Electromagnetic optics

• Optics IS a branch of electromagnetism

• Electromagnetic optics include both
  – Wave optics (scalar approximation)
  – Ray optics (limit when $\lambda \to 0$)

• Can also describes the propagation of wave in media characterized by their e.m. properties.
Maxwell’s equations

- Macroscopy Maxwell’s equations

\[
\begin{align*}
\nabla \cdot \vec{D} &= \rho, \text{ Coulomb law} \\
\nabla \cdot \vec{B} &= 0, \text{ no magnetic charges} \\
\nabla \times \vec{E} &= -\partial_t \vec{B}, \text{ Faraday law} \\
\nabla \times \vec{H} &= \vec{J} + \partial_t \vec{D}, \text{ Ampère- Maxwell law}
\end{align*}
\]

\( \vec{D} \equiv \varepsilon \vec{E} + \vec{P} \) is the electric displacement, \( \varepsilon \) the medium electric permittivity, \( \vec{E} \) the electric field \( \vec{P} \) is the polarization.
\( \rho \), is the charge density.
\( B \) is the induction, \( \vec{H} \equiv \vec{B} / \mu - \vec{M} \) is the magnetic field, \( \mu \) the magnetic permeability of the medium and \( \vec{M} \) the magnetization.
\( \vec{J} \) is the current.
Maxwell’s equations II

- Consider a linear, homogenous, isotropous, and non dispersive medium then Maxwell’s equations simplify to

\[
\begin{align*}
\nabla \cdot \mathbf{D} &= 0 \Rightarrow \nabla \cdot \mathbf{E} = 0; \\
\nabla \cdot \mathbf{B} &= 0; \\
\n\nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0; \\
\n\nabla \times \mathbf{H} - \partial_t \mathbf{D} &= 0 \Rightarrow \nabla \times \mathbf{B} - \mu \varepsilon \partial_t \mathbf{E} = 0.
\end{align*}
\]

- For now on we consider non-magnetic media ($M=0$) and $\mu=\mu_0$. 
Wave equation (revisited)

- From Faraday’s law

\[
\nabla \times (\nabla \times \vec{E}) + \partial_t \nabla \times \vec{B} = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} + \mu \varepsilon \partial_t^2 \vec{E} \\
= -\nabla^2 \vec{E} + \mu \varepsilon \partial_t^2 \vec{E} = 0
\n\]

- From Ampere’s law

\[
\nabla \times (\nabla \times \vec{B}) - \mu \varepsilon \partial_t \nabla \times \vec{E} = \nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B} + \mu \varepsilon \partial_t^2 \vec{B} \\
= -\nabla^2 \vec{B} + \mu \varepsilon \partial_t^2 \vec{B} = 0
\n\]

- So both equation can be cast in the form

\[
\begin{pmatrix}
\nabla^2 - \frac{1}{c^2} \partial_t^2 & \\
\vec{E} & \vec{B}
\end{pmatrix} = 0. \\
c \equiv \frac{1}{\sqrt{\varepsilon \mu}}
\]
Dielectric Media

- A dielectric medium is characterized by the medium equation, i.e. the relation between polarization and electric field vectors.

- Can also be viewed as a system with $E$ as input and $P$ as output characterized by an impulse-response function $h$

- The E-field generates electron oscillations in the medium that are responsible for polarization (and medium-induced E-fields)
Dielectric Media: some definitions

- **Linear media**: medium equation is of the form $\vec{P} \propto \vec{E}$

- **Nondispersive media**: $P(t)$ depends on $E(t)$ [at the SAME time]: the time-history of $E$ is not involved (assumed infinitely fast material, this is an idealization, but good approximation)

- **Homogenous media**: the medium equation $P = f(E)$ is independent of the position $r$

- **Isotropic media**: the medium $P = f(E)$ equation is independent of the direction

- **Spatially nondispersive**: the medium equation is local
Linear, homogeneous, isotropy, nondispersive media

• Medium equation is given by $P = \epsilon_0 \chi E$

• So electric displacement is $\vec{D} = \epsilon \vec{E}$

where $\epsilon = \epsilon_0 (1 + \chi)$ and the scalar $\chi$ is the electric susceptibility

• Maxwell’s equations are

  \begin{align*}
    \nabla \cdot \vec{E} &= 0; \\
    \nabla \cdot \vec{B} &= 0; \\
    \nabla \times \vec{E} &= -\partial_t \vec{B}; \\
    \nabla \times \vec{B} &= \epsilon \mu_0 \partial_t \vec{E}.
  \end{align*}

• The wave equation is

  $$\left[ \nabla^2 - \mu_0 \epsilon \partial_t^2 \right] \vec{E} = 0.$$ 

• Can define

  $$c \equiv \frac{c_0}{n} = \frac{1}{\sqrt{\epsilon \mu_0}}; \quad \text{and} \quad n = \left[ \frac{\epsilon}{\epsilon_0} \right]^{1/2} = [1 + \chi]^{1/2}$$

P. Piot, PHYS 630 – Fall 2008
Inhomogeneous media I

- Only inhomogeneity is considered (all the other properties are the same as in the previous slide)
  \[ P = \varepsilon_0 \chi \vec{E} \quad D = \varepsilon \vec{E} \]

- But now
  \[ \chi(\vec{r}) \quad \varepsilon(\vec{r}) \quad n = n(\vec{r}) \]

- If we consider the medium to be locally homogeneous (i.e. \( |\delta \vec{r}| \sim \lambda \)) then
  \[
  \left[ \nabla^2 - \frac{1}{c(\vec{r})} \frac{\partial^2}{\partial t^2} \right] \vec{E} = 0.
  \]
  \[ c(\vec{r}) \equiv c_0/n(\vec{r}) \]

Velocity of light is position dependent.
Inhomogeneous media II

- In the most general case (non-locally homogeneous)
- Start with
  \[
  \nabla \times (\nabla \times \vec{E}) + \partial_t \nabla \times \vec{B} = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} + \mu \epsilon \partial_t^2 \vec{E}
  \]
- Using
  \[
  \nabla \cdot \vec{D} = 0 \Rightarrow \nabla \cdot (\epsilon \vec{E}) = 0
  \Rightarrow \epsilon \nabla \cdot \vec{E} + \vec{E} \cdot \nabla \epsilon
  \]
- The wave equation is
  \[
  \nabla^2 \vec{E} - \frac{1}{c^2(r)} \partial_t^2 \vec{E} + \nabla \left( \frac{1}{\epsilon(r)} \vec{E} \cdot \nabla \epsilon \right) = 0
  \]

Velocity of light is position dependent

Can be neglected if \( \epsilon(r) \) varies over scales smaller than \( \lambda \)
Anisotropic media

• Only anisotropy is considered

• The polarization and dielectric displacement are now given by

\[ P_i = \sum_i \epsilon_0 \chi_{ij} E_j \quad \text{and} \quad D_i = \sum_i \epsilon_{ij} E_j \]

wherein \( \chi \) is a tensor (= a 3x3 array)
Dispersive media

- Only dispersion is considered

- The polarization and dielectric displacement are generally related via a differential equation of the form

\[ \alpha_1 \frac{\partial^2 \vec{P}}{\partial t^2} + \alpha_2 \frac{\partial \vec{P}}{\partial t} + \alpha_3 \vec{P} = \vec{E} \]

- The polarization can be written in term of an impulse-response function

\[ \vec{P}(t) = \varepsilon_0 \int_{-\infty}^{+\infty} h(t - t') \vec{E}(t') \, dt' \]
Nonlinear media

• Only nonlinearity is considered

• The polarization and dielectric displacement are generally related via a nonlinear equation

\[ \vec{P} = a_1 \vec{E} + a_2 \vec{E}^2 + a_3 \vec{E}^2 + \ldots \]

• Starting with

\[ \nabla \times (\nabla \times \vec{E}) + \partial_t \nabla \times \vec{B} = \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} + \mu_0 \partial_t^2 \vec{D} \]

and using \( \vec{D} = \varepsilon \vec{E} + \vec{P} \) the “wave equation” writes

\[ \nabla^2 \vec{E} - \frac{1}{c_0^2} \partial_t^2 \vec{E} = \mu_0 \partial_t^2 \vec{P} \]
Monochromatic waves I

- Up to now did not say anything about the electromagnetic field, we assumed they were real.
- It simplifies things (really) to write them in complex notation as a spatial dependence times an harmonic time function:

\[
\begin{align*}
\vec{E}(\vec{r}, t) &= \Re[\vec{E}(\vec{r}) \exp(i\omega t)] \\
\vec{B}(\vec{r}, t) &= \Re[\vec{B}(\vec{r}) \exp(i\omega t)]
\end{align*}
\]

- Then Maxwell’s equations (in a non-magnetic medium) become:

\[
\begin{align*}
\nabla \cdot \vec{D} &= 0; \\
\nabla \cdot \vec{B} &= 0; \\
\n\nabla \times \vec{E} &= -i\omega \vec{B}; \\
\n\nabla \times \vec{H} &= i\omega \vec{D}.
\end{align*}
\]
The optical intensity is given by the Real part of the time-average complex Poynting’s vector

\[ \overrightarrow{S} = \frac{1}{2\mu_0} \overrightarrow{E} \times \overrightarrow{B} \]

For linear, homogeneous, non-isotropic, non-dispersive media, Maxwell’s equations are:

\[ \nabla \cdot \overrightarrow{E} = 0; \]
\[ \nabla \cdot \overrightarrow{B} = 0; \]
\[ \nabla \times \overrightarrow{E} = -i\omega \overrightarrow{B}; \]
\[ \nabla \times \overrightarrow{B} = i\omega \epsilon_0 \mu_0 \overrightarrow{E}. \]

For dispersive media, the medium equation becomes

\[ \overrightarrow{P} = \epsilon_0 \chi(\nu) \overrightarrow{E} \quad \text{where} \quad \chi(\nu) = \int_{-\infty}^{+\infty} h(t) \exp[-2i\pi \nu t] dt \]
\[ \overrightarrow{D} = \epsilon(\nu) \overrightarrow{E} \quad \text{with} \quad \epsilon(\nu) = \epsilon_0 [1 + \chi(\nu)] \]