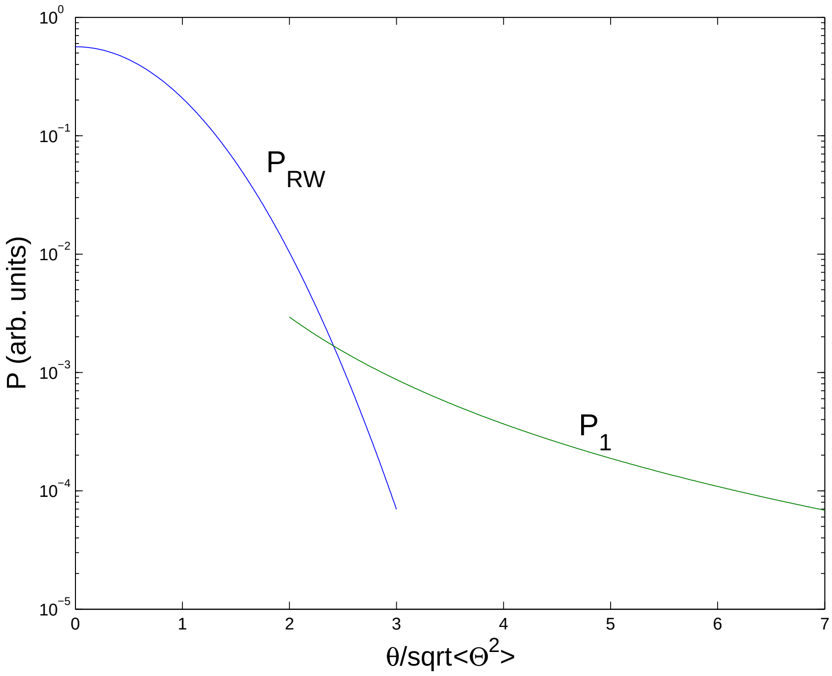

Figure 5.7: ???

Chapter 6

Standard Formula Sheet

Definite Integrals
For $n =$ non-negative integer,

$$\int_0^\infty x^n e^{-x}\,dx = n!$$
$$\int_0^\infty x^{2n} e^{-\beta x^2}\,dx = \frac{(2n)!\,\sqrt{\pi}}{n!\,(2\sqrt{\beta})^{2n+1}}$$
$$\int_0^\infty x^{2n+1} e^{-\beta x^2}\,dx = \frac{n!}{2\,\beta^{n+1}}$$

where $0! = 1$ and $n! = 1 \cdot 2 \ldots (n-1) \cdot n$.

Stirling's Approximation:

$$n! = \sqrt{2\pi}\, n^{n+1/2} \exp\left(-n + \frac{1}{12n} + \mathcal{O}(1/n^2)\right)$$
$$\ln(n!) = n\ln(n) - n \qquad \text{(for } n \gg 1\text{)}$$

Legendre Polynomials:

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3x)/2$,
$P_4(x) = (35x^4 - 30x^2 + 3)/8$, $P_5(x) = (63x^5 - 70x^3 + 15x)/8$

Spherical Harmonics:

$\ell = 0:$ $$Y_0^0(\theta,\phi) = \sqrt{\frac{1}{4\pi}}$$

$\ell = 1:$ $$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}\sin\theta$$

$\ell = 2:$ $$Y_2^0(\theta,\phi) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1) \qquad Y_2^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}e^{\pm i\phi}\sin\theta\cos\theta$$
$$Y_2^{\pm 2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}e^{\pm 2i\phi}\sin^2\theta$$

For all ℓ: $$Y_\ell^0(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$$

Numerical Constants:

$\hbar = 1.05 \times 10^{-27}\,\text{erg sec} = 1.05 \times 10^{-34}\,\text{J sec}$ $\qquad$ $a_0 = 0.529 \times 10^{-8}\,\text{cm} = 0.529 \times 10^{-10}\,\text{m}$

$\hbar c = 1.97 \times 10^{-7}\,\text{eV m}$ $\qquad$ $hc = 1.24 \times 10^{-6}\,\text{eV m}$

$e = 4.80 \times 10^{-10}\,\text{esu} = 1.60 \times 10^{-19}\,\text{C}$ $\qquad$ $c = 3.00 \times 10^{10}\,\text{cm/sec} = 3.00 \times 10^{8}\,\text{m/sec}$

$\epsilon_0 = 8.85 \times 10^{-12}\,\text{C}^2/\text{N m}^2$ $\qquad$ $\mu_0 = 4\pi \times 10^{-7}\,\text{N/A}^2$

$m_e = 9.11 \times 10^{-28}\,\text{g} = 9.11 \times 10^{-31}\,\text{kg} = 0.511\,\text{MeV}/c^2$

$m_p = 1.67262 \times 10^{-24}\,\text{g} = 1.67262 \times 10^{-27}\,\text{kg} = 938.272\,\text{MeV}/c^2$

$m_n = 1.67492 \times 10^{-24}\,\text{g} = 1.67492 \times 10^{-27}\,\text{kg} = 939.565\,\text{MeV}/c^2$

$N_0 = 6.02 \times 10^{23}\,\text{particles/mole}$

$k_B = 1.38 \times 10^{-23}\,\text{J K}^{-1} = 1.38 \times 10^{-16}\,\text{erg K}^{-1} = 8.62 \times 10^{-5}\,\text{eV K}^{-1}$

Spherical Coordinates (r, θ, ϕ)

Relations to rectangular (Cartesian) coordinates and unit vectors:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \qquad \begin{aligned} \hat{x} &= \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi \\ \hat{y} &= \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi \\ \hat{z} &= \hat{r}\cos\theta - \hat{\theta}\sin\theta \end{aligned}$$

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi &= \tan^{-1}(y/x) \end{aligned} \qquad \begin{aligned} \hat{r} &= \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta \\ \hat{\theta} &= \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta \\ \hat{\phi} &= -\hat{x}\sin\phi + \hat{y}\cos\phi \end{aligned}$$

Line element: $$d\vec{\ell} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

Volume element: $$d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$$

Gradient: $$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\,\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$$

Divergence: $$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\,\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,v_\theta) + \frac{1}{r\,\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

Curl: $$\begin{aligned}\vec{\nabla}\times\vec{v} = {} & \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r\,v_\phi)\right]\hat{\theta} \\ & + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\phi}\end{aligned}$$

Laplacian: $$\nabla^2 f = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2\,\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\,\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Cylindrical Coordinates (r, ϕ, z)

Relations to rectangular (Cartesian) coordinates and unit vectors:

$$
\begin{aligned}
x &= r\cos\phi & \hat{x} &= \hat{r}\cos\phi - \hat{\phi}\sin\phi \\
y &= r\sin\phi & \hat{y} &= \hat{r}\sin\phi + \hat{\phi}\cos\phi \\
z &= z & \hat{z} &= \hat{z}
\end{aligned}
$$

$$
\begin{aligned}
r &= \sqrt{x^2+y^2} & \hat{r} &= \hat{x}\cos\phi + \hat{y}\sin\phi \\
\phi &= \tan^{-1}(y/x) & \hat{\phi} &= -\hat{x}\sin\phi + \hat{y}\cos\phi \\
z &= z & \hat{z} &= \hat{z}
\end{aligned}
$$

Line element: $d\vec{\ell} = \hat{r}\,dr + \hat{\phi}\,r\,d\phi + \hat{z}\,dz$

Volume element: $d\tau = r\,dr\,d\phi\,dz$

Gradient: $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$

Divergence: $\vec{\nabla}\cdot\vec{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rv_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$

Laplacian: $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$

Vector Formulae

In the following formulae, $\vec{A}$ and $\vec{B}$ are vector functions and ψ is a scalar function.

$$
\begin{aligned}
\vec{\nabla}\times(\vec{\nabla}\psi) &= 0 \\
\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) &= 0 \\
\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) &= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} \\
\vec{\nabla}(\vec{A}\cdot\vec{B}) &= (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A} + \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) \\
\vec{\nabla}\cdot(\vec{A}\times\vec{B}) &= \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B}) \\
\vec{\nabla}\times(\vec{A}\times\vec{B}) &= \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} \\
\vec{\nabla}\cdot(\psi\vec{A}) &= \vec{A}\cdot\vec{\nabla}\psi + \psi\vec{\nabla}\cdot\vec{A} \\
\vec{\nabla}\times(\psi\vec{A}) &= \psi\vec{\nabla}\times\vec{A} - \vec{A}\times\vec{\nabla}\psi
\end{aligned}
$$