PHYS 630: Homework III

due date: Tuesday, October 21st, 2008 at class meeting.

You are welcome to work together. If you partially use work from other (e.g. something you might have found in a book or a journal paper), you should properly credit the author by citing the material used.

1. Optics in a doubly-negative (left handed) meta-material (50 pts)

[Problem from Candidacy exam of September 2008]: In man-made dielectric and magnetic material unusual situations arise when the electric permittivity and magnetic permeability are both real and negative. We consider a plane wave with electric field \( E = E_0 \exp[-i(kr - \omega t)] \) and magnetic field \( H = H_0 \exp[-i(kr - \omega t)] \).

(a) Let first consider a standard material with permittivity \( \epsilon \) and magnetic permeability \( \mu \) both real and positive. Find the Maxwell equations satisfied by \( E_0 \) and \( H_0 \) in such a medium, the velocity of light in such a medium, and what combination of \( E_0 \), \( k \) and \( H_0 \) form a right-handed set. (10 pts)

(b) From now on we consider a doubly-negative meta-material and write \( \epsilon = -|\epsilon| \) and \( \mu = -|\mu| \). Write the Maxwell equations satisfied by \( E_0 \) and \( H_0 \) in the meta-material and show that the role of \( E_0 \) and \( H_0 \) are interchanged compared to the standard case explored in (1). (10 pts)

(c) What can you state about the set \( (E_0, H_0, k) \)? (5 pts)

(d) We now specialize to a plane wave propagating along the z axis and take \( E = E_0 \exp[-i(kz - \omega t)] \hat{x} \) and \( H = H_0 \exp[-i(kz - \omega t)] \hat{y} \) (where \( \hat{u} \) stands for the unit vector along the \( u \) direction). Find the Poynting vector \( \vec{S} \) in the meta-material and compare its direction to \( k \). (10 pts)

(e) Discuss the implication of (d) to the sign of the index of refraction. Consider an incoming optical ray propagating at the interface of a standard material (with index of refraction \( n_1 > 0 \)) and a meta-material (with index of refraction \( n_2 = -|n_2| < 0 \)). Take the incident and refracted angles to be respectively \( \theta_1 \) and \( \theta_2 \). Write Snells refraction law at the interface and draw a schematic stressing the difference(s) with the usual situation of refraction at the interface between two standard materials. (15 pts)
2. **Angular momentum beams as secure information carriers (20 pts):**

Read the attached paper from *New Scientist* and explain why light beam carrying angular momentum can be used in secure communications.

3. **Stokes parameters to characterize the polarization of a wave (30 pts):**

   (a) Using the $x$ and $y$ linearly polarized vectors

   \[
   \left\{ \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}
   \]

   as an expansion basis, the expansion coefficient for the Jones vector

   \[
   \vec{J} = \begin{bmatrix} A_x \\ A_y \end{bmatrix}
   \]

   are by definition $\alpha_1 = A_x$ and $\alpha_2 = A_y$. Find the expansion coefficient in the basis

   \[
   \left\{ \hat{e}'_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \hat{e}'_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}
   \]

   which will be called $\alpha_{45}$ and $\alpha_{135}$, i.e., $\vec{J} = \alpha_{45} \hat{e}'_1 + \alpha_{135} \hat{e}'_2$.

   (b) Similarly, find the expansion coefficient in the basis

   \[
   \left\{ \hat{e}''_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \hat{e}''_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\}
   \]

   which will be called $\alpha_R$ and $\alpha_L$, i.e., $\vec{J} = \alpha_R \hat{e}''_1 + \alpha_L \hat{e}''_2$.

   (c) Show that the $S_1$, $S_2$, and $S_3$ Stokes parameters (see Lesson 12) associated to the vector $\vec{J}$ can be written

   \[
   S_1 = |A_x|^2 - |A_y|^2,
   \]

   \[
   S_2 = |\alpha_{45}|^2 - |\alpha_{135}|^2, \text{ and } S_3 = |\alpha_R|^2 - |\alpha_L|^2
   \]

   note that the "0" Stokes parameters is $S_0 = |A_x|^2 + |A_y|^2$